

Three Tuttles in a Tet trip Tait

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1 Introduction: enter Tait

Historical Remark 1 In 1884, Tait conjectured the following (paraphrasing in modern terminology).

Tait's Conjecture 'Every 3-connected planar cubic graph is Hamiltonian.

Structure 1 This is about graphs in the sense of Graph Theory.

Definition 1 A *cubic* graph is one in which every vertex has precisely 3 edges emanating from it. I.e. each vertex is of degree 3, a 'Chemical' alias for which is 'of valency 3' or 'trivalent'. A *planar* graph is one that can be drawn in the plane without any edges crossing. Finally, a 3-connected graph is one that remains connected upon deleting 3 of its vertices. This is clearly a network robustness criterion.

Remark 1 If this conjecture is true, then the famous 4-colour conjecture for maps in the plane would be proven.

4-colour conjecture [Guthrie 1852]. " ≤ 4 colors are needed for no 2 connected regions in any map to need to have the same colour."

The stepping stone between these is as follows.

"Bridgeless cubic planar graphs possess 3-edge-colourings."

Definition 2 A graph is *bridgeless* if indeed it contains no bridges. In the sense of no edges the cutting of 1 of which disconnects the graph. An *edge-colouring* is indeed a colouring of the edges. The 3 here indicates the number of distinct colours of edges emanating from each vertex.

Remark 2 So given that our graph is cubic, all edges about each vertex have a different colour. This and the 4-colour problem's colorations are related by face-vertex duality. I.e. the map's regions – faces – become the graph's vertices.

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2 Enter Tutte

2.1 The Tet chassis

Structure 1 But Tutte built a counterexample in 1946 [2] on a matching number of 46 vertices. Let us here give a conceptual account of this. Whether or not Tutte thought about it in this way (which we shall be checking shortly).

Take the standard planar presentation of the tetrahedron graph, Tet (Fig 1.a).

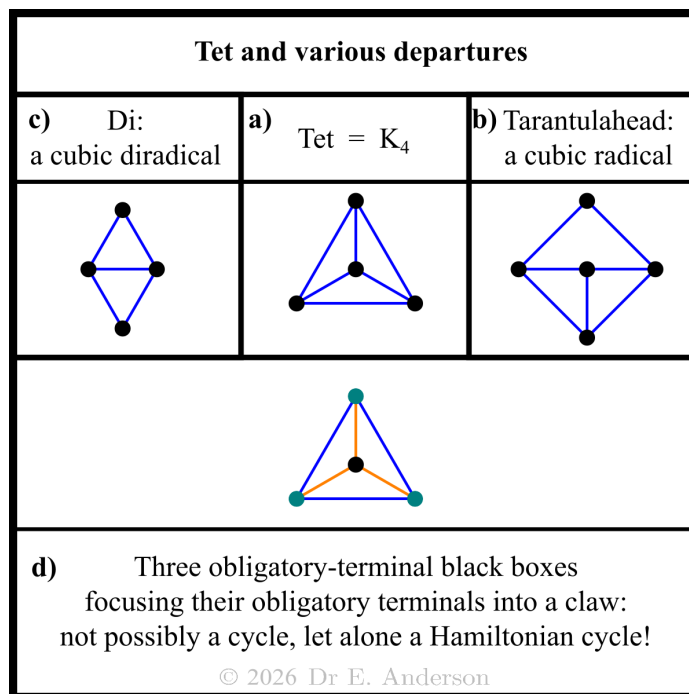


Figure 1:

Let each peripheral vertex be a black box. That has an obligatory terminus. In the sense that there are 3 edge fragments connecting it to the exterior, labelled A , B and C . Such that one can go from A to B or from B to C . But not from A to C . So all paths meandering through all the vertices in the black box are forced to enter or exit it through B .

Tet is cubic, so if the interior of the black box is cubic, then the whole object is cubic. So does cubic-model-land possess a black box with the above traversability properties? As realized by Hamiltonian paths: sweeping through every vertex in our box!

2.2 Fragments and interconversion with radicals

Remark 1 This reduces to not finding a cubic graph black box but a cubic graph *fragment* black box. In the sense that Tet 's first homeomorph Tarantulahead (Subfig b) is a cubic graph fragment; a Chemist would say 'radical'! And Tet 's first edge-deletion Di (Subfig c), commonly alias Diamond, is a di-radical. Our black box is then to be a tri-radical, since it has 3 attachment points. Alternatively, one could stylize the fragment to have a loose edge emanating from each of the attachment points.

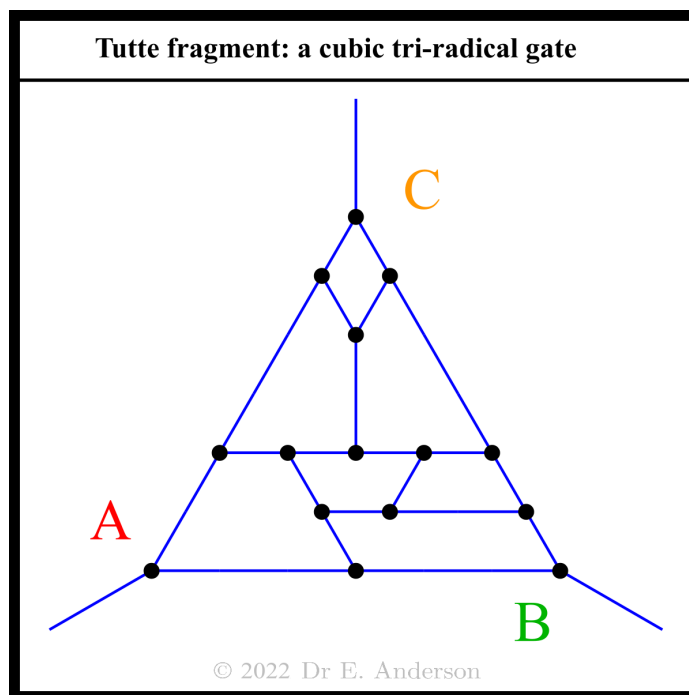


Figure 2:

Naming and Notational Remark 1 Di , Tet and Tarantulahead are names-and-notations coined by S. Sánchez [13]. On the premises that irreducibles should have short names-and-notations. Be those homeomorph irreducibles or double irreducibles: homeomorph-and-foliation irreducibles [15]. Their first few homeomorphs and foliations would then be longer phonetic variants of their irreducible whenever practicable [18]. For today, then, the first homeomorph of Tet is named Tarantulahead . Di and Tarantulahead are moreover both close relations of Tet that play a key role in constructing the cubeomorph irreducibles [8, 16].

Naming Remark 2 The difference between a fragment and a radical is as follows. The fragment attaches an extra edge to the radical’s under-regular vertex. With no vertex yet at its other end. Doing this to Tarantulahead yields the append- Tarantulahead operation [13, 16]. While doing this to Di ’s 2 radical vertices yields the ‘thread- Di ’ operation [13, 16].

2.3 the Tutte fragment

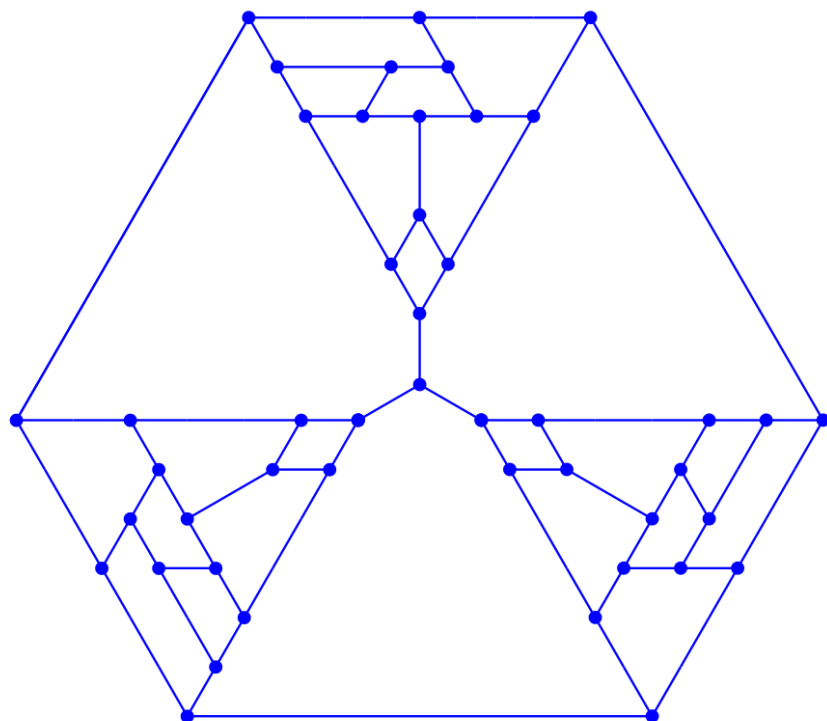
Structure 2 Tutte found such a black box on 14 vertices (Fig 2).

Naming Remark 3 It has subsequently come to be known as the *Tutte fragment*. With *Tutte gate* a truer name from the Mechanical, Electronics and Hardware points of view. Thus the titular three Tutties in a Tet trips Tait!

This can be interpreted as factorizing the synthesis of a large molecule into smaller pieces. In the present case with homogeneous pieces. Or also as rephrasing a hitherto hard conjecture as the finding of a rather smaller machine within the cubic-model-land arena.

An even truer name for the edge-removed version is as follows. *Smallest Hamiltonian-path terminus-forcing cubic tri-radical!*

Tutte 46-graph



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Figure 3:

Pedagogical Remark 1 This formulation of ours is rooted on Tutte starting off by reading Chemistry at Cambridge. Quite soon after, he cracked the Wehrmacht high command's 'Tunny code' without any Ally ever having seen the 'Lorenz cypher' machine involved... Let us leave it to someone else to figure out whether Tutte's Chemical beginnings, or his secret life as the most significant code-breaker of the 20th century, played any part in his finding this counterexample to the Tait conjecture...

3 Enter Barnette, as part of tightening to the minimum counterexample to Tait

Exercise 2 Show that the Tutte fragment is the unique smallest machine with the above properties. Adhering to the Tait conjecture’s modelling assumptions included!

Pointer 1 Tutte’s counterexample is not however minimum. It has

$$N = (3 \text{ Tutes in a Tet}) = 3 \times 14 + 4 = 42 + 4 = 46 .$$

vertices as claimed. So, parametrizing by the cubic graph theorists’ adapted variable

$$m := \frac{N}{2} ,$$

it has

$$m = \frac{46}{2} = 23 .$$

While subsequent work found examples going down to

$$m = 19 .$$

A 19 was first found by Barnette, whose work is at the centre of the current family of short articles. Via intermediary cases found by Grinberg, or by Faulkner–Younger.

In fact, Bosák and Lederberg each separately also found the same first instance of a 19 at around the same time as Barnette. So this is known as the Barnette–Bosák–Lederberg (BBL) graph. Incidentally, Lederberg’s principal occupation was as a Theoretical Chemist. So in a sense, the Tait counterexamples started and ended with Chemical input...

What took several further decades to work out was that this is the minimum counterexample. It is furthermore known to occur with multiplicity 6 . I.e. that the BBL graph has 5 friends which share the honour of minimality. They can be jointly described as follows [13].

“Two Tutes in a pentagonal prism saturates, sporting sixfold stuffing-ambiguity.”

Remark 1 This reflects that all the minimum counterexamples can be viewed as stuffing 1 less Tutte in than before. But now into a larger chassis than Tet: the pentagonal prism of Fig 4.a) I.e. with $m = 5$ rather than $m = 2$. By which a factorization using just 2 copies of Tutte’s gate is also possible.

This more complicated stuffing then turns out to support a 6-fold ambiguity. While no other $m = 19$ graph manages to share the honour of possessing the problem’s minimality property.

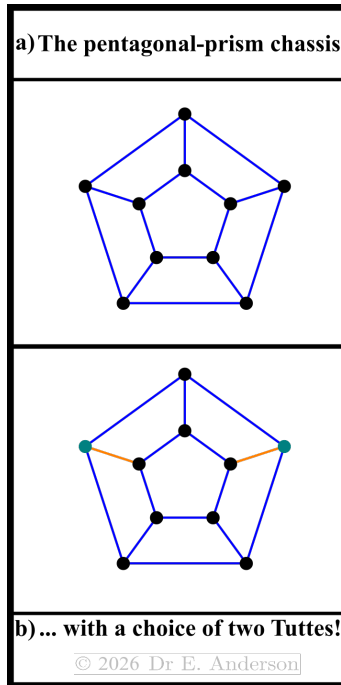


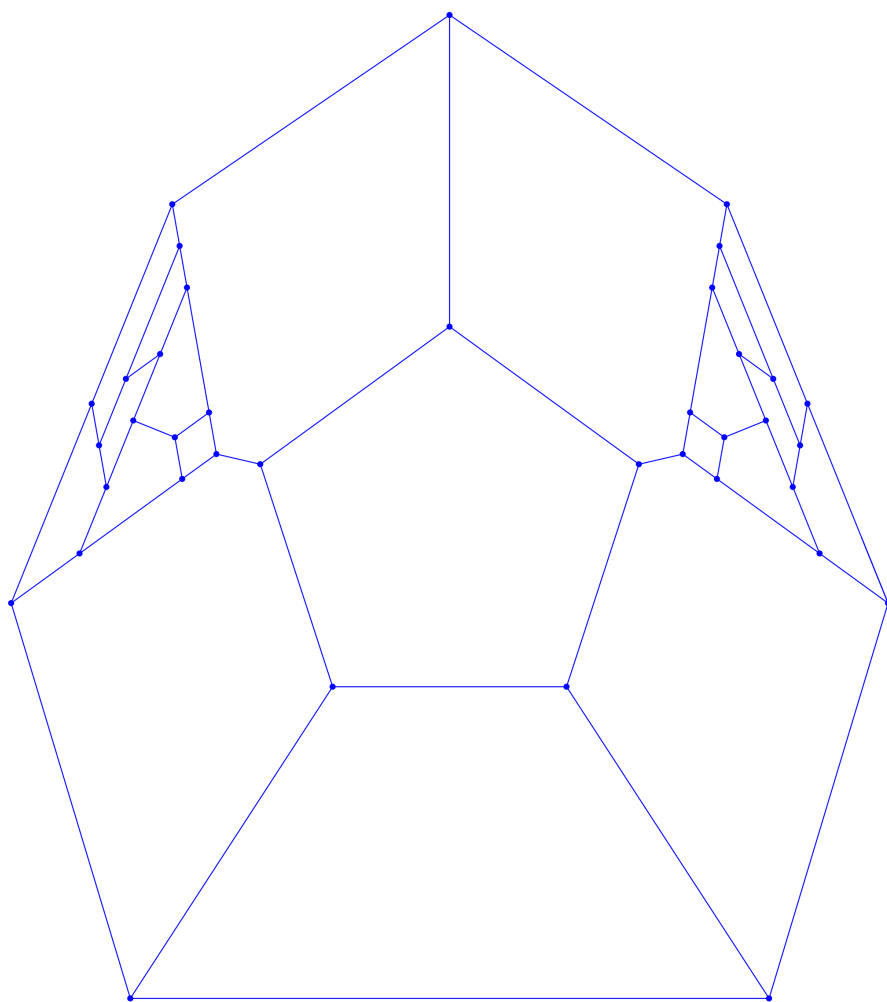
Figure 4:

Exercise 1 a) Show that the graph depicted in Fig 5 is not Hamiltonian.

b) Use Subfig 4.a)'s pentagonal-prism chassis. With a pair of turquoise vertices for Tutte gates as for example in Subfig b), forming Fig 5's case. Each with external edges labelled A, B, C . So as to establish the following. That the ambiguity in stuffing two Tutes into a pentagonal prism to trip Tait is 6-fold.

Remark 2 The intermediary Grinberg and Faulkner–Younger graphs however make no use of Tutte fragments at all! In particular, Grinberg's were enabled by his famous theorem [3, 14]. Which amounts to a Topologically distinct approach.

2 Tutes in a pentagonal prism



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Figure 5:

4 Barnette's conjecture

Remark 1 The reason that Barnette is currently topical is that his work on 3-connected cubic Hamiltonian graphs extended into forming the following conjecture.

Barnette's conjecture "Every 3-connected planar cubic bipartite graph is Hamiltonian."

Amounting to adding a judicious adjective to see if this fixes Tait. It would not however fix the 4-colour problem. Though by today, Appel and Haken had long resolved this problem by other means [7]...

Remark 2 The Barnette conjecture is one of over 700 conjectures that were proven, claimed or substantially advanced by Open AI in October 2026. The Barnette conjecture was specifically stated as affirmatively settled, with Lean certificate included. Thus vindicating Barnette's combination of adjectives. Illustrating furthermore how carefully thinking about a subject area in the wake of settling a major conjecture can produce further major conjectures. Which is firstly a desirable feature. And secondly, provided that motivation and capacity for this feature persist, the world shall never run out of new Mathematics to investigate.

Pointer 2 Past attempts at the Barnette conjecture, and commentary on Open AI's recent work on the Barnette Conjecture, shall form most of the current series' other articles.

5 Completing the square (Order-Theoretically!)

Remark 1 Bipartiteness also had separate history within Tutte's own work on connected cubic Hamiltonian graphs.

Tutte's Conjecture [1969-1971] [5]. "Every 3-connected cubic bipartite graph is Hamiltonian."

Structure 1 This is false. In this case, the historical counterexample was the *Horton graph* [6].

Remark 2 So part of the good taste in Barnette's choice of extra adjective comes from Tutte's previously non-superposed use in this subject area. In this case, more minimal counterexamples have also been found. But no proven minimal counterexample has been reached. Given the current development, it would be quite timely to sort out this neighbouring unfinished corner. This may well eventually motivate a small 'Tutte to Horton and simplifications' Article in [19].

Open Question "What is the minimum counterexample to Tutte's conjecture? And with what multiplicity is this realized?."

Structure 2 The corresponding *Wille concept lattice* [9, 11] of Hamiltonian conjectures for types of 3-connected graph forms the following. The square in Fig 6 that is realized as an Order-Theoretic diamond. Observe that this carries a second attribution to Tait, since he did not ab initio entertain planarity. It arises from treating 3-connected cubicness as a common package of concepts throughout. While then permitting the power set of possibilities as regards including or excluding planarity and bipartiteness. Indeed, the square is the third smallest power set (or the fourth is the unset's unpower set is included...) Power set orders are in a sense too dull for much Order-Theoretic study [13, 14] out of their being too regular...

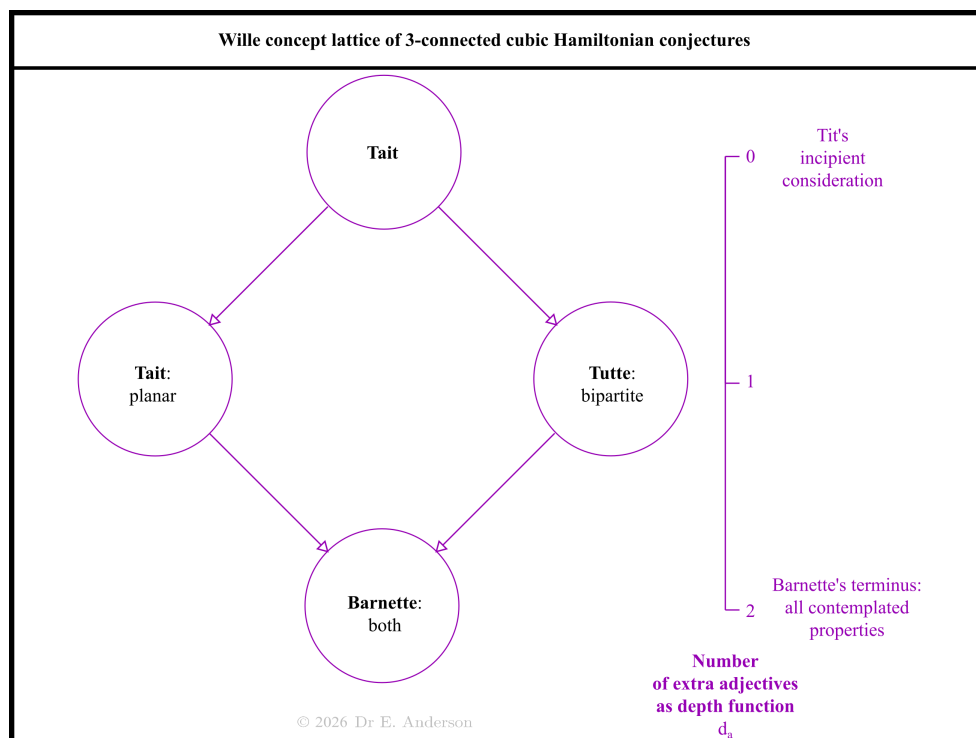


Figure 6:

End Remark 1 Tait and Tutte are both flawed Hamiltonianness criteria. Their superposition -Barnette- now has a claimed proof of being an actual Hamiltonianness criterion. Thus killing two birds with one stone. From Tait and Tutte’s ‘wood’ the ‘marble’ of Barnette has arisen.

Open-ended Project [S. Sánchez 2018]. Use modern Graph Drawing and Visualization [?] on all useful graphs, starting with the most beautiful ones. And the smallest: often easiest to draw well. And the existing subset of this for poset drawing, or an expansion thereof that [19] shall eventually provide.

End Remark 2 In fact, S. Sánchez more generally values irregularity, so larger and regular is less beautiful. This accounts for S. Sánchez drawing the Tutte fragment, but neither the Tutte graph nor the BBL graph and its 5 friends. In the Drawing and Visualization style laid out in Chapter 46 of [14]. So blame the Author for the choice of style used for these in the current Article. And for any non-optimality in the currently presented figures. We would like to eventually display a systematic mosaic of GDV style subfigures for every beautiful or highly technically significant graph!

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References

- [1] P.G. Tait, "Remarks on the Colouring of Maps", Proc. Royal Soc. Edinburgh **10** 729 (1880).
- [2] W.T. Tutte, "On Hamiltonian Circuits", J. London Math. Soc. **21** 98 (1946).
- [3] E.J. Grinberg, "Plane Homogeneous Graphs of Degree Three without Hamiltonian Circuits", Latvian Math. Yearbook, Izdat. Zinatne, Riga **4** 51 (1968).
- [4] *Recent Progress in Combinatorics: Proceedings of the Third Waterloo Conference on Combinatorics May 1968* ed. W. T. Tutte (A.P. New York 1969): Conjecture 5 in Appendix, attributed to Barnette.
- [5] W.T. Tutte, "On the 2-Factors of 3-Connected Graphs", in the Many Facets of Graph Theory ed. G. Chartrand and S.F. Kapoor, 243–248. (Springer, Berlin 1969);
"On the 2-Factors of Bicubic Graphs", Discrete Math, **1** 203 (1971).
- [6] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications* (Macmillan, London: Macmillan, 1976).
This is the first place that the Horton graph appeared in print.
- [7] K. Appel and W. Haken "Every Planar Map Is Four Colorable. Part I: Discharging", Illinois Journal of Mathematics **21** 429 (1977);
K. Appel, W. Haken, and J. Koch, "Part II: Reducibility" *ibid.* 491.
- [8] N.C. Wormald, "Classifying k -Connected Cubic Graphs", in Combinatorial Mathematics VI, ed. D.A. Holton and Jennifer Seberry (Springer, Berlin 1979).
- [9] R. Wille, "Restructuring lattice theory: an approach based on Hierarchies of Concepts" in Ordered Sets, ed. I. Rival, 445 (Reidel, Dordrecht 1982).
- [10] D.B. West, *Introduction to Graph Theory* (Prentice–Hall, Upper Saddle River N.J. 2001).
- [11] B.A. Davey and Hilary A. Priestley, *Introduction to Lattices and Order* (C.U.P, Cambridge 2002).
- [12] *Handbook of Graph Drawing and Visualization* ed. R. Tamassia (Taylor & Francis, Boca Raton, Fl. 2014).
- [13] Combinatorial discussions between S. Sánchez and E. Anderson (2017-2020).
- [14] E. Anderson, *Applied Combinatorics*, Widely-Applicable Mathematics Series. A. Improving understanding of everything with a pinch of Combinatorics. **0**, (2022). Made freely available in response to the pandemic here: institute-theory-stem.org/combinatorics/ .
- [15] E. Anderson and A. Ford, "Simple Graphs' 8 Double-Irreducibility Classes", in [19] (2026).
- [16] E. Anderson, "The Smallest Cubic Graphs and their Cubeomorph Poset Arena" Online Encyclopaedia of Applied Graph and Order Theory, institute-theory-stem.org/oeagot-graphs-smallest-cubic-arenas/ (2026).
- [17] <https://github.com/openai/math> , "Paired States and Hamiltonian Cycles in Cubic Bipartite Planar Graphs" (2026).
- [18] A. Ford, "The smallest simple graphs: naming and notation systems on ≤ 5 vertices", Online Encyclopaedia of Applied Graph and Order Theory institute-theory-stem.org/oeagot-0-5-graph-naming-notation/ (2026).
- [19] Online Encyclopaedia of Applied Graph and Order Theory, institute-theory-stem.org/online-encyclopaedia-of-graphs-and-orders/ .