

The Moser Spindle

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Abstract

We review the Moser spindle graph [Spindle](#) and a its fortiori unit-distance presentation. We include multiple routes for arriving at this object. And multiple conceptually unrelated applications of it. Between these routes and applications, it is verily a citizen of Kallista. We elect it for current review due to recent advances in the Hadwiger–Nelson problem for Euclidean-plane unit-distance colouring rendering it topical.

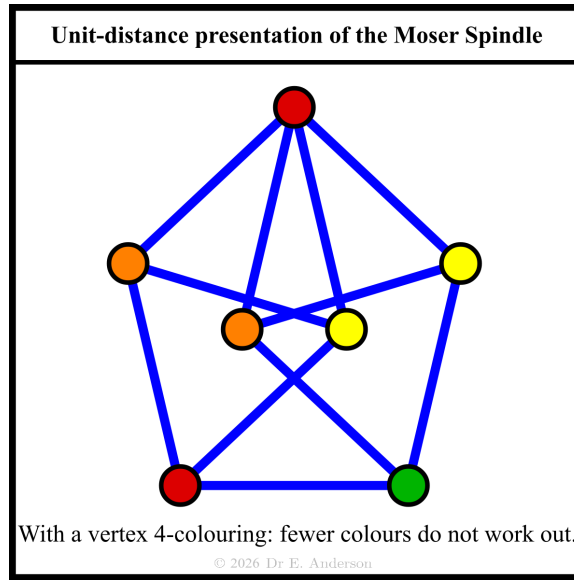


Figure 1:

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1 Introduction

1.1 3 heirs of Tet graph

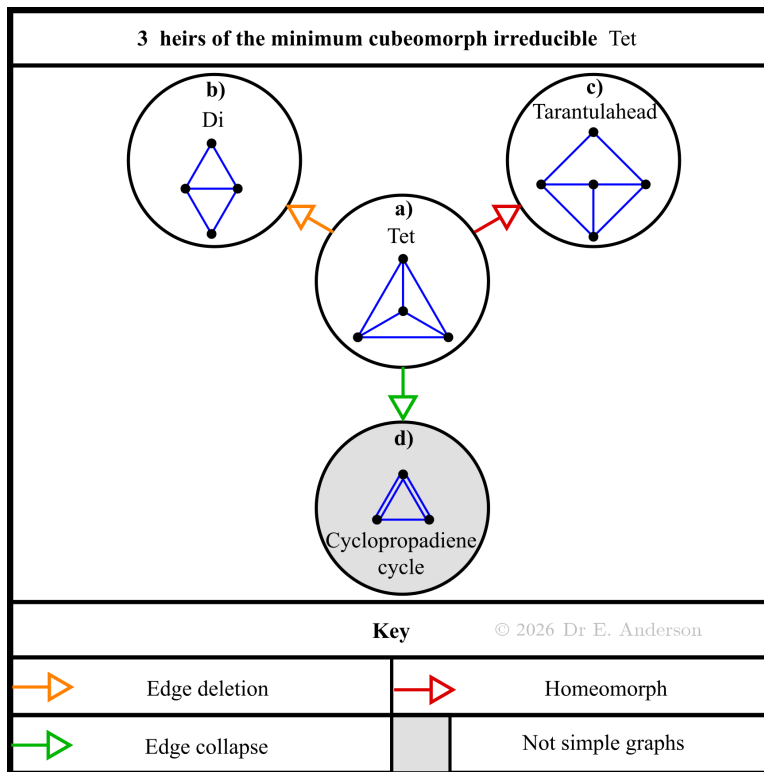


Figure 2:

Structure 1 The *tetrahedron graph*

$$\text{Tet} = K_4$$

(Fig 1.a) is the smallest cubic irreducible. Two simple moves on this are *edge-deletion*, returning Di(amond): Subfig b) And the *single homeomorph*, returning Tarantulahead : Subfig c). These play key roles in constructing further cubeomorph irreducibles (CIs). See [22] for definitions of cubeomorph and CI; the current Article is written as a straightforward extension to this Invited Review.

The current Article concerns a third move: *collapse edge*. Acting on Tet , this kicks one out of the simple graphs, to the ‘cyclopropadiene’ cycle of Subfig d). Accounting for this move not featuring in [22]’s analysis of Tet .

1.2 9 heirs of diDi

Structure 1 The *dimer of diamonds*, at ours both alias and denoted by

diDi

(Fig 2.a), is the second-smallest connected CI.

Exercise 1– Why is ‘connected’ a necessary stipulation here?

Structure 2 continued Now edge-deletion, single-homeomorph and edge-collapse each produce 3 outcomes (Subfigs b-j). This is on account of diDi being less symmetrically privileged than Tet. More generally, Tet is a stronger citizen of Kallista [16, 17, 22] than diDi. But the current Article is about a different way in which diDi gains some further pieces of calyxtenic strength. Through its Spindle heir i).

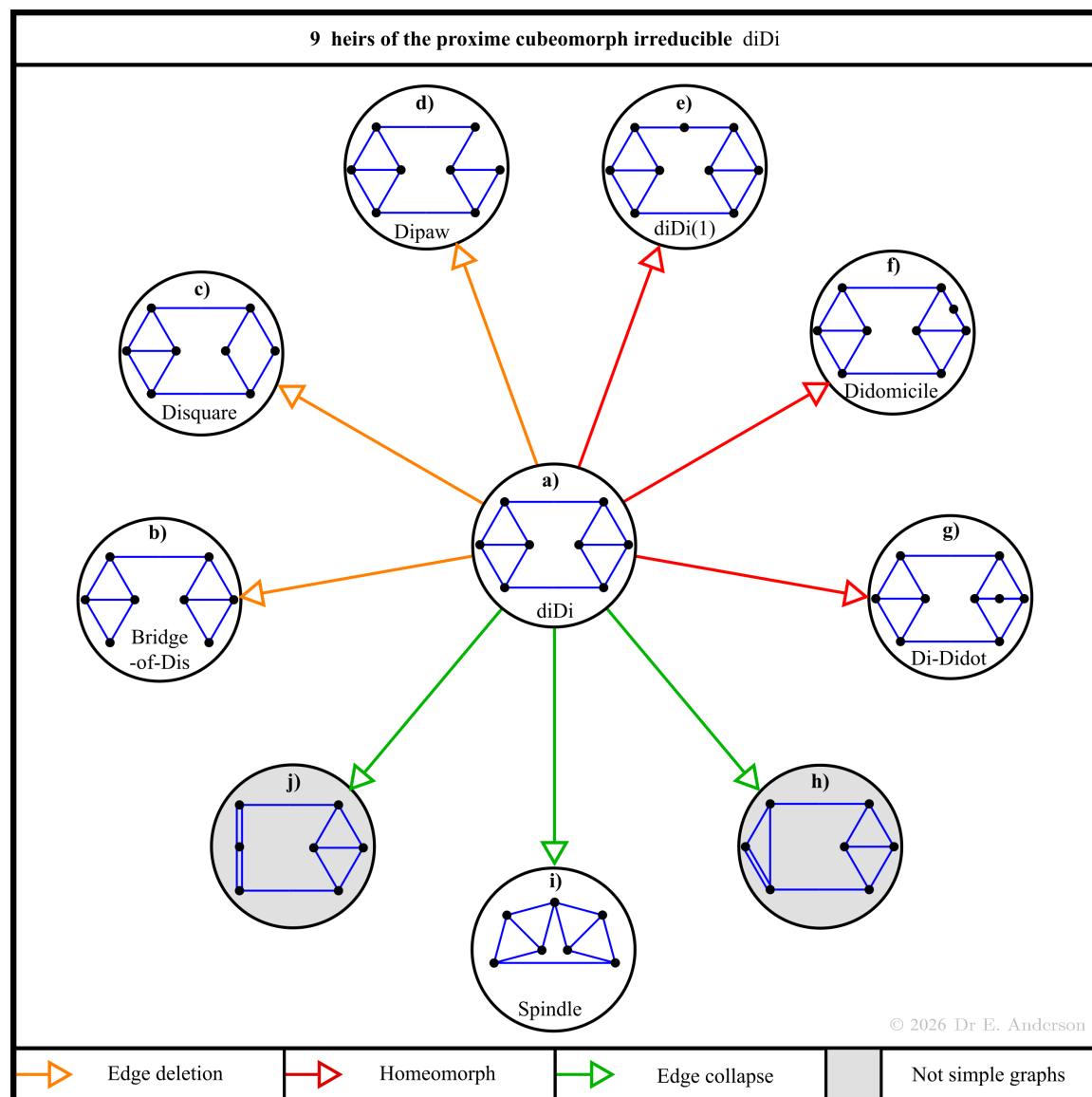


Figure 3:

2 Spindle as a graph

2.1 Some basic counts

Remark 1 $V = 7, E = 11, F = 6$ including the outer face. So indeed

$$V - E + F = 7 - 11 + 6 = 2, \text{ the Euler characteristic of } \mathbb{S}^2.$$

Remark 2 The corresponding symmetry group is

$$\text{Sym}(\text{Spindle}) = C_2^3. \quad (1)$$

So

$$|\text{Sym}(\text{Spindle})| = |C_2^3| = 2^3 = 8. \quad (2)$$

2.2 Planarity

Remark 1 We already exhibited a nice planar presentation in Fig 3.i). See Fig 4 for another with a Graph Drawing and Visualization feature marked. ‘Strongly fitting’ here means that [16] both the vertices and the edges fit the underlying tessellation in the given presentation.

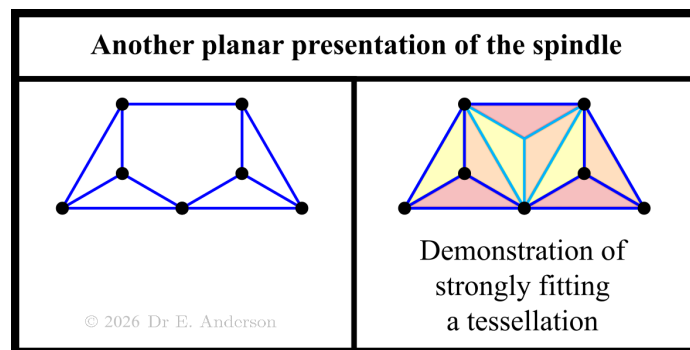


Figure 4:

2.3 Hamiltonianness

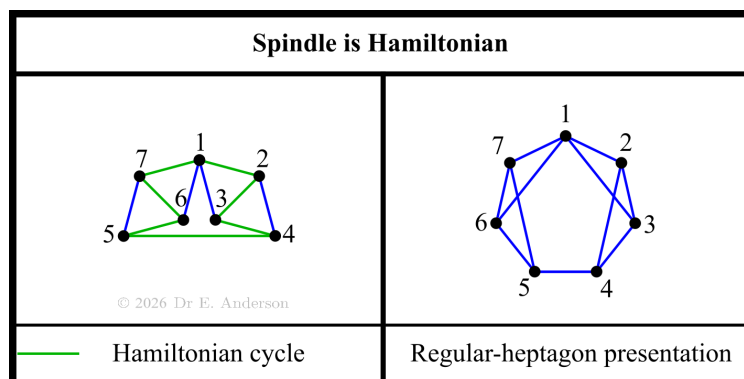


Figure 5:

Remark 1 It is Hamiltonian. See Fig 5.a) for a cycle. And b) for a regular-heptagon manifestly-Hamiltonian presentation.

2.4 Complements

Remark 1 The corresponding complement is

$$\overline{\text{Spindle}} = \text{Utilities}(1) . \quad (3)$$

I.e. the sole first homeomorph supported by the utilities graph, alias complete bipartite graph $K_{3,3}$. We work this out in Fig 6, via the Ramsey presentation on the K_7 chassis. This is furthermore a profitable complement in the sense that

$$|\overline{\text{Spindle}}| = 10 < 11 = |\text{Spindle}| . \quad (4)$$

A useful check here is that

$$E_{\max}(7) = \binom{7}{2} = 21 = 11 + 10 .$$

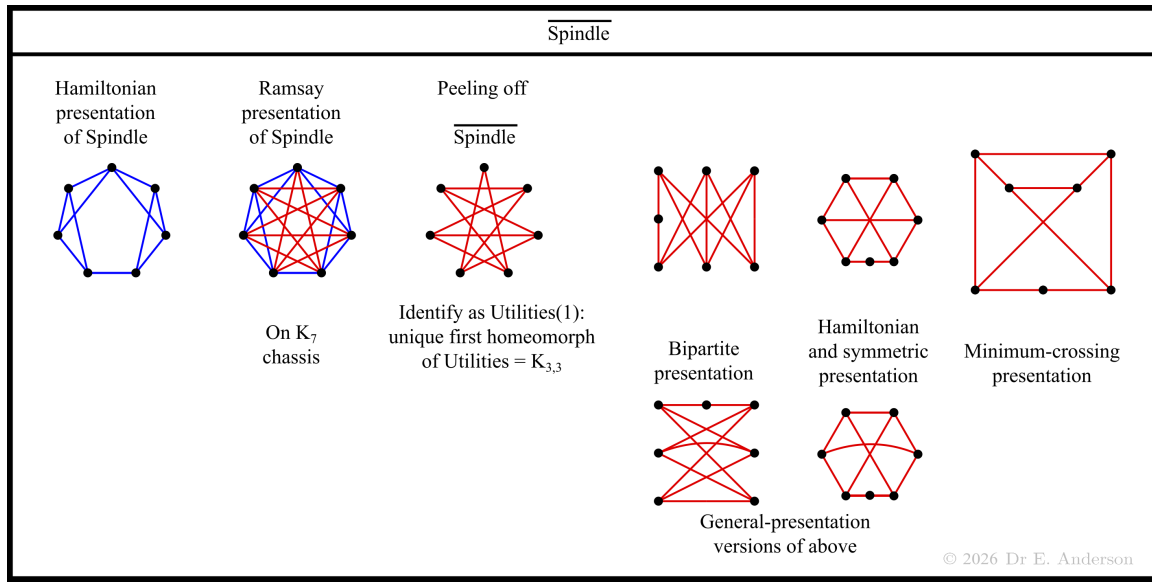


Figure 6:

2.5 Colourability

Remark 1 Spindle is 4-colourable and 4-edge colourable.

A 4-colouring is provided in the cover figure, while a 4-edge colouring is provided in Fig 7.

Remark 1 The above vertex colouring result is crucial below, so let us derive it here.

In any 3-colouring of 1 of its 2 constituent Di , the following occurs. 2 acute-angled vertices of the Di are forced to share colour. But suppose that the shared vertex of the 2 Di shares colour with the 2 opposite acute-angled vertices. Then these 2 vertices are forced to share colour too. But this is in breach of the colouring rule. Contradiction!

Thus ≥ 4 -colouring is forced. But the cover figure exhibits a 4-colouring, so the answer is 4.

Exercise 2 Is Spindle uniquely colourable? Uniquely edge-colourable? If not, what colourability posets [21] are formed?

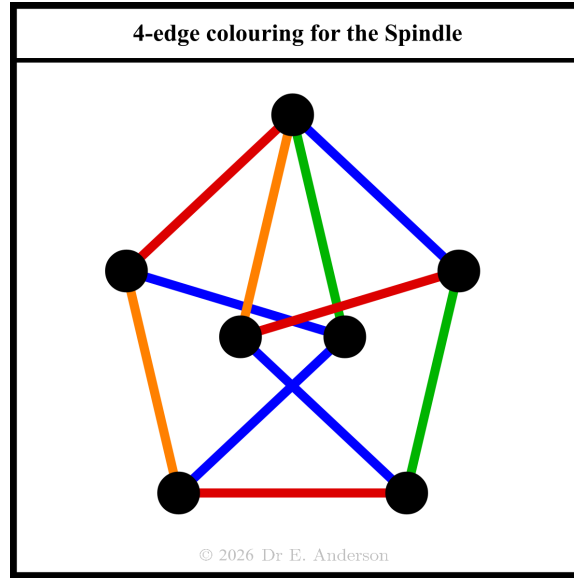


Figure 7:

2.6 Further aliases

Naming Remark 1 The alternative name of *Mosers' spindle* has merit, since it is by the Moser brothers [5] Leo and William, also not to be confused with the great Mathematician Jürgen Moser...

The name *Hajós graph* has also on occasion been used. This is with reference to the *Hajós moves* [6] for constructing k -colourable graphs from complete graphs. Which are of some merit, even if we ourselves choose to proceed instead with the following elementary moves in our simple (quasi-)cubic context. CI-generating moves, cubeomorphs and the tricolour of arrows' moves in Fig 2 (then replayed in Fig 3). We do not however recommend the Hajós graph name, since it has unrelated homonyms.

In the current Article, we just use the name and notation *Spindle* . As we shall see below, *unit-distance high-colourability subgraph* is a merited truer name. More specifically, it is the smallest such, rather than the unique independent such.

3 Spindle’s traditional role in unit-distance graph colouration

3.1 Unit-distance graphs

Definition 1 A *unit-distance graph* is one admitting at least 1 presentation in which all the edges are of the same length. The more usual context for this, which is also the one used in the current Article, is for graphs embedded in the Euclidean plane, $\langle \mathbb{R}^2, || || \rangle$. Also let

$$\mathcal{UD}$$

denote the arena of unit-distance graphs as embedded in the plane. Where ‘arena of’ is shorthand for ‘the Mathematical space formed by the totality of...

Example 1 Only two of Tet and its heirs in Fig 2 admit unit-distance presentations. These are depicted in Fig 8. On this occasion, 2 are found to be already in unit-distance form.

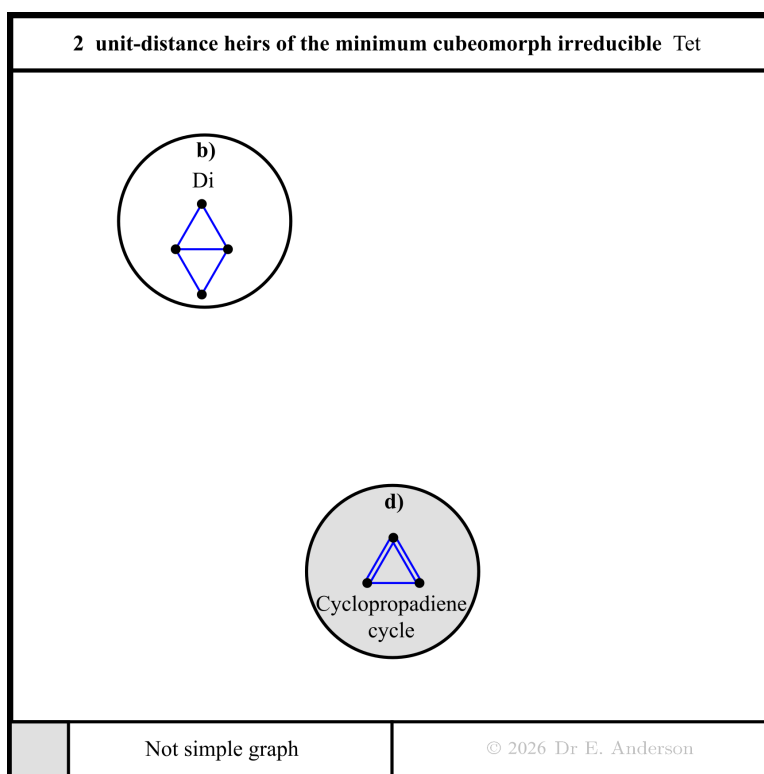


Figure 8:

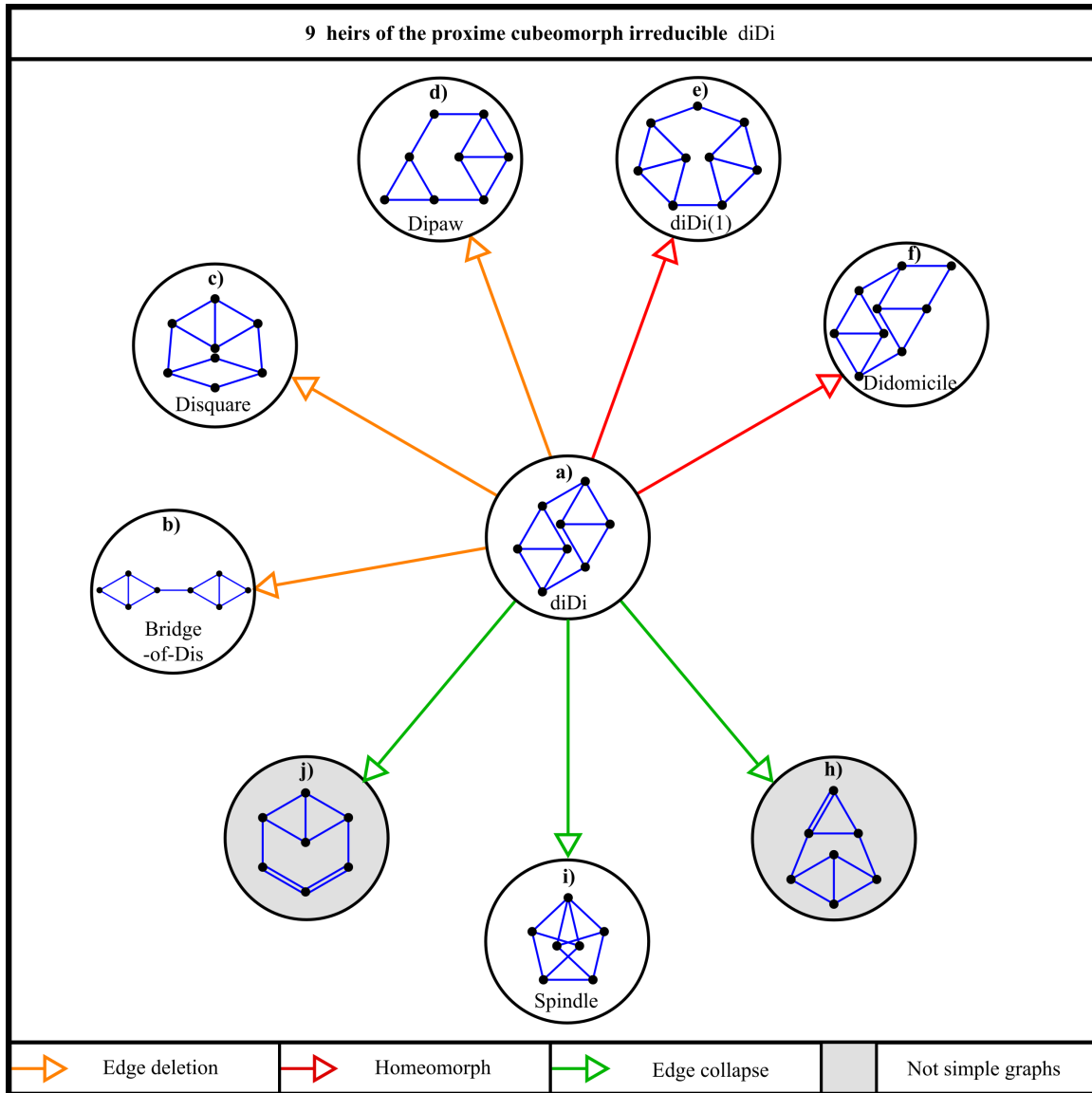


Figure 9:

Example 2 diDi and 8 of its 9 heirs in Fig 3 are unit-distance. We exhibit these in 9; now quite a few require redrawing to manifest the unit-distance property. In particular, the titular Spindle is unit-distance.

Remark 2 We also established that $\chi(\text{Spindle}) = 4$. It is furthermore the unique minimum example of a unit-distance graph with chromatic number 4.

Exercise 3 Prove that this is a minimum, and then that it is the unique such.

3.2 Spindle as a unit-distance graph

Motivation 1 One setting in which the Spindle has multiple calyxenic properties is that of the unit-distance graphs. This is one of the earlier parts of Graph Drawing and Visualization [13] to be widely recognized. And to have lists of Geometrical Combinatorics posed for it.

Remark 1 Here is a first construction of the unit-distance presentation of Spindle . Start with 2 equilateral-triangle Dis with a common hinge vertex. Swing about the hinge to the angle at which the final edge is unit.

The *swinging angle* α is the same as the following. The angle between the constructed midlines of the 2 Di (Fig 10.a). But this midline's length is twice the height of the unit equilateral triangle:

$$2 \frac{\sqrt{3}}{2} = \sqrt{3} .$$

These 2 midlines and the unit base form an isosceles triangle. Produce the perpendicular side bisector corresponding to the base. Then the definition of sine gives that

$$\alpha = 2 \arcsin \left(\frac{1}{2\sqrt{3}} \right) .$$

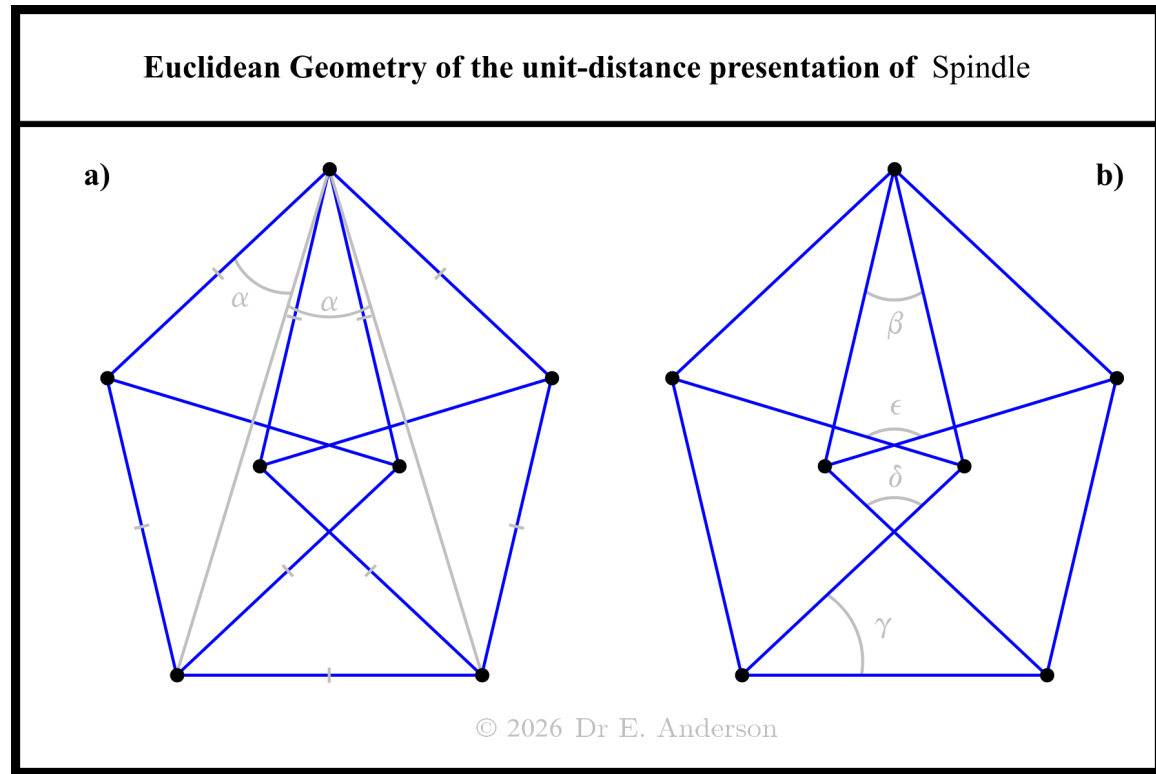


Figure 10:

Exercise 4 Angle-chase to find the $\beta, \gamma, \delta, \epsilon$ marked on Subfig b).

Motivation 1 continued Spindle places a bound on the colourability of unit-distance graphs. By the colouring marked on the cover figure being minimal. This leaves

$$4 \leq \chi(\mathcal{UD}) . \quad (5)$$

3.3 The Hadwiger–Nelson problem

Hadwiger—Nelson Question [10, 20, 2, 4] How many colours are required to colour the points of the Euclidean plane? Such that each unit-distance pair of points are assigned different colours. Equivalently, what is the (vertex-)chromatic number of the following infinite graph. Its vertices are all the points in the plane. While its edges are all unit-distance pairs of points.

Naming Remark 2 Edward Nelson never published this question; it however firstly originated in his student days. Hadwiger had already previously considered the faces-for-vertices dual map problem. A second reason to append Nelson’s name is that this provides a useful distinction since Hadwiger made further conjectures...

A truer name is afforded by the description *Euclidean-plane unit-number chromatic-number problem* (*FIC*). Which has the good fortune of sharing acronym with the front end of its first word: *EUC*.

3.4 Hadwiger’s hexagonal-tiling bound

Lemma 1 In this context, the Euclidean plane’s top regular tessellation – by hexagons – imposes the following basic bound.

$$\chi(\mathbb{E}^2) \leq 7$$

For any given tile has 6 neighbours, and must be coloured differently from them. Leading to

$$1 + 6 = 7$$

colours being required. Here

$$\mathbb{E}^2 := \langle \mathbb{R}^2, || || \rangle : \text{ the Euclidean plane .} \quad (6)$$

3.5 The de Bruijn–Erdős bridge

de Bruijn–Erdős theorem [3] Assuming that the Axiom of Choice holds, the following are equivalent.

- i) An infinite graph is k -colourable.
- ii) Every finite subgraph of it is k -colourable.

Naming Remark 3 A truer name is *finite-heredity of infinite graph colorability*.

3.6 Spindle provides a lower bound

Corollary 1 Spindle imposes the following bound.

$$4 \leq \chi(\mathbb{E}^2) . \quad (7)$$

Remark 1 The following Spindle and hexagonal-tiling sandwich then applies.

$$4 \leq \chi(\mathbb{E}^2) \leq 7 \text{ in the Human era .} \quad (8)$$

With reference to how computer-unassisted humans never managed to improve on either bound. In particular, 5-colourable unit-distance graphs turned out to be too difficult to find.

Exercise 4 Reprove that Spindle is 4-colourable by using an independent sets argument.

4 Spindle's modern role in unit-distance graph colouration

4.1 Enter de Grey

Structure 1 In the intermediary age of computer-assisted proofs, de Grey [15] forced revision of the lower bound from 4 to 5. With a 1581-vertex graph far too large to draw presentations of here. Thus

$$5 \leq \chi(\mathbb{E}^2) \leq 7 \text{ in the intermediary age.} \quad (9)$$

This example contains many Spindle subgraphs.

Remark 1 This counterexample is far from minimal; smaller counterexamples have since been found. Nor has this minimization yet concluded, providing the following topic for further research.

Open Question 1 Find the minimum example of 5-colourable unit-distance graph. Is this example unique? If not, what is its multiplicity?

4.2 Enter Open AI

Claimed Theorem

$$6 \leq \chi(\mathbb{E}^2) \leq 7 \text{ in the AI age.} \quad (10)$$

Remark 1 This was arrived at by [25] a composition making multiple uses of Spindle. Two pieces of Geometry-in-detail enter. That of Spindle. And that of a region that this has the good fortune of fitting within. Multiple copies of which are then used.

Pointer 1 Let us leave this repeated region, and a substantial amount of techniques drawn from elsewhere in Mathematics, for a subsequent occasion. I.e. a conceptual description of the new claimed proof.

4.3 For the future

Open Question 0 Is the answer 6 or 7? The second of these would render Hadwiger's hexagonal-tiling upper bound tight...

Open Question 2 Soifer [20] asked whether 5-colourability can be attained in the absense of Spindle subgraphs. The answer is yes [24] My follow-up question is whether the final minimum 5-colourable unit-distance graphs will involve Spindle. Or not so. Or, via having multiplicity, represent both cases...

Question 3 At least I do not presently know the answer to the following. What is the smallest known unit-distance graph to require 6 colours?

Open Question 4 Minimize this example: the 5 to 6 parallel of Open Question 1.

Open Question 5 Can 6-colourability can be reached without use of Spindle subgraphs? I.e. the 5 to 6 parallel of Open Question 2. To eventually include addressing the counterpart of my follow-up...

5 Some further routes to Spindle

5.1 As an heir of diDi

Structure 1 One of diDi's edge collapses produces the Moser spindle Spindle (Fig 3). And among graphs embedded in the Euclidean plane, this graph admits a presentation – that is usually called Moser spindle itself – that has further calyxenic properties.

Interlude 1 Let us first begin comparison with diDi's 8 other heirs. The first 3 contain 2 vertices 1 short of being cubic. The next 3 contain 1 such vertex each. Only the Spindle (Subfig i), and Subfigs b)-g), are simple graphs.

5.2 As an quasi-cubic graph and CI

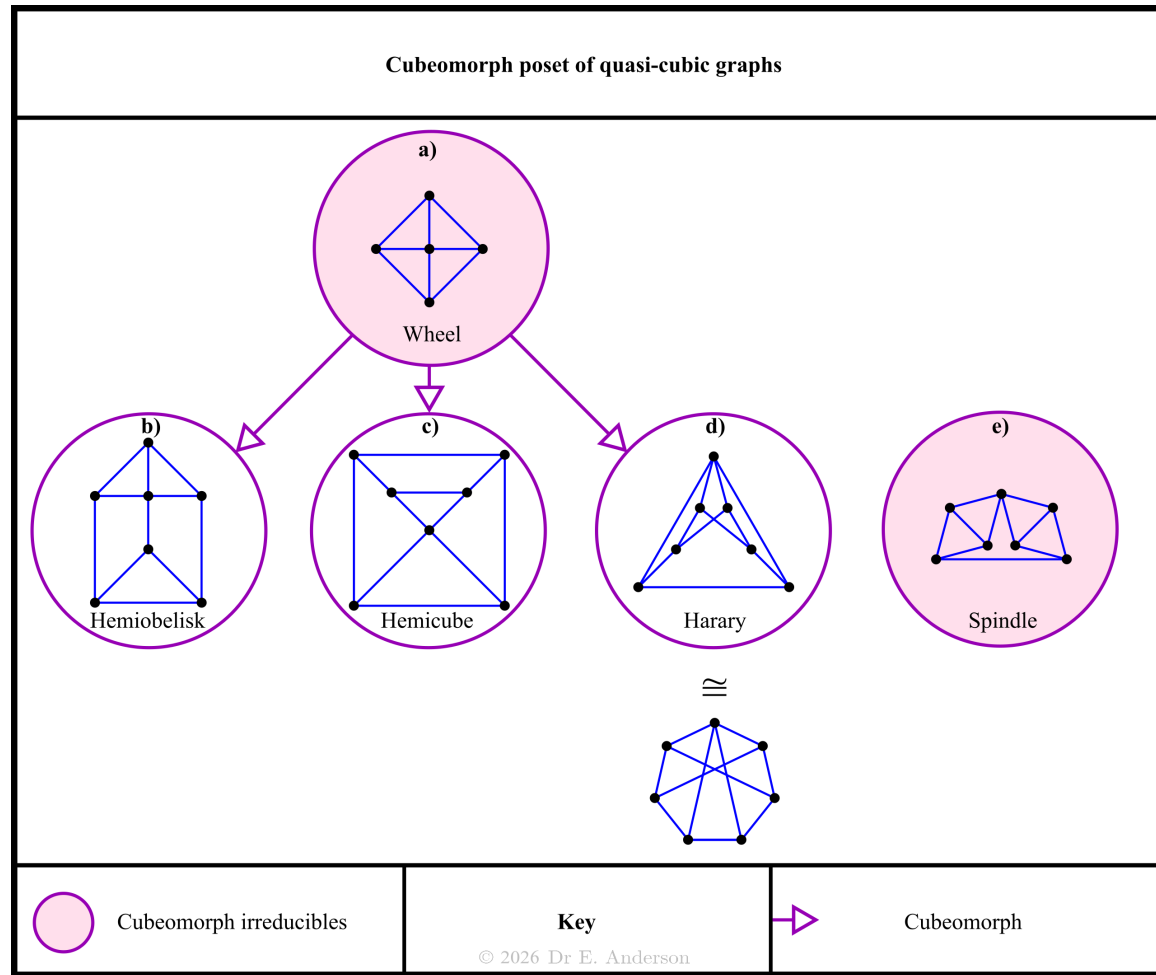


Figure 11:

Structure 1 A *quasi-cubic graph* is a subcase of quasi-regular graph [18, 19]. I.e. a graph such that the vertex degree [17] of every vertex is the same δ . Bar a single vertex whose vertex degree is

$$\Delta = \delta + 1.$$

I.e. $(\delta, \delta + 1)$ -biregular with just the 1 ‘nexus’: maximum-degree vertex

Remark 1 The bottom 3 graphs in Fig 3 are quasi-cubic. Though among these, Spindle is the only simple graph.

In fact, this is among the second-smallest quasi-cubic graphs. For the handshake lemma forces quasi-cubic graphs to have an odd number of vertices.

Tet is the smallest cubic graph. So Fig 11.a)'s Wheel on 5-vertices is the minimum quasi-cubic graph.

Leaving 7 vertices as the second-smallest graph order supported. The multiplicity of this second-smallest case is laid out in the rest of this Figure. In the manner of the poset under cubeomorphs [22]. Among these, Wheel is the minimum quasi-cubic irreducible (MQCI). To Spindle the unique PQCI (proxime). The other three are the possible first cubeomorphs of Wheel.

5.3 The matchstick property

Definition 1 A *matchstick graph* [8, 9] is one that admits ≥ 1 presentation that is manifestly-planar and unit-distance.

Remark 1 In the previous Section, we exhibited Spindle in piecemeal manifestly-planar and unit-distance presentations. And yet planar- and unit-distance-presentabilities turn out to be fully incompatible constraints in the case of Spindle. Rendering it a planar unit-distance graph that does not however meet the definition of *matchstick graph*.

Naming Remark 4 In contrast, diDi is a matchstick graph. Indeed another facet-and-name for diDi is *MCMG: minimum cubic matchstick graph*!

5.4 The (minimally-)rigid planar property

Definition 1 A *rigid planar graph* is one whose edges form a rigid system of rods. A *minimally-rigid planar graph*² is one whose edges minimally form such a system.

Remark 1 Spindle is minimally-rigid planar. Rendering it a minimally-rigid non-matchstick graph! In contrast, diDi is a flexible matchstick graph. For one has some angle parameter freedom of rotation about the Di's acute angle.

Exercise 5 Is Spindle the smallest quasi-cubic graph with the second sentence's minimality property?

Exercise 6 What are the 5 smallest minimally-rigid matchstick graph? The smallest cubic such?

Remark 2 See [23, 26] for interplay between matchstick and minimally-rigid graphs.

5.5 Two further extremal properties

Remark 1 Adding any edge to Spindle breaks the unit-distance property. Thus it is an extremal-edge maximum among the 7-graphs.

Remark 2 Spindle is furthermore a *unidihomeomorph irreducible* and a *unidicubeomorph irreducible* in S. Sánchez' parlance [16]. I.e. no simplifying unit-distance preserving homeomorph is available for it. Nor is any simplifying unit-distance preserving cubeomorph.

²Often alias *Laman graph* [7]. Or, more correctly, *Geiringer–Laman graph* [1] along the lines of building up what turned out to be a rediscovered notion.

6 Further applications

6.1 Testing $2-d$ unit-distance presentability

Remark 1 The Previous SSec’s two properties of `Spindle` have found the following application.

Proposition 1 [11] Testing whether a given graph admits a $2-d$ unit-distance presentation is NP -hard.

Remark 2 The proof amounts to a reduction from Boolean 3-satisfiability. In which `Spindle` plays the central role of truth-setting device.

6.2 Euclidean Ramsey Theory

Proposition 2 Let T be any triangle in the plane. Bipartition the points of the plane into pinks and purples. Then the following alternative holds.

Either there is a purple translate (shifted copy) of T .

Or there is a unit-distance pair of pink points.

Exercise 7⁺ Prove this by making use of `Spindle`.

End Remark 1 The above extra routes, and applications to a wider range of subjects, serve to illustrate that `Spindle` has further calyxenic strength.

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