

Grinberg's Theorem as the Zero-Index Planar Hamiltonian Necessity Theorem. II. As a Corollary to Euler's Index Theorem.

Edward Anderson¹

Abstract

Grinberg's theorem provides a necessary condition for planar graphs to be Hamiltonian. It has recently been cast in zero-index theorem form. In the sense that the Topological side of this index theorem is just 0. To be a contentful index theorem, however, we need to find at least some cases in which the Topological side is switched on. The current Article performs this role, by latching on to a well-known result of Euler. In the process, it pins an index theorem interpretation on an already-known generalization of Grinberg's criterion to graphs on oriented 2-surfaces.

Cite as E. Anderson "Grinberg's Theorem as the Zero-Index Planar Hamiltonian Necessity Theorem. II. As a Corollary to Euler's Index Theorem.", Online Encyclopaedia of Applied Graph and Order Theory, institute-theory-stem.org/oeagot-grinberg-as-toy-index-theorem-2 (2026).

1 Introduction

Grinberg's theorem [4, 5, 14, 16, 8, 24] is a necessary condition for planar graphs to be Hamiltonian. S. Sánchez and I pointed out in 2019 [22] that this can be viewed as a zero-index case of a toy index theorem. We thus renamed it *ZIPHoN theorem* [22, 24]. Standing for 'zero-index planar Hamiltonian necessity'.

It corresponds to building up a bipartition of one's input planar graph. Into yellow and blue outerplanar [24, 16] strips.

And then reconceptualizing in terms of the below *proto-index*. We take this term to mean some *reflective difference*

$$\Delta P := P_+ - P_- . \quad (1)$$

This being $\mathbb{Z}$ -valued is a further useful feature for an index to have.

Our specific ZIPHoN proto-index is

$$\Delta T = T_+ - T_- = T_{\text{blue}} - T_{\text{yellow}} . \quad (2)$$

Which is a difference of triangulation counts. Corresponding to how many triangles are contained in each of the 2 outerplanar strips. This description [25] includes that if any larger polygon face is encountered in the process, then it is to be triangulated.

Finally, the ZIPHoN theorem is that in planar graphs that are Hamiltonian, these 2 triangle counts must be the same. I.e.

$$\Delta T = 0 . \quad (3)$$

'Yellow' and 'blue' are distinguishable and yet individually-meaningless labels. So they obey exchange symmetry. And thus we attain a reflective difference. $\mathbb{Z}$ -valuedness then follows from $0 \in \mathbb{Z}$. Which fixed value furthermore provides the qualifier 'zero' in our name. See Article I for further details about the Introduction's material so far.

¹Dr.E.Anderson.Maths.Physics *at* protonmail.com Institute for the Theory of STEM. Date stamp: v1: 19-10-2026. Copyright of Dr E. Anderson.

The current Article serves to furthermore justify ZIPHoN being an actual toy [18, 12, 1, 2, 21, 27] index theorem [3, 11, 7, 10, 13]. This is attained by broadening our scope to nonplanar graphs, in the sense of those that can be embedded instead on some oriented 2-surface. And finding therein a toy index theorem of Euler form that reduces to ZIPHoN in its planar subcase.

Notational aside Combinatorialists usually use ‘black’ and ‘white’. Our shortly forthcoming pedagogical introduction changes these meaningless labels to ‘yellow’ and ‘blue’. Using signs instead of colours places us in a reflective-symmetric notation as found in examples that are already well-known to be index theorems.

2 Euler’s theorem

2.1 Planar version

Theorem 0 (Euler) For a simplicial complex in the plane,

$$V - E + F = 2 . \quad (4)$$

Remark 1 Here $V = \#(\text{vertices})$, $E = \#(\text{edges})$ and $F = \#(\text{faces})$ of a 2-simplicial complex model for our 2-surface. So that the left-hand side is Combinatorial in nature: a simple case of Analytic.

2.2 Oriented 2-surface version

Theorem 1 (Euler) For a simplicial complex on an oriented 2-surface,

$$V - E + F = \chi . \quad (5)$$

Remark 2 This generalized right-hand side consists of the Euler characteristic χ which is Topological.

One way to espy its Topological nature is its further presence in the Gauss–Bonnet [19, 18] and Poincaré–Hopf [20, 1] theorems. Which are themselves slightly larger toy index theorems.

Another is through its reparametrization as

$$\chi = 2(1 - g) . \quad (6)$$

Where g is the *genus* of our 2-surface. This counts the number of holes. Which in turn uniquely characterizes oriented 2-surfaces, by the basic classification theorem for these [9].

Algebraic Topology [9, 15, 17] can furthermore be qualified as various ways of studying miscellaneous kinds of holes in spaces.

The genus is then indeed a simple example of a Topological property of a kind that Algebraic Topology is designed to detect.

Structure 1 An index theorem’s conceptual type is a bridge between two parts of Mathematics, formed by finding an unexpected equality between an Analytic object and a Topological object.

Remark 3 The Euler characteristic generalizes for instance via the Gauss–Bonnet theorem’s higher-dimensional descendants [11, 1].

Or via more general diversity of notions of hole afforded by higher- d spaces. Which can be tracked using more specialized Algebraic Topology.

This can furthermore also be tracked by *characteristic class* [11] information. Which now concerns the Topology in the twisting of fibre bundles over our spaces.

Various more sophisticated notions [11] of characteristic class then enter full-blown Index Theorems. So the above Euler relation is not only a toy index theorem. But also can be technically generalized into full-blown index theorems. Indeed, the famous *Atiyah–Singer index theorem* [3, 11, 7, 10, 13] can be viewed as a unification of the following. The Gauss–Bonnet theorem [19, 18] of Surface Geometry. And the Riemann–Roch theorem [12] of Complex Analysis.

Remark 4 Our left-hand side’s conceptual type generalizes to *Schläfli’s alternating sum*

$$\sum_{i=0}^d (-1)^i S_i \quad (7)$$

for a d -simplicial complex. The generalization that we require is however instead to a 2-surface chunk, in the sense of a 2-surface with boundary, as follows.

2.3 Oriented 2-surface chunk version

Theorem 1’

$$V - E + F + B = \chi . \quad (8)$$

Remark 5 B is here a count of the external faces. Which for many simple examples is 1, corresponding to just the one external face.

Remark 6 Let us now return to the planar context. In effect, we have two notions of F , the internal F , and the total $F_T = F + B$.

According to the *closed-surface interpretation*, theorem 1 uses F . While theorem 1’ extends this to F_T which is then split up into F and B .

In contrast, according to the *chunk-in-the-plane interpretation*, theorem 1 can already be viewed as the $F = F_T$ version. Theorem 1’ is then just a split of this without prior extension.

3 Generalized ZIPHoN as a Corollary

Remark 1 Our ZIPHoN reformulation can be interpreted within the following generalized modelling context. Form the reflective difference of $+$ and $-$ copies of (8). Now relative to a partitioning cycle S . In general, this cuts up our chunk of surface into components: some blue and some yellow.

Remark 2 Summing over components and multiplying by -2 to display precisely our ΔT , we obtain the following [22].

$$\begin{aligned} -2\Delta V + \Delta T + \Delta R - 2\Delta B &= 4\Delta \sum_c g_c . \\ \Rightarrow \underbrace{\Delta [T - 2(V + B + r)]}_{\text{Combinatorial}} &= \underbrace{4\Delta \sum_c g_c}_{\text{Topological}} . \end{aligned} \quad (9)$$

Where c runs over components. And the relation between the two remainder terms is

$$r = -\frac{R}{2} .$$

Remark 3 Finally, in versions of this work from February 2019 onward, we adopted [23]’s decomposition of our remainder term,

$$\Delta r = -(2\Delta C + \Delta d) .$$

Here

$$C = \sum_c C_c ,$$

where each C_c counts out a component. Also

$$d = \sum_c d_c ,$$

where each d_c is the *boundary deficit* of that component. I.e.

$$d_c := |E_c \cap E_S| - |V_c \cap C_S| .$$

In this way, we arrive at the following.

Corollary 1

$$\underbrace{\Delta [T + 2 (2C + d - V - B)]}_{\text{Combinatorial}} = \underbrace{4 \Delta \sum_c g_c}_{\text{Topological}} . \quad (10)$$

4 Discussion

Remark 1 This is not of the simplest Index Theorem form²

$$\underbrace{\Delta A}_{\text{Analytic}} = \underbrace{\mathcal{T}}_{\text{Topological}} \quad (11)$$

But rather of the form

$$\underbrace{\Delta A + \sum_o \Delta O_o}_{\text{Analytic}} = \underbrace{\mathcal{T}}_{\text{Topological}} \quad (12)$$

for further Analytic proto-indices O_o .

Specifically

$$\underbrace{\underbrace{\Delta T}_{\text{motivating proto-index}} + \underbrace{2 (2 \Delta C + \Delta d - \Delta V - \Delta B)}_{\sum_o \Delta O_o}}_{\text{Combinatorial}} = \underbrace{4 \Delta \sum_c g_c}_{\text{Topological}} . \quad (13)$$

Interpretation 1 Not all index theorems are of simplest form. A more general form is [27]

$$\sum_a \Delta A_a = \sum_t T_t .$$

Whether or not the T_t are themselves of the form ΔV_t . In the current Article the one type of Topological object present is of this form.

In discussing this [22], I was able to point to another well-known example (and a variant) with multiple Analytic objects to 1 Topological object. With the distinction that now just 1 object – Analytic – is of reflective-difference form. Though since these results have not yet been publicly envisaged as index theorems, I leave it to another day soon to present them to the world as such.

Interpretation 2 S. Sánchez formed the following Combinatorial adapted variable.

$$S := \frac{1}{4} [T + 2C + d - V - B] . \quad (14)$$

²[27] shall classify known (toy) index theorems by form.

Then

$$\Delta S = \underbrace{\Delta \sum_c g_c}_{\text{Topological}}. \quad (15)$$

For the planar case without boundary, neither the genus contributes:

$$\Delta S = 0. \quad (16)$$

Nor does the triangle-counting variable T get corrected. Unrescaling by a factor of 4, we thus return back to ZIPHoN:

$$\Delta T = 0.$$

5 Conclusion

Naming Remark 1 Reconceptualize our ZIPHoN name as follows [22].

$$\underbrace{\text{Zero-index}}_{\text{Index}} \underbrace{\text{planar}}_{\text{2-surface}} \underbrace{\overbrace{\text{Hamiltonian} \text{ necessity}}^{\text{Functionality}}}_{\text{cycle partition}} \underbrace{\text{theorem}}_{\text{theorem}}$$

In the current Article’s more general context, next identify the index theorem present while reordering at will.

2-surface cycle-partition corollary of Euler’s index theorem with boundary.

Where ‘with boundary’ both qualifies Euler’s theorem and the 2-surface. Next, we take 2-surface to itself be subsumable by Euler’s index theorem. For > 2 would require ‘generalized Euler’ or ‘Schläfli’ or some qualifier from the related generalized Gauss–Bonnet theorem. While < 2 would require a lesser result. And so we coin the following name for the current Article’s corollary. *Cycle-partition corollary of Euler’s index theorem with boundary.*

Remark 1 The theorem’s content is not new to [22] or the current Article due to [6, 8, 23]. The insight that it is an index theorem is however new. As is the considerable simplification afforded by the Δ notation.

Remark 2 Viewing Grinberg as the corresponding zero-index theorem – ZIPHoN – is in the process justified, fulfilling the current Article’s aim. A new interpretation of ZIPHoN is afforded. By passing from our triangle count variable T for Grinberg’s diagnostic quantity to Sánchez’ adapted variable S . The two coincide in Grinberg’s case in the plane. And yet S extends domain of validity to partitioning cycles in graphs embedded on oriented 2-surfaces.

Acknowledgments I thank S. Sánchez for this discussion, and for other matters of wisdom, understanding and leadership. A. Ford for proofreading an earlier version of the manuscript. And the Applied Combinatorics and Topology Discussion Group’s members.

References

- [1] S.-s. Chern, "Historical remarks on Gauss-Bonnet, Analysis, et cetera", *Volume in Honor of Jürgen Moser*, (Acad. P., 1990); Reprinted in *A Mathematician and his Mathematical Work, Selected Papers of S. S. Chern* ed. S.Y. Cheng, P. Li and G. Tian, (World Scientific, Singapore 1996). The original work started to be published in 1944.
- [2] F. Hirzebruch, "The Signature Theorem. Reminiscences and Recreation", *Prospects in Mathematics*, Ann. Math. Studies, Band **70** (1971). The theorem dates back to 1954.
- [3] M.F. Atiyah and I.M. Singer, "The Index of Elliptic Operators on Compact Manifolds", Bull. Amer. Math. Soc., **69** 422 (1963).
- [4] E.J. Grinberg, "Plane Homogeneous Graphs of Degree Three without Hamiltonian Circuits", *Latvian Math. Yearbook* **4** 51 (1968); for an English translation, see arXiv:0908.2563 .
- [5] R. Honsberger, *Mathematical Gems. I* (M.A.A., Providence RI 1974).
- [6] Y. Shimamoto, "On an Extension of the Grinberg Theorem", J. Combi. Theory **B 24** 169 (1978).
- [7] P. Shanahan, *The Atiyah-Singer Index Theorem: an Introduction*, Lecture Notes in Mathematics **638** (Springer, Berlin 1978).
- [8] J. Zaks, "Extending an Extension of Grinberg's Theorem", J. Combi. Theory **B 32** 95 (1982).
- [9] M.A. Armstrong, *Basic Topology* (Springer-Verlag, N.Y. 1983).
- [10] P.B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem* (Wilmington, DE. 1984.)
- [11] M. Nakahara, *Geometry, Topology and Physics* (I.o.P. Publishing, London 1990).
- [12] Frances Kirwan, *Complex Algebraic Curves* (C.U.P., Cambridge New York, 1992).
- [13] R. Melrose, *The Atiyah-Patodi-Singer Index Theorem* (A.K. Peters, Wellesley, Mass. 1993).
- [14] V.K. Balakrishnan, *Graph Theory* (McGraw-Hill, New York 1997).
- [15] J.R. Munkres, *Topology* (Prentice-Hall, Upper Saddle River, NJ 2000).
- [16] D.B. West, *Introduction to Graph Theory* (Prentice-Hall, Upper Saddle River N.J. 2001).
- [17] A. Hatcher, *Algebraic Topology* (C.U.P., Cambridge 2001).
- [18] H. Wu, "Historical Development of the Gauss-Bonnet Theorem", *Science in China Series A: Mathematics* **51** 777 (2008).
- [19] A.N. Pressley, *Elementary Differential Geometry* (Springer, London 2010).
- [20] T. Frankel, *The Geometry of Physics: An Introduction* (C.U.P., Cambridge 2011).
- [21] R. Ghrist, *Elementary Applied Topology* (2014).
- [22] Applied Topology discussions between E.A. and S. Sánchez (2019).
- [23] G. Brinkmann and Carol Zamfirescu, "Grinberg's Criterion", *European J. Combi.* **75** 32 (2019).
- [24] E. Anderson, *Applied Combinatorics*, Widely-Applicable Mathematics Series. A. Improving understanding of everything with a pinch of Combinatorics. **0**, (2022). Made freely available in response to the pandemic here: institute-theory-stem.org/combinatorics/ .
- [25] "Grinberg's Theorem as the Zero-Index Planar Hamiltonian Necessity Theorem. I. Outerplanar-Strip, Triangle-Count and Zero-Index Reformulations", forthcoming (2026), for [26], also referred to as Article I.
- [26] Online Encyclopaedia of Applied Graph and Order Theory, institute-theory-stem.org/online-encyclopaedia-of-graphs-and-orders/ .
- [27] E. Anderson, "Toy Index Theorems", Invited Review for Periodically Updating Reviews in Applied Topology, forthcoming (2027). For now, see homepage of 2027 Summer School on Toy Index Theorems, institute-theory-stem.org/toy-index-theorems-2027/ .