Unicorn Fountains

Edward Anderson[1]

Abstract

Here is a start on nomenclature for the AI age. In which automated research assistants are available to everybody, and at least frontier models incur proof-plentifulness in Mathematics. A basic unicorn involves applying a strand of second-year Mathematics to another such so as to obtain new research. These used to be scarce. But systematic convex span sweeping renders these plentiful, and AI is well suited to scale this venture.

In the new age of plentiful unicorns, these can arise in batches: unicorn fountains. In numbers warranting a comparative taxonomy, such as unicorn depth and fraction menus provided in the current Article. And adjectives for unicorns, in which regard we entertain basic, beautiful, hidden, aesthetic, arenaic and toy. Beautiful unicorns enable a directed lode-mining counterpart to sweeping. All in all, one has *convex span minesweeping*: with the last word meant in the new sense of directed mining as well as undirected searches.

We furthermore consider some interplays between the above adjectives, many of which can be illustrated using just Heron's formula. Another concrete toy in the current Article is Grinberg as a toy for Barnette. From the point of view that both of these planar graph Hamiltonian necessity theorems share a hidden beauty: index theorems. The Steinberg–Saxl staircase unicorn–toy unicorn pair is also highlighted as doubly beautiful. This bridges between symmetric group and Lie algebra Representation Theory.

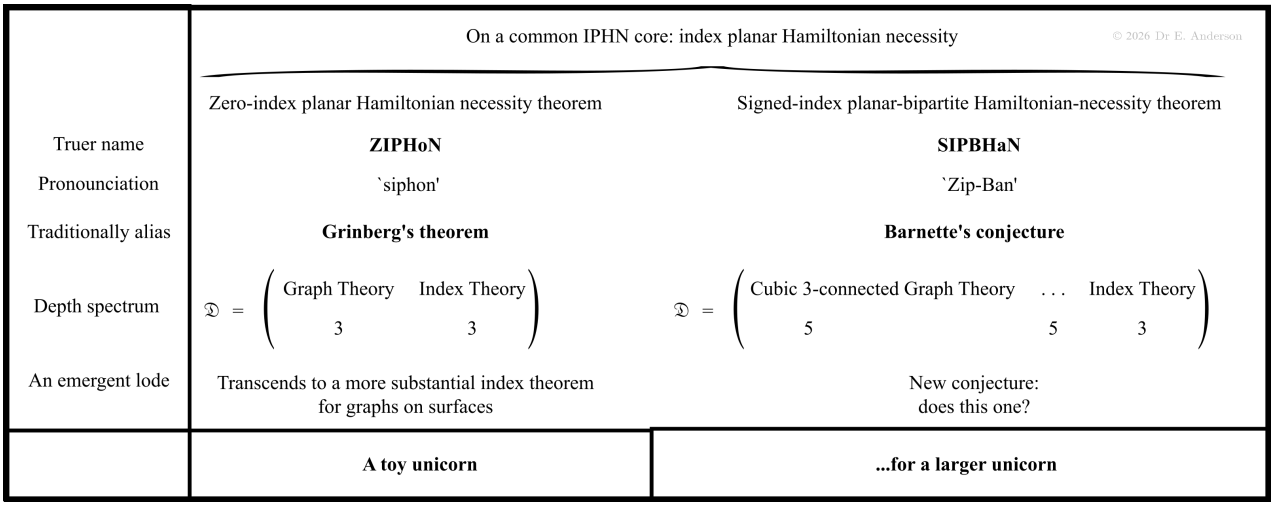


Figure 1:

Cite as E. Anderson, "Unicorn Fountains", Online Encyclopaedia of Applied Graph and Order Theory, institute-theory-stem.org/oeagot-ai-age-unicorn-fountains (2026).

[1]Dr.E.Anderson.Maths.Physics *at* protonmail.com Institute for the Theory of STEM.

This work is ⟨3⟩ : understandable by third-year undergraduates.

Date stamp: v1: 18-11-2026.

Copyright of Dr E. Anderson.

1 Introduction

1.1 Convex span sweeping

Motivation 1 Let us entertain weaving together threads from different branches of Mathematics. Certain combinations have long been considered, typically at the graduate level or higher: $\langle \geq 4 \rangle$. For instance applying Differential Geometry ideas to Lie Theory. Or paralleling finite-group Representation Theory with that of compact Lie groups.

Principle 1 Such weaving is universal: *any combination of no matter how many threads* is capable of Mathematical usefulness.

Naming Remark 1 One name for this is linear span sweeping.

Remark 1 This is a cornerstone of our part of the work that the current Article is about. Much of which was done in 2018-2020. And around two-thirds of which has been written up over 2017-2026. We have furthermore added around an extra fifth, which is being written up over 2024-2028 alongside the remaining third of the original.

Naming Remark 2 A technically harder, and yet in part technically more correct name is *convex hull filling* [87].

Harder because convex is two leaps from linear; the stepping stone is affine.

More correct since convex combinations are bounded above and below whereas linear ones are not. At precisely 0 and 1, bounding the proportion of the subject used. Which cannot be negative or in excess of the full subject.

Pedagogical Remark 1 The unnecessary jargon is 'hull', since this is just a synonym of span. With $O(100)$ more students having seen and understood linear spans than convex hulls, "hull" obstructs pedagogical access. So 'linear span sweeping' immediately gets through to $O(100)$ as many students.

Naming Remark 2 continued While at a more nuanced level, one should explain that linear (Chapter 9 of [72]) generalizes to affine (Chapter 10) and then to convex (Chapter 11). With all of combinations, (in)dependence, span and basis having a version in each of these somewhat analogous types of Mathematics. Among which it is convexity that offers a better model, for its precisely matching the above bounds. And yet the word 'span' remains useful. Hence our verdict for now: *convex span sweeping*!

Why 'sweeping' and not 'filling' [66]? Because sweeping moves around a temporary probe to find useful cases. As opposed to occupying all possibilities without discerning between useful cases and others not known to be useful. Compare minesweeping! Where the objective is to find all the mines, not to fill up the sea full of minesweeper boats...

1.2 Proof scarcity versus proof plentifulness regimes

Remark 1 By itself, convex span sweeping increases not only the repertoire of known interconnections between fields of STEM. But also the number of proofs.

Increased compute also acts in this direction, and the two reinforce each other.

By now, OpenAI has publicly declared [88] a large number of claims about proofs of, or substantial pieces of progress toward, longstanding conjectures in Mathematics. For now, only a proportion of these proofs are Lean-certified, nor is Lean certification by itself conceptually watertight. Nonetheless, many of these claims will be debated for some years, and it is conceivable that a sizeable proportion of them will pass. Such declarations are also beginning to branch out into other STEM subjects.

Remark 2 So Mathematics can no longer feasibly be run as a proof scarcity regime. In any case, such scarcity is not a property of the Platonic space of Mathematically-sharp concepts and their unexpected inter-relations.

Firstly, because convex span sweeping permits posing and investigating many new questions.

Secondly, because tying hire, funding and promotion of Mathematicians to proving results from certain problem lists, rather than to discovering and posing their own problems, has ceased to be viable. For quite a large proportion of these well-known, listed problems has suddenly been claimed to be solved. This additionally creates uncertainty as regards whether AI will continue to crack the Mathematicians' traditional prescribed list of problems. Creating practical need to discover and pose further problems. Indeed including making new conjectures in reaction to the changing landscape of resolved conjectures. We leave for another occasion how AI has changed which kinds of problem are likely to remain longstanding...

Naming Remark 3 Instead, today we point out that the proof-plentiful era requires its own vocabulary. Due to qualitative differences from the proof-scarcity era.

This is why we put in some detailed Naming work above to arrive at 'convex span sweeping'.

The above two regimes are themselves further such. Let us here coin *Probatiocopia* and *Probatiopenuria* for this qualitative distinction between the present and future on the one hand, and the past on the other. With common root the Latin for 'proof'. Compounded either with the 'copia' of 'cornucopia', or indeed 'copious'. And 'penuria', which is common in French as 'pénurie' while having itself become far more scarce in English!

1.3 Unicorn fountains

Naming Remark 4 The titular 'unicorn fountain' is indeed yet another such. And one that we shall be styling adjectives for below.
For now let a *basic unicorn* be the following. Applying at-most second-year Mathematics to at most second-year Mathematics to obtain new research. I.e. a convex span sweep of to most depth $\langle 2 \rangle$.

A *basic unicorn fountain* is then a procedure that creates a batch of basic unicorns.

Open-endedly upon say isotropically extending to $\langle a \rangle$-*unicorns* for increasing a .

1.4 Sweeping fraction menus

Structure 1 This is a $2 \times c$ array with subjects on the top row. With the second row consisting of numbers p_i in the continuous interval $(0, 1]$. Such that

$$\sum_i p_i = 1 .$$

These are 'the coefficients in the convex combination'. This is a first simple navigational tool for convex span sweeping results.

Naming Remark 5 Let us explain this use of "partial" or "fraction" via motivating examples.

1) *Partial masses* as normalized by the total mass of a system.

2) *Partial moments of inertia* as normalized by the total moment of inertia of a system.

3) *Partial pressures* as normalized by the total pressure of a sample.

4) *Partial entropies* as normalized by total entropy of a system.

5) *Partial molarity* as normalized by total molarity of a sample.

All of these can also be termed X fractions in place of partial X.

1-4) are Physical. 5) is mostly contemplated by Chemists. Though all the others in this list play a role in "Theoretical Chemistry" or "Molecular Physics". For instance, equating partial vapour pressures to partial molarity constitutes a fractional formulation of Raoult's law.

Incidentally, this pins convexity interpretations on all of the above notions, resting on the non-negativity of each property. So e.g. in contrast 'partial charge' is *not* a convex notion.

Remark 1

$$p_i = 0$$

is excluded as one of the many subjects not contributing to a result. Though not contributing to one proof of a result does not imply that it does not contribute to some other proof. This gives a reason to operate proof by proof, not result by result...

Naming Remark 6

$$p_i = 1,$$

is however usefully included as a pure-subject result. To be contrasted with multiple nonzero p_i being involved. Comparing this pair with pure versus mixed states in QM [33, 61] makes for an interesting natural square. Indeed motivating our names 'pure' and 'mixed'. Which apply both to statements of results, and separately to each specific proof.

Remark 2 The symmetric formulation

$$p_i \in (0, 1)$$

comprises all of the unicorns. Now excluding the useful pure limiting case as well as the useless irrelevant case. While the symmetric formulation

$$p_i \in [0, 1]$$

keeps the baby at the expense of also keeping the bathwater.

Caveat 1 Returning to the current application, however, it is not clear how to quantify the proportion of each subject in a mixed-subject result. Should this go as the number of notions used from that subject? The number of lemmas used that the subject contributes to? The number of lines that the subject contributes to a minimum-length Lean code? Because of such operational ambiguities, we do not use unicorn fractions in the current Article. Pointing out this matter instead as a limitation on quantifying convex span sweeping! [2]

[2] This caveat updates part of [66] to include the subsequent topical example of Lean proofs.

1.5 Sweeping depth menus

Remark 1 Let us now turn to the generic situation of anisotropic unicorns, starting with $\langle a, b\rangle$-*unicorns*. So as to emphasize using a relatively harder tool to solve an a priori simpler problem or vice versa. As extending to the following more detailed lists.

Notational Remark 1 The *sweeping depth menu* $\mathfrak{D}$ is another $2 \times c$ array. Whose first row is again a subject list. But now whose second row is the maximum level $\in \mathbb{N}_0$ to which each subject strand is taken. This is a second simple navigational tool for convex span sweeping results. Indicating how far along one's studies in the underlying 'standard convex basis' of subjects one needs to go to encounter each of the result's strands' definitions and techniques.

Example 0 So say a Generalist who is isotropically $\langle 4\rangle$ – i.e. has mastered all STEM that should plausibly be taught up to fourth year – is at least piecemeal-ready to handle a $\langle 4\rangle$-unicorn. And likewise for a Research Group who between them provide an isotropic cover of STEM theory up to $\langle 4\rangle$. For all that piecemeal-readiness need not imply composition comprehensibility: the derivation's total complexity may well exceed that of its constituent parts.

Of course, it may be that a person happens to know and understand the particular Group Theory $\langle 4\rangle$ technique required, rather than all of Group Theory up to this level. Or, if not, that they can identify and learn the specific technique missing from their repertoire. However, STEM Theory is built up sequentially. So a person who is $\langle 2\rangle$ at Group Theory may well find that mastering a $\langle 4\rangle$ technique in Group Theory requires reverse-engineering that one needs a large chunk of $\langle 3\rangle$ Group Theory. With both the reverse-engineering and the largeness of the chunk slowing down one's response time to the innovation. In this way, the sweeping depth menu indicates who is immediately ready to face the intricacies of the new combination without having to preliminarily invest in its strands. Including from the point of view of arranging collaborations on the basis of forming a team who jointly pass this prerequisite.

2 Basic unicorns

2.1 Unicorn calibration

Remark 1 Our standardization that a basic unicorn is shorthand for a $\langle 2 \rangle$-unicorn is arbitrary and yet somewhat useful. It corresponds to the adage that at least since the 1970s, almost all PhD studies in STEM Theory involve zero unicorns. Whether found by the student's own volition, or by their PhD supervisors passing them one to develop.

$\langle 1 \rangle$-unicorns and $\langle 0 \rangle$-unicorns are expected to be much rarer. Not least out of those 'lower reaches of $\mathfrak{P}$latonia ' having been exposed to centuries of the best minds. From say Euclid, Archimedes, Apollonius, Ptolemy, Pappus and Diophantus. Via say Newton, Leibniz, Euler, Lagrange, Cauchy, Gauss and Riemann. To the many luminaries operating between around 1880 and 1970.

Let us introduce $\mathfrak{B}$asica our standardization's truncation of $\mathfrak{P}$latonia . I.e.

$$\mathfrak{B}\text{asica} \ = \ \mathfrak{P}\text{latonia}|_{\langle \, \leq \, 2 \rangle} \ .$$

2.2 Explosion in Mathematics induced a rupture in systematic sweeping

Remark 1 However, historically something started to go wrong in the 1930s as regards systematic sweeping. Mathematics, and STEM theory more generally, suddenly got much larger, and huge public polymaths essentially ran out with von Neumann.

Example 1. Flat Geometry and Linear Algebra. Nobody systematically back-tracked what Linear Algebra *in detail* does for Flat Geometry. Be this Flat Geometry to second-year university standard, or first-year , or Olympiad, or even basic school. I.e.

$$\mathfrak{D} \ = \ \begin{pmatrix} \text{Flat Geometry} & \text{Linear Algebra} \\ \text{0-2} & \text{1-3} \end{pmatrix} \ .$$

Many would have presumed that this had already been done. And some of the arguments for this would have lacked nuance.

For instance, determinants *had* been applied in detail to all of STEM theory in say 1850-1930, from Sylvester and Dodgson (alias Lewis Carroll) to Muir. But the matrices that these determinants correspond to *are not the only ones* usefully attributable to Flat Geometry. For instance, there is a quadratic-forms implementation in which the Heron matrix is 3×3 [29]. Whereas the well-known Heron determinant is 4×4 ...

This kind of approach is further obscured by its Linear Mathematics being built on top of substituting the natural variables of Flat Geometry for adapted variables that are squares of the originals. I.e. *weak Conway variables* [96]. By which observation, sporadic previous use of quadratic forms [29] became systematized. Returning a large number of interesting Flat Geometry results based on applying Linear Mathematics-in-detail. As we worked out a first batch of in 2017-2024.

Other of these unicorns are more anisotropic, say $\langle 1, 3 \rangle$ or $\langle 0, 3 \rangle$. Because 'Linear Algebra in detail' is reasonably capable of transcending from what would be in the currently standard second-year course to an almost everywhere hypothetical third-year course [72] extending the second year course. Say to the level of [47, 48, 72].

A further element of some of the above unicorns is Geometry to N-body problem fusion. I.e. that intercalating elements from these two previously distinct subjects is not only internally consistent but also strengthens both of them, in the manner of a unified theory. See Sec 7 for a few specifics.

2.3 Further examples

Example 2. Combinatorics of everything. Another hole is that it became standard to dismiss problems across STEM that reduced to basic Combinatorics as essentially solved. Without actually writing out the answer or its basic Combinatorics derivation. This often did not appear to be necessary for 'publishability'. Non-Combinatorialists may have been reluctant to be explicit out of concern for pitfalls that Combinatorialists could easily spot. While Combinatorialists did not view it as part of their job to systematically review other subjects. In the process synthesizing all the implied and yet not explicitly written down applications of Combinatorics ⟨0-2⟩ to that subject.

And so in 2018 [66] we estimated that it would take a Human research group around 15 years to novelly systematically apply Combinatorics to the rest of basic STEM Theory. We have had a reasonable go at this, firstly finding a way to apply Combinatorics to itself in [69], further detailed in Sec 7. And to the Flat Geometry-N-body complex in e.g. [63, 76]. And also to Linear-Algebra-in-detail in e.g. [72, 86, 84]. So far, these are

$$\mathfrak{D} = \begin{pmatrix} \text{Anything} & \text{Combinatorics} \\ \text{0-2} & \text{1-2} \end{pmatrix}.$$

Group Theory [52, 50], Representation Theory [37] and Lie Theory [49] have already Combinatorialized themselves relatively effectively. So new patches here are smaller. Differential Geometry, somewhat less so; take [49, 84] as preliminary indicators of some directions here.

Example 3. Adapted variables for everything. A reverse strike is to import *adapted variables* – common in ⟨ $\geq$ 3⟩ Physics [6, 36] – into not only Flat Geometry and Differential Geometry but also Combinatorics. For instance, Sec 1.4's fraction variables and Sec 2.2's weak-Conway variables are simple examples of adapted variables. The usefulness of this strand depresses the likes of relative differences, and forming dimensionless ratios, into several lectures that are indispensable in the new world's second-year or even first-year courses. Here

$$\mathfrak{D} = \begin{pmatrix} \text{Anything} & \text{Adapted variables} \\ \text{0-2} & \text{2} \end{pmatrix}.$$

Various Combinatorial unicorns of this type are displayed in [69]. The anisotropy of a partition's Young diagram's hugging rectangle is a marquee example.

Pointer 1 By now, each of [69, 72, 96] now carries, or points to further reading about, around 40 basic unicorns apiece.

2.4 Unicorn lodes and convex span minesweeping

Naming Remark 7 We coin *unicorn lodes* to describe how quite deep understanding of basics permits extension away from a first success. This is a mining word, now in the mineral rather than naval sense... By which a new meaning for the word '*minesweeping*' encompasses all of the following. Complete coverage and random searches: sweeping. And directed searches following basics – the current section – or beauty – the next section – i.e. types of mining.

Example 4 For instance, in Geometry, from Heron into Brahmagupta and Bretschneider for cyclic and convex quadrilateral counterparts. Or from Apollonius to Stewart: medians to arbitrary Cevians. And then using Jacobi trees – mapping to at-most binary (AMB) trees [76] – to place infinite-family extensions on these [77]. In the process finding that Euler's quadrilateral theorem is a nearest neighbour for Apollonius' theorem. And yet only one of two: the H-shaped one as opposed to the K-shaped one.

3 Beautiful unicorns

Examples We elect to explain this organically: with examples first. Some examples of basic and yet exhalted *citizens of* $\mathfrak{K}$*allista* are as follows.

1) The *Petersen graph* $\langle 3 \rangle$: counterexample to almost everything [30]. Among many other things, this graph is the 5-cage: smallest cubic graph of girth 6 .

2) The *Fano plane* $\langle 4 \rightarrow 1 \rangle$ [80], which is the smallest Projective configuration and indeed the smallest Projective plane. Whose incidence graph is alias the *Heawood graph*: key example in colouring surfaces for the properties that it confers to the torus. And is also the 6-cage.

3) The *Pappus* and *Desargues configurations* $\langle 2 \rangle$ [7], underpinning the two key structural theorems for Projective planes. Their configuration graphs and their incidence graphs $\langle 3 \rangle$ [81].

4) The *integer partitions* and the *Combinatorial Young diagrams* for these $\langle$1-2$\rangle$ [69, 84].

5) The *symmetric groups* $\langle$1-2$\rangle$ [37].

6) The *semi-simple Lie algebras* $\langle$3-4$\rangle$ [8].

7) The Young diagrams again, and their decoration to form the *Young tableaux* [37], for the representations of the last two $\langle$4-5$\rangle$.

8) The *Hopf maps* $\langle 3 \rightarrow 2 \rangle$, such as Hopf's little map [46] – smallest non-trivial fibre bundle in many ways – and Hopf's generalized map $\langle$4-5 $\rightarrow 3\rangle$ [24].

Pedagogical Remark 1 The above are almost always major sins by omission or by minimization in undergraduate curricula. Minimized, rather than entirely suppressed, cases are as follows. The symmetric groups. The integer partitions. And piecemeal use of some Lie algebras, such as $so(n)$ and $su(n)$.

Some other citizens of $\mathfrak{K}$allista are however entirely standard and widely used from the point of view of undergraduate curricula. At $\langle$1-2$\rangle$ these include $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ and the Platonic solids.

Motivation 1 There are various reasons why people pursuing Mathematics in the AI age would do well to explore *all of* the basic beauties. They *heavily* recur in a priori unrelated parts of STEM theory. Each citizen *arises naturally along many conceptually distinct lines of thought*. As a concrete basic example, the Heron matrix is also the cycle of cosine rules matrix and the cycle of triangle inequalities matrix (Article 2 on [64]).

Motivation 2 Citizens of $\mathfrak{K}$allista *accumulate many interesting properties*, including *realizing multiple higher notions of symmetry*. In other words, they are uncannily widely applicable, multi-faceted and have long property lists. By which they *interconnect subjects*, and feature as *dominant objects in many a Wille concept lattice* $\langle 4\text{-}5 \rightarrow 2\rangle$ ([43] and Sec 7.5). *Which is a key technique for systematic study of conceptual variety in the AI age, and conversely also serves as a detector of citizens of Kallista...*

Exercise 1 Can you think of any further basic and yet exhalted citizens of $\mathfrak{K}$allista ?

Definition 1 A *beautiful* alias *callyxtine unicorn* is one that manages to attribute a new feature, or a new application, to a citizen of $\mathfrak{K}$allista . By combining it with some basic subject, be it as a new use of a tool, or as a new setting. For instance, at the basic level,

$$\mathfrak{D} = \begin{pmatrix} \mathfrak{K}\text{allista} & \text{Any tool or any setting} \\ \text{0-3} & \text{0-3} \end{pmatrix} .$$

Recommendation 1 We very strongly recommend that everyone reading this who has freedom of choice of topic for a dissertation or a PhD to pursue the following recommendation, or one of the other recommendations listed below. And likewise in the context of distinctive open-ended unscripted problems to work on at Tech companies.

Use essentially only basic beauties as examples.

['Essentially only?' Trivial cases, and cases too small to be beautiful are of some preliminary value...]

Have this piece of work of yours be entirely about new facets of basic beauties. Find new heirs of basic beauties, or other middlingly-basic beauties, using AI. Check carefully that what you find in this regard does indeed possess the intended properties.

Remark 1 Let us postpone specific examples of beautiful unicorns to the first and third sections below...

4 Hidden unicorns

Example 1. Hidden basic. The triangle inequality in squared variables returns Heron's formula. With the following hidden extra strand: that area non-negativity encodes spacelike separation vectors. And null edge cases (Article 2 on [64]). This is how one of my ab initio 2-strand unicorns became 3-strand.

In Montgomery's alternative route [78], he geometrized Heron's formula to obtain a copy of the Minkowski spacetime of Special Relativity. Apparently without realizing its possessing an even more basic strand: the triangle inequality. For treating the N-body problem separately from Flat Geometry has not yet found Flat Geometry's weak Conway variables as a heavily recurring trick.

Example 2. Hidden beauty. The eigentheory of the Heron matrix provides its own derivation [64] of Hopf's little map. And two further little beauties, which we present three sections down.

5 Less basic unicorns

Remark 1 Our own group's publicly declared selection of such is as follows.

1) The cubic graphs of 'higher Graph Theory' [79] $\langle \geq 3 \rangle$. As opposed to 'basic Graph Theory's' graphs whose vertices have whichever degree.

Exercise 2 Many of the smaller citizens of $\mathfrak{K}$allista are cubic graphs. Which of 1)-8) in the previous section are cubic graphs? Also see if you can find 10 further cubic-graph citizens of $\mathfrak{K}$allista !

2) A further type of multi-faceted beauty: Index Theorems, which bridge between Analytic and Topological information. Toys included. The Rank–Nullity Theorem of Linear Algebra is a well-known toy ([56], chapter 17 of [72]), as is the Gauss–Bonnet Theorem [85]...

3), 4) The topics covered in the next two sections conclude our group's current publicly declared list.

6 Aesthetic unicorns

6.1 Graph Drawing and Visualization (GDaV) is relatively mature

Remark 1 (GDaV) has come to be formalized into a series of styles. Each optimizing for features that extremely seldom can be co-optimized. For instance, maximizing a graph's presentation's symmetry. Versus planar presentation, or, that failing, minimum-crossing presentation. Versus Hamiltonian presentation, or, that failing, maximum-cycle presentation. Versus unit-distance edge presentation, or, that failing, minimum edge-distance number presentation. Versus fitting a grid, or possibly a tessellation.

Many of these styles were detailed in the Handbook of Graph Drawing and Visualization edited by Roberto Tamassia $\langle$6-8 $\rightarrow$ 4-5$\rangle$ [57]. Not all, however, with say some aspects of Tessellation-fitting and minimum edge-distance number our own group's 'extra chapters' [82].

Example 1. Basic GDaV unicorns. Target basic graphs specifically with Tamassia-level GDaV.

$$\mathfrak{D} = \begin{pmatrix} \text{Anything} & \text{GDaV} \\ \text{0-2} & \text{4-5} \end{pmatrix} .$$

Our group picked out around 40 further unicorns in this way.

Example 2. Beautiful GDaV unicorns. Target beautiful graphs specifically with Tamassia-level GDaV.

$$\mathfrak{D} = \begin{pmatrix} \mathfrak{K}\text{allista} & \text{GDaV} \\ \text{0-3} & \text{4-5} \end{pmatrix} .$$

Our group has publicly declared around 40 of these unicorns. See [82] for most of both (and more eventually all the ones that we found!)

Example 3 GDaV unicorns also feature in developer-facing observability tools for tracking LLM pipelines.

Recommendation 2 Involve GDaV in your work of choice. If it is in Graph Theory. *Or* if you find your work of choice is afforded some useful insights when viewed as a piece of Applied Graph Theory.

Remark 2 Example 3 is an active technical use in Coding and Machines. GDaV however has a separate use in Human interpretation of Machine output. Namely, Humans are more likely to be able to follow, find errors in, and find suboptimalities in not preprint writeups or Lean proofs about Graph Theory but GDaV output... This is because many Humans skilled in Graph Theory are rather visual operatives...

6.2 Optical debuggers for navigating unicorn fountains

Naming Remark 8 It is worth pointing out that unicorns' multi-strand nature itself sources a need for DaV. Namely in the form of *optical debuggers*. That pick out each strand in a separate colour.

Pointer 2 Our group has been operating unicorn fountains since at least as far back as 2017 [64, 69]. With Graph Theory and Riemannian Geometry precursors on a smaller scale to rather less depth for substantially longer than this, as outlined in the next section. As such, it is unsurprising that we already have 3 optical debuggers on various patches of what we do.

Example O.1 The "Tensor Networks throughout the Basic Theory of STEM" series that [84] belongs to is deploying an optical debugger called 'Rainbow Penrose'. This uses tensor network diagrams, colouring both the boxes around tensor symbols and the unusually thick legs that replace indices in this diagrammatic approach. This has various useful and reuseable features.

Firstly, background boxes rather than coloured text or symbols. For given multi-strand objects, it is far easier to stack, splay or fan background boxes than it is to multi-colour symbols. Both as regards production and for Human navigation.

Secondly, the thickness of the legs is motivated to support candy-patterning. By which subtypes of index can be grouped by colour and yet be further distinguished by patterning.

Example O.2 In the current context of unicorn strand tracking, one possibility is to use 1 colour per branch of Mathematics involved. Followed by *candy-patterning* to track levels. If this were done, then say the Geometers would know to hunt for all the purple boxes in a preprint drop. The Topologists, for the blue. But the Geometers whose systematic study of Topology got up to say $\langle 4 \rangle$ could do the following. Filter out from all the purple-containing preprints the ones that contain blue candy $\langle \geq 5 \rangle$.

It clearly makes sense for this kind of optical debugging to be very straightforwardly adjustable by the reader. As extension of the mainstream proposal for the personalization of study function of AI in Education.

6.3 Whichever other DaV

Recommendation 3 Extend the range of DaVs available to Humanity. So as to extend GDaV's advantages to further branches of Mathematics; see [94] for further details.

Remark 1 Aside from GDaV and tensor networks, let us highlight the following.

Pointer 3 Poset DaV is a subcase of GDaV, and Wille concept lattice DaV a subset of that. Given what else we say about these structures, they are particularly deserving of an expansion pack of DaV styles.

Pointer 4 Tensor networks DaV is another subcase of GDaV. Some of the above developer-facing observability tools are of this kind. See [83] for a recent public demonstration of style, and [60, 67] for some more historical back-tracking.

Pointer 5 Integer partitions are *well-DaVed* by Young diagrams. For all that further layers of DaV can be built thereupon [69, 84].

7 Arenaic unicorns

7.1 Arenas

Structure 1 Given some particular type of Mathematically well-defined object o , the corresponding arena $\mathfrak{O}$ as follows. The Mathematical space formed by the totality of these objects.

These $\mathfrak{O}$ are not just sets; one should ask what natural Topologies they carry.

The following turn out to be quite widespread in basic examples.

A) Orders.

B) Analysis or Geometric metrics.

C) Measures.

D) Rings.

Example 1 Configuration spaces in Physics [5] and Dynamics [21] are an archetype.

Application 1 Isham [23, 26, 27, 28] set about Quantizing everything.

Motivation 1 Quantum Mechanics furthermore unfolds on configuration space. Arenas are the basis of another perspective on Classical Dynamics [35, 78] and a meaningful Probability Theory as per Kendall [42] as well as of Quantization. So the Author asked what *all* the generalized configuration spaces of STEM objects are. And proceeded to systematically work this out over $\mathfrak{B}$asica .

7.2 Avoiding being lost

Remark 1 One practical observation here is that Topological study, followed by A)-D) viewed as an open list of finer levels of structure, is *vastly* more productive than anything in the previous sentence. I would go as far as calling the pursuit of anything in the anti-penultimate sentence, without the penultimate sentence's program, as *ill-conceived*. As an extension of Galileo's adage '

"*He who attempts Natural Philosophy without Geometry is lost*".

To the Modern Applied Topology Ages, as follows.

"*Those who attempt the Theory of STEM without Modern Applied Topology are lost.*" [66]

7.3 Aside on Modern Applied Topology Ages versus previous Ages

Remark 1 Where the Modern Applied Topology Ages so far could be viewed as follows.

1) The first Modern Applied Topology Age, whose guidebook is Ghrist [56].

2) The *Big Data age*, in which Big Data and successor fields attained maturity [58].

3) The *LLM age*, with these freely available and widely useful, which started in March 2023 [71].

4) The *AI age*, in which AI frontier models ushered in proof-plentifulness in Mathematics. Which became undeniable on October 6 2026 [88].

Motivation 2 Such Ages are qualitatively distinct from each other. Compare the well-known expressions Stone Age, Bronze Age and Iron Age. Also the Renaissance, the Age of Universal Law alias Scientific Revolution, centred about Newton, the Enlightenment and the Steam Age alias Industrial Revolution. And finally with the Automobile, Air, Computer and Internet Ages.

7.4 Arenization as a map that is generative

Remark 1 Arenization is a map, *Arenize* , from the theory of STEM to itself.

Remark 2 *Arenize* often kicks one out of one branch of Mathematics into another.

Remark 3 *Arenize* is *generative*: mapping out of a given era's *known* STEM Theory. A heavy-handed proof of concept here is as follows. That the first way in which Topos Theory was arrived at by Grothendieck [19]: as the totality of all sheaves.

7.5 Arenization fixed points and other subject-pair collapses

Pedagogical Remark 1 The matters outlined in the current Section render arenas so significant that not only should configuration space be taught to all second years. But also generalized configuration spaces and arenization should be widely exemplified in second and third year courses.

Setting up access to the following 3-strand unicorns.

$$\mathfrak{D} = \begin{pmatrix} \text{Anything } O & \text{Configuration space} & \text{Anything } A \\ \text{0-2} & \text{4-5} & \text{variable} \end{pmatrix} .$$

Where A is the piecemeal answer for what each object type O maps to under *Arenize* .

Example 1. A basic collapse.

$$\mathfrak{D} = \begin{pmatrix} \text{Combinatorics} & \text{Configuration space} \\ \text{0-3} & \text{2} \end{pmatrix} .$$

Where now *Arenize* just tamely maps a piece of Combinatorics to another piece of Combinatorics.

Example 2 As an actual arenization fixed point, Gromov [22] showed the following.

The arena of metric spaces is itself naturally and usefully a metric space. Namely that carrying the Gromov–Hausdorff metric.

Example 3 While at the opposing 'discrete rather than continuum' 'easy end' of Topology, S. Sánchez provided the following [66, 69].

The arena of posets is itself naturally and usefully a poset.

This is nested within Example 1's *merely localized collapse*.

Example 4. A heterogeneous basic.

$$\mathfrak{D} = \begin{pmatrix} \text{Linear Algebra} & \text{Configuration space} & \text{Combinatorics} \\ \text{1-3} & \text{2} & \text{0-3} \end{pmatrix} .$$

Remark 1 The Author accumulated a large reservoir of Graph Theory treatments of arenas along the above two lines. S. Sánchez then showed me that upgrading these to Order-Theory treatments considerably strengthens the material. And so the Author wrote the Book [69] to cover Example 1. [86, 84] and more eventually links on [72]'s homepage are accommodating Example 2, which was left as an Exercise.

Example 5 Aside from Isham's works and mentoring, the Author was also influenced in the early parts of the above venture by the following abstraction from Kendall's work [42].

$$\mathfrak{D} = \begin{pmatrix} \text{Flat Geometry} & \text{Configuration space} & \text{Differentiable Manifolds} \\ \text{0-2} & \text{2-3} & \text{2-4} \end{pmatrix} .$$

Example 6 A. Ford and the Author [63, 76, 64, 96] subsequently compiled a number of complementary examples for the Flat-Geometry N-body problem complex.

$$\mathfrak{D} = \begin{pmatrix} \text{Flat Geometry–}N\text{-body} & \text{Configuration space} & \text{Combinatorics} \\ \text{1-3} & \text{2} & \text{0-3} \end{pmatrix} .$$

Here is a concrete example of further flat Geometry unicorns of this kind. For triangles, the following matrices are multiplicatively compatible. The CoM-removing projector – 3-body subcase of an N-body input – the Apollonius involutor and the Heron matrix. Thus they define an algebra: the *triangle matrix algebra* (Articles 3-4 on [64]). Which on inspection is commutative, and share all eigenspaces.

Example 7 The Wille collapse is to arenize any subject's objects, with attribute lists. As a *Wille concept lattice*, so the target space is a piece of Order Theory and thus of Combinatorics. The range of source spaces for this arenaic map is universal (rather than even restricted to some part of the Theory of STEM).

$$\mathfrak{D} = \begin{pmatrix} \text{Anything's objects and attributes} & \text{Combinatorics} \\ - & 3 \end{pmatrix} .$$

7.6 Arenization's interplay with beauty

Motivation 2 S. Sánchez [66] also pointed the Author at the strategy of centering arenizeation about the beautiful objects. These last two approaches intersect to give the program of arenizing all the basic beauties. Placing the basic beauties in larger arenas of heirs, and within arenas of meaningful types of basic object.

Remark 1 Some arenas are themselves further citizens of $\mathfrak{K}$allista , such as the Young lattice [40, 37]. Or indeed the topoi as at least historically top-notch citizens...

Remark 2 Some arenas constitute another find of a citizen that had previously been found to arise elsewhere. Such as the power sets of all labelled subsets of a finite set. Which occur elsewhere as say Boolean algebras or groups all of whose non-identity elements are of order 2 .

Remark 3 Consider embedding arenas in larger arenas. Then sufficiently large embedding arenas include uninteresting objects and have some tendency to be Mathematically regular. Both of which are undesirable. While quotienting arenas down to smaller arenas sharpens what objects remain included and distinguishable. And causes all of Mathematical irregularity, Mathematical intractability and structural sparseness. This conceptualization is moreover a Goldilocks: the most interesting 'porridge' is to be found in the middle. I.e. neither 'too sweet' in the sense of sentence 2 nor 'too salty' in the sense of sentence 5.

Remark 4 Citizens of $\mathfrak{K}$allista are rare. Considering nothing but these provides irregular and yet sparse arenas: 'too salty'. So embedding these to some extent is more likely to produce arenas that are 'just right'. Take embedding too far, however, and 'excess sweetness' impinges.

As an example, the N-body problem starts with the $\mathbb{R}^{d\,N}$ arena. Of N points, particles or vertices in dimension d . This contains unphysical information and is too regular besides. Quotienting out the similarity group in 1-d produces spheres. And in

$$\mathbb{CP}^{N\,-\,2}$$

in 2-d : the Smale and Kendall results. With

$$\mathbb{CP}^1 \;=\; \mathbb{S}^2$$

conferring various abovementioned benefits and extra beauty.

But in $\geq$ 3-d , one gets nontrivial stratified manifolds. Next suppose that one goes as far as quotienting out Affine or Projective transformations. As one would like for Image Analysis or Computer Vision. Then ensuing stratified manifolds manifest various Topological-level catastrophes. In the current paragraph, then, 'excess salt' takes over.

Remark 5 Hardy [2] proclaimed

"*The mathematician's patterns, like the painter's or the poet's, must be beautiful... Beauty is the first test: there is no permanent place in the world for ugly mathematics.*"

A follow-up of this is what S. Sánchez replied [66] once we had found *many* unicorns, and so were in need of stringent selection principle.

" *Arenize can map beautiful objects to ugly arenas and vice versa. So in arenized Mathematics, double-beauty provides a more stringent test. There is a more permanent place in the world for beautiful objects which map to beautiful arenas.*"

Naming Remark 9 Let us call this the *bicallyxta principle*. Integer partitions and their Young lattice, and individual Lie algebras and their Cartan classification, are examples. Using Young diagrams to track symmetric group representations, and Young tableaux to track Lie algebra representations, are further adjacent such. These are the start of the *bicallyxtine highway*: one of the most prominent features in

$$\mathfrak{K}\text{allista}|_{\langle \leq 3 \rangle} .$$

Remark 6 *Grand arenas* can also be defined. A well-known formal example is to not only consider Wheeler's superspace arena for General Relativity [10], of 3-metrics on a fixed spatial topology. But also the union of these over all spatial topologies, thus enabling Topology change. A name for this is Grand Superspace... The problem with n-callyxta tower generalizations of bicallyxta is that these swiftly transition from a selection principle to a near-total extinction event for types of Mathematics..

7.7 Arenization in other parts of Mathematics and in Theoretical Computer Science

Motivation 4 Linear Algebra has been quite well arenized. Readers will likely have previously encountered vector spaces, eigenspectra, eigenspaces [3] and various matrix groups [8]. All of which are among Linear Algebra's arenas. Graph Theory, Topology, Abstract Algebra and Functional Analysis have also been quite thoroughly arenized. Many other subjects have not yet been.

Motivation 5 $\mathfrak{P}$latonia, $\mathfrak{B}$asica and $\mathfrak{K}$allista are themselves, at least formally, arenas.

Motivation 6 For further inspiration, some Computer Science examples are as follows. Trees as database arenas, on which sort and search operate [39]. The arena of all barcodes [70]. The idea of modelling an LLM's answers as a Topos, due to the Topos' second facet [16] as a context-dependent logic is theoretically interesting, in part due to the reasons for its impracticality.

Exercise 3 Spot 10 further arenas of note that are already present in first courses on their corresponding subject.

Recommendation 4 Pick an arenization topic.

7.8 Arenization as a map that may be wild

Question [Anderson and Sánchez 2019] [66] *Is arenization generically a wild map*?

Remark 1 How to Mathematically-sharply define 'generically' and 'wild' here is itself an interesting and ambiguity-inducing question.

Pointer 5 One joint implication of the Platonic view of Mathematics and wildness is that higher-level arenaic unicorns would then constitute an infinite fountain.

8 Beautiful unicorn lodes: fragile versus persistent

Remark 1 Beauty is capable of generating unicorn lodes. Fragile beauty places hard limits on the size of such unicorn lodes. And yet some instances of beauty persist under some kinds of generalization.

Example 1 S. Sánchez pointed out that many nice Linear Algebra results about Flat-Geometry triangles (articles 1-4 on [64]'s publication page) break down for quadrilaterals.

1) The advent of multiplicative incompatibility between quadrilateral or higher versions of triangle matrices. For instance the AMB tree indexed extension of Apollonius' theorem is multiplicatively incompatible with the CoM-removing projector. This is a type of fragility.

2) Hopf's little map's ties Heron's formula to Smale and Kendall's little theorems [64, 73, 96]. Whose configuration space spheres are simpler and support extra swathes of techniques. And which are the two other little results promised in Sec 4.

3) Along the simplices, Cayley–Menger determinants generalize the Heron determinant. Also topological-level sphericity persists due to Casson's theorem [42]. More strongly, Hopf's generalized map persists to all N-a-gons. Paving a route to Smale's generalized theorem [14] and Kendall's generalized theorem [42]. And yet this is cut off from generalizations of Heron's formula. Also there is no counterpart for any fixed dimension ≥ 3 . The current paragraph encompasses a weaker lode direction, a more beautiful and stronger lode direction and a fragility no-go.

4) 2) and 3) were already known. Excepting the cut off, since proving the little trio of Hopf, Smale and Kendall from Heron was itself new.

While 1) is new, because applying Linear Algebra in detail to N-a-gons via weak-Conway variables and quadratic forms is new. The Author was left to work out the details. With K. Everard, an unexpected feature was found [74]. Namely that Ptolemy's theorem steps up to provide a new source of involutors that are multiplicatively compatible with the CoM-removing projector. This system furthermore extends to inclusion of a Brahamagupta matrix to form the *cyclic quadrilateral matrix algebra*. The Author also alternatively extended with the Bretschneider matrix to form the *convex quadrilateral matrix algebra* [95].

5) S. Sánchez further pointed out that not all Heron heirs are area formulae like Brahmagupta or Bretschneider. Or p-volume formulae like Cayley–Menger via the much longer-known della Francesca–Tartaglia volume formula for tetrahedra. Another is the square root of the square of the sum of subsystem areas. This was known from Molecular Physics [34]. But S. Sánchez further pointed out that this is the true Casimir heir of Heron's formula's fake Casimir. In direct correspondence to 3-d total angular momentum versus the mere square of the angular momentum in 2-d . So there is furthermore a persistent lode of [lead $SO(N-1)$ subgroup] Casimirs on all N-a-gons.

6) A. Ford pointed out that the triangle matrix algebra generalizes arenaically in a different way. Namely that (Articles 5 to 8 on [64]) its Algebraic properties generalize to all algebras of Combinatorial Matrices [38]. Some interplay with 4) is then that Ptolemy, Brahmagpta and Bretschneider matrices fail to fit this rubric. Whether these usefully admit a parallel rubric of larger matrix arenas remains an open question.

7) Ptolemy matrices return a more troublesome counterpart to the Heron–Minkowski connection. Namely that Ptolemy's inequality encodes not the hyperbolic-signature Minkowski metric of the triangle inequality but an ultrahyperbolic one. Also while Brahmagupta looks more symmetric than Heron in plain algebra, at the level of matrices it is doubly degenerate. These observations load further problems on the quadrilaterals. Finally pentagons and hexagons illustrate further losses of control. Both are needed since some matters decouple into odd and even N-a-gon results.

9 OpenAI's less basic unicorns

Remark 1 A much larger number of threads are used here. Including a much larger number of less basic threads, say $\langle$3-6$\rangle$. The number of threads used in a particular project, at least as captured by preprint summary highlights, is seldom particularly large. So say many 2 to 4 course meals from an $O(200)$ sized menu.[3]

[3]These menu size M , combination size C and level $\langle L \rangle$ estimates will likely vary as we further analyze OpenAI preprints. An estimate of the proportions M, N of OpenAI preprints which are *not* convex span sweeps shall in the process also emerge. For M the mines and N not spans at all. Take this footnote to contain a first simple model for a useful parametrization for a large preprint drop. A second slightly less simple model is of course the *arena of minesweeping depth menus* for the drop.

10 Toy unicorns

Remark 1 The idea here is to use

$$\mathfrak{D} = \begin{pmatrix} A & B \\ u & v \end{pmatrix} .$$

As a toy-model unicorn for

$$\mathfrak{D} = \begin{pmatrix} C & D \\ x & y \end{pmatrix} .$$

If whichever of the following holds.

$$u < x \text{ and either } A = C \text{ or } A \text{ is a simplified version of } C .$$

$$v < y \text{ and either } B = B \text{ or } B \text{ is a simplified version of } D .$$

Example T.1 OpenAI's claims include [89] a proof of the long-standing *Barnette conjecture*. That cubic 3-connected graphs which are both A) planar and B) bipartite must necessarily be Hamiltonian. In the context that Tait assumed A) and Tutte hit this with a counterexample [4, 91]. While Tutte [13] assumed B) and Horton hit this with a counterexample [20].

This proof is moreover both also a hidden-extra-strand and a hidden-beauty unicorn. The extra – and beautiful – strand is that the contradictory linear equations derived are a fortiori *toy index theorems*... OpenAI's write-up missed this.

I immediately spotted this because my discussions with S. Sánchez [66] included the following. I identified *Grinberg's planar Hamiltonian-necessity theorem* to be a null-index theorem [69]. To which S. Sánchez retorted that this became a more substantial index theorem for graphs embedded on a class of surfaces. With reference to a known result, that had not however yet been interpreted as an index theorem... Due to my Barnette proof interpretation rendering this interpretation topical, I made this interpretation public [93] several days later.

See Fig 1 for a brief conceptual analysis of this pair of unicorns.

Example T.2 OpenAI's claims also include an affirmative proof [90] of the *Saxl conjecture* [55]. A toy unicorn pairing here is between the staircase partitions in symmetric group Representation Theory. And the Steinberg representations in Lie Representation Theory. The shared underlying conceptual class is *tensor square covering problems*. The bridge holding this pairing together was built in [54]. A strong reason for looking at this result and claim pair is provided in the Conclusion.

Recommendation 5 Set up toy unicorns. Can any other OpenAI unicorn be accorded a pedagogically-useful toy? A technically-insightful toy?

Recommendation 6 Suppose that you have the good fortune of being able to match an OpenAI preprint's sweeping depth menu. Either right now or well within a project's lifetime. Then aim to detailedly analyze it. Do not risk this for ones as yet without Lean verification. And always allow for the possibility that you may need to pivot from interpreting a such, to dismantling it.

11 Conclusion

Remark 1 Targetting known conjectures will miss many hidden-strand and hidden-beauty results. We illustrated this firstly with Heron's formula. Secondly, with the toy index theorem content of both Grinberg's theorem and the claimed proof of Barnette's conjecture.

Remark 2 Consider the following as an example of a further conjecture ensuing from identifying an extra strand.

Does Barnette's theorem generalize away from the plane to some class of surface? In the manner of a more substantial index theorem, as occurs for Grinberg's theorem?

Remark 3 The current Article has illustrated that the Modern Applied Topology Age already had its own smaller unicorn fountains. With toy unicorns for larger unicorns now of interest, finding basic unicorns can acquire higher significance.

Remark 4 The current Article has, naturally, applied the callyxtine and even bicallyxtine selection principles to OpenAI's preprint drop.

In the sense that the Barnette unicorn, and its pairing with the Grinberg toy unicorn, is callyxtine.

While the Saxl and Steinberg pair are denizens of the bicallyxta highway. Here it is less clear in this case which of the two is the toy (if any).

On the one hand, the Saxl half was posed later and hitherto resisted proof.

On the other hand, historical ordering and technical complexity are not necessarily aligned.

Not however in the sense of claiming that we have yet found all the callyxtine advances in the preprint drop. With indeed our allusion to hidden beauties rendering such a task highly nontrivial and ongoing...

Remark 5 The current Article is not claimed to be an exhaustive classification of known types of unicorn fountain. Much less of AI Age fountain of plenty. It does however represent a start on such matters. May more follow.

Pointer 6 Follow [82] to see the heirs of Petersen form a comparable sparse and relatively basic tangle of callyxtine objects to that outlined for the Heron–Hopf complex's heirs in the current Article. And more eventually for the heirs of Fano forming a considerably denser and deeper tangle. Reflecting Fano's role as the most exhalted citizen of $\mathfrak{K}\text{allista}|_{\langle \leq 3 \rangle}$, at least among known such citizens.

Remark 6 In the course of writing the current Article, 6 recommendations have been provided. To choose among for future dissertation and PhD thesis topics. Or for distinctive open-ended unscripted problems to work on at Tech companies.

Acknowledgments I thank S. Sánchez, Chris Isham, Don Page, Jan Saxl A. Ford and K. Everard for discussions long past. And the last two for proofreading an earlier unreleased draft of the current Article.

References

[1] G. Galilei, *Dialogue Concerning the Two Chief World Systems* (1632).

[2] G.H. Hardy, *A Mathematician's Apology* (CUP, Cambridge 1940).

[3] P. Halmos, *Finite-Dimensional Vector Spaces*, (Springer-Verlag, N.Y. 1942).

[4] W.T. Tutte, "On Hamiltonian Circuits", J. London Math. Soc. **21** 98 (1946).

[5] C. Lanczos, *The Variational Principles of Mechanics* (U. Toronto P., Toronto 1949).

[6] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Butterworth–Heinemann, Oxford 1987; original dates to 1959).

[7] H.S.M. Coxeter, *Projective Geometry* (Blaisdell 1964; 2003).

[8] J-P. Serre, *Lie Algebras and Lie Groups*, (Springer, 1965).

[9] V.I. Arnol'd, "Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits." Ann. Inst. Fourier, **16** 319 (1966).

[10] J.A. Wheeler, "Superspace and the Nature of Quantum Geometrodynamics", in *Battelle Rencontres: 1967 Lectures in Mathematics and Physics* ed. C. DeWitt and J.A. Wheeler (Benjamin, New York 1968).

[11] B.S. DeWitt, "Quantum Theory of Gravity. I. The Canonical Theory." Phys. Rev. **160** 1113 (1967).

[12] E.J. Grinberg, "Plane Homogeneous Graphs of Degree Three without Hamiltonian Circuits", Latvian Math. Yearbook, Izdat. Zinatne, Riga **4** 51 (1968).

[13] W.T. Tutte, "On the 2-Factors of 3-Connected Graphs", in *The Many Facets of Graph Theory* ed. G. Chartrand and S.F. Kapoor (Springer, Berlin 1969).

[14] S. Smale "Topology and Mechanics. II. The Planar N-Body Problem", Invent. Math. **11** 45 (1970).

[15] A.E. Fischer, ' "The Theory of Superspace." In *Relativity* ed. M. Carmeli, S.I. Fickler and L. Witten (Plenum Press 1970)

[16] F.W. Lawvere "Quantifiers and Sheaves" in *Actes du Congrès International des Mathématiciens (Nice, 1970)* (Gauthier-Villars 1971).

[17] W.T. Tutte, "On the 2-Factors of Bicubic Graphs", Discrete Math **1** 203 (1971).

[18] C.W. Misner, "Minisuperspace", in *Magic Without Magic: John Archibald Wheeler* ed. J. Klauder (Freeman, San Francisco 1972).

[19] M. Artin, A. Grothendieck, and J-L. Verdier, *Théorie des Topos et Cohomologie Étale des Schémas (SGA 4), Tome 1: Théorie des Topos* (Lecture Notes in Mathematics **269** (Springer-Verlag, 1972) Write-up of material dating back to 1963-64. Without the name 'Topos' yet in circulation, some of this material originated in 4 seminal papers of Grothendieck on separate topics, all from 1957.

[20] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications* (Macmillan, London: Macmillan, 1976). This is the first place that the Horton graph appeared in print.

[21] V.I. Arnol'd, *Mathematical Methods of Classical Mechanics* (Springer, New York 1978).

[22] M. Gromov, *Groups of Polynomial Growth and Expanding Maps*, Publications Mathematiques I.H.E.S. **53** (1981).

[23] C.J. Isham, "Topological and Global Aspects of Quantum Theory" in *Relativity, Groups and Topology II* ed. B.S. DeWitt and R. Stora, (North-Holland 1984).

[24] H. Gluck, F. Warner and W. Ziller, "The Geometry of the Hopf Fibrations", Enseign. Math. **32** 173 (1986).

[25] W. Kondracki and J. Rogulski, "On the Stratification of the Orbit Space for the Action of Automorphisms on Connections", Diss. Math. **250** (1986).

[26] C.J. Isham, "Quantum Topology and Quantization on the Lattice of Topologies", Class. Quan. Grav **6** 1509 (1989); "Quantization on the Lattice of Topologies", in *Florence 1989, Proceedings, Knots, Topology and Quantum Field Theories* ed. L. Lusanna (World Scientific, Singapore 1989).

[27] "An Introduction To General Topology And Quantum Topology", unpublished, Lectures given at Banff in 1989 (and available on the KEK archive).

[28] "Canonical Groups And The Quantization Of Geometry And Topology", in *Conceptual Problems of Quantum Gravity* ed. A. Ashtekar and J. Stachel (Birkhäuser, Boston, 1991).

[29] R.H. Buchholz, "Perfect Pyramids", Bull. Austral. Math. Soc. **45** 353 (1992).

[30] D.A. Holton and J. Sheehan, *The Petersen Graph* (C.U.P., Cambridge 1993).

[31] R. Montgomery, "Gauge Theory of the Falling Cat", in *Dynamics and Control of Mechanical Systems* Fields Institute Communications **1** ed M.J. Enos (AMS 1993).

[32] D.C. Lay, *Linear Algebra and Its Applications* (Addison-Wesley 1994).

[33] C.J. Isham, *Lectures on Quantum Theory* (Imperial College P., London 1995).

[34] R.G. Littlejohn and M. Reinsch, "Internal or Shape Coordinates in the N-body Problem", Phys. Rev. **A52** 2035 (1995).

[35] "Gauge Fields in the Separation of Rotations and Internal Motions in the N-Body Problem", Rev. Mod. Phys. **69** 213 (1997).

[36] P. Cvitanovič, *Universality in Chaos* (Hilge 1997).

[37] W. Fulton, *Young Tableaux: with Applications to Representation Theory and Geometry* (CUP, Cambridge 1997).

[38] D.E. Knuth, *The Art of Computer Programming* **1** *Fundamental Algorithms* (Addison-Wesley, 1997).

[39] **3** *Sorting and Searching*, (ibid. 1998)

[40] R.P. Stanley, *Enumerative Combinatorics* **2** (CUP, Cambridge 1999).

[41] C.J. Isham, *Modern Differential Geometry for Physicists* (World Scientific, Singapore 1999).

[42] D.G. Kendall, D. Barden, T.K. Carne and Huiling Le, *Shape and Shape Theory* (Wiley 1999).

[43] B.A. Davey and Hilary A. Priestley, *Introduction to Lattices and Order* (C.U.P, Cambridge 2002).

[44] G. Rudolph, M. Schmidt and I.P. Volobuev, "On the Gauge Orbit Space Stratification: a Review", J. Phys. A. Math. Gen. **35** R1 (2002).

[45] M. Schmidt, "How to Study the Physical Relevance of Gauge Orbit Space Singularities?", Rep. Math. Phys. **51** 325 (2003).

[46] H.K. Urbantke, "The Hopf Fibration Seven Times in Physics", J. Geom. Phys. **46** 125 (2003).

[47] H. Dym, *Linear Algebra in Action* (A.M.S, Providence RI 2006).

[48] S. Roman, *Advanced Linear Algebra* (Springer 2007).

[49] P. Cvitanović, *Group Theory, Birdtracks, Lie's, and Exceptional Groups* (P.U.P., Princeton NJ 2008).

[50] J. Meier, *Groups, Graphs and Trees: An Introduction to the Geometry of Infinite Groups* (CUP, Cambridge 2008).

[51] M. Farber, *Invitation to Topological Robotics* (EMS, Zurich 2008).

[52] N.C. Carter, *Visual Group Theory* (MAA, Washington, DC 2009).

[53] D. Giulini, "The Superspace of Geometrodynamics", Gen. Rel. Grav. **41** 785 (2009)

[54] G. Heide, J. Saxl, P.H. Tiep and A.E. Zalesski, "Conjugacy Action, Induced Representations and the Steinberg Square for Simple Groups of Lie type", Proc. Lond. Math. Soc. **106** 908 (2013), arXiv:1209.1768.

[55] I Pak, Greta Panova and E. Vallejo, "Kronecker Products, Characters, Partitions and the Tensor Square Conjectures" Adv. Math. **288** 702 (2016), arXiv:1304.0738.

[56] R. Ghrist, *Elementary Applied Topology* (2014).

[57] *Handbook of Graph Drawing and Visualization* ed. R. Tamassia (Taylor & Francis, Boca Raton, Fl. 2014).

[58] M. Chen, S. Mao, and Y. Liu, "Big Data: A Survey". Mobile Networks and Applications, **19** 171 (2014).

[59] V. Patrangenaru and L. Ellingson, *Nonparametric Statistics on Manifolds and Their Applications to Object Data Analysis* (CRC Press 2016).

[60] M. Abadi et al, "TensorFlow: a System for Large-Scale Machine Learning", arxiv:1605:08965.

[61] I. Bengtsson, and K. Życzkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement* (CUP, Cambridge 2017, considerably expanding on 2006 edition).

[62] E. Anderson, *The Problem of Time. Quantum Mechanics versus General Relativity*, (Springer International 2017) Fundam. Theor. Phys. **190** (2017) 1-920 DOI: 10.1007/978-3-319-58848-3. Free access to its extensive Appendices is at https://link.springer.com/content/pdf/bbm%3A978-3-319-58848-3%2F1.pdf . And it is part of these that you want for the review of configuration spaces!

[63] "The Smallest Shape Spaces. I. Shape Theory Posed, with Example of 3 Points on the Line", Accepted as Periodically-Updating Reviews in Applied Topology **1** Preprint: arXiv:1711.10054.

[64] "Two New Perspectives on Heron's Formula", Ann. Inst. Theory STEM **1** 1 (2026), institute-theory-stem.org/annals-of-the-institute-of-stem/ Preprint: arXiv:1712.01441 Updated preprint: wordpress.com/page/conceptsofshape.space/1244 (2024).

This reference also services the next 7 Articles on the publication webpage.

[65] G. Strang, *Linear Algebra and Learning from Data*, (Wellesley-Cambridge P., Cambridge 2019).

[66] Combinatorial and Geometrical discussions between E.A. and S. Sánchez (2018-2020). Some furthermore involved A. Ford.

[67] C. Roberts et al, "TensorNetwork: A Library for Physics and Machine Learning", arxiv:1905:01330.

[68] V.I. Arnol'd and B.A. Khesin, *Topological Methods in Hydrodynamics*, (Springer 2021; previous edition dates to 1998).

[69] E. Anderson, *Applied Combinatorics*, (2022). Made freely available in response to the pandemic here: institute-theory-stem.org/combinatorics/ .

[70] G. Carlsson and M. Vejdemo-Johansson, *Topological Data Analysis with Applications* (C.U.P., Cambridge 2022).

[71] W.X. Zhao et al, "A Survey of Large Language Models", arXiv:2303.18223. This is a periodically-updating review.

[72] E. Anderson, *Linear Mathematics* Book in preparation. Homepage at: conceptsofshape.space/linear-mathematics/ (2023; 2025).

[73] "Quadrilaterals in Shape Theory. II. Alternative Derivations of Shape Space: Successes and Limitations", conceptsofshape.space/does-the-heron-proof-of-kendalls-little-theorem-generalize-to-other-area-or-volume-formulae/ (2024).

[74] E. Anderson and K. Everard (she/her), "Linear Algebra of Cyclic Quadrilaterals: Ptolemy, Diagonal, Area, and Circumradius Formulae", Preprint: conceptsofshape.space/linear-algebra-of-cyclic-quadrilaterals-ptolemy-diagonal-area-and-circumradius-formulae/

[75] E. Anderson, "An Infinite Series of Generalizations of Apollonius' Theorem", conceptsofshape.space/infinite-series-of-generalizations-of-apollonius-theorem/ (2025).

[76] E. Anderson and A. Ford (she/her), "Graph and Order Theory of N-Body Problem's Eigenclustering Networks", Periodically-Updating Reviews in Applied Topology **2** (2026) institute-theory-stem.org/puriat-2/ Preprint: wordpress.com/page/conceptsofshape.space/2090 (2025).

(2024).

[77] 3 preprints by E. Anderson, A. Ford and K. Everard, and one by E. Anderson, at institute-theory-stem.org/two-infinite-families-of-generalizations-of-stewarts-theorem/ (2025).

[78] R. Montgomery, *Four Open Questions for the N-Body Problem* (C.U.P., Cambridge 2025).

[79] E. Anderson, *The Smallest Cubic Graphs and their Cubeomorph Poset Arena*, https://institute-theory-stem.org/oeagot-graphs-smallest-cubic-arenas/ (2026).

[80] "The Fano Configuration, Plane and Graph", Online Encyclopaedia of Applied Graph and Order Theory, institute-theory-stem.org/oeagot-fano/ (2026);

"The Fano Incidence Graph, alias Heawood Graph and $(6, 3)$-Cage, Online Encyclopaedia of Applied Graph and Order Theory, institute-theory-stem.org/oeagot-fano-incidence-graph/ (2026).

[81] "The Pappus Configuration, Theorem and Graph" Online Encyclopaedia of Applied Graph and Order Theory, institute-theory-stem.org/oeagot-pappus-graph/ (2026);

"The Pappus Incidence Graph", Online Encyclopaedia of Applied Graph and Order Theory, institute-theory-stem.org/oeagot-pappus-incidence-graph/ (2026);

"The Desargues Configuration, Theorem and Graph", Online Encyclopaedia of Applied Graph and Order Theory, institute-theory-stem.org/oeagot-desargues-graph/ (2026);

"The Desargues Incidence Graph", Online Encyclopaedia of Applied Graph and Order Theory, institute-theory-stem.org/oeagot-desargues-incidence-graph/ (2026).

[82] Online Encyclopaedia of Applied Graph and Order Theory, institute-theory-stem.org/online-encyclopaedia-of-graphs-and-orders/ .

[83] Beheshteh T. Rakhshan and G. Rabusseau, "Tensor Cookbook: Mastering Tensors through Diagrams", arXiv:2605.16610.

[84] E. Anderson, *Tensor Networks throughout the Basic Theory of STEM. IV. Cartesian Tensors* institute-theory-stem.org/oetoma-tensor-networks-for-cartesian-tensors/ (2026).

[85] *The Gauss–Bonnet Theorem* Accepted as Periodically-Updating Reviews in Applied Topology **4** (2026) Preprint: institute-theory-stem.org/from-differential-geometry-to-topology-summer-school-2026/ .

[86] E. Anderson and K. Everard, *Eigentheory of* $\mathbb{R}$*-Symmetric Matrices from a Graph and Order Theory point of view* institute-theory-stem.org/eigentheory-of-r-symmetric-matrices-from-a-combinatorial-pov/ (2026).

[87] Á. Lozano-Robledo, "What should we tell our Students?", https://terrytao.wordpress.com/2026/10/08/what-should-we-tell-our-students/ (October 8 2026).

[88] OpenAI, https://github.com/openai/math (2026).

[89] OpenAI, "Paired States and Hamiltonian Cycles in Cubic Bipartite Planar Graphs", https://github.com/openai/math (2026).

[90] OpenAI, "A Cyclic Polytabloid Proof of Saxl's Conjecture", https://github.com/openai/math (2026).

[91] E. Anderson, "Three Tuttes in a Tet trips Tait", institute-theory-stem.org/oeagot-tait-tutte/ (2026).

[92] "The Barnette Proof as a Contradiction between Toy Index Theorems" institute-theory-stem.org/oeagot-barnette-as-index/ (2026).

[93] "Grinberg's Theorem as the Zero-Index Planar Hamiltonian Necessity Theorem. II. As a Corollary to Euler's Index Theorem", institute-theory-stem.org/oeagot-grinberg-as-toy-index-theorem-2/ (2026).

[94] Forthcoming 2026.

[95] "Convex Quadrilaterals' Ptolemy–Lagrange–Bretschneider Algebras", forthcoming 2027, as a preprint at https://wordpress.com/page/conceptsofshape.space/2278 .

[96] *The Structure of Flat Geometry*, forthcoming 2027. With homepage conceptsofshape.space/geometry/ .

For a list of the articles and preprints in this program, see pp 2-6 of [74] (2024; 2025).