The Barnette Proof as a Contradiction between Toy-Index Equations

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Abstract

We here offer further interpretation to Open-AI's recent proof of the long-standing Barnette conjecture . We point out that adapted variables scale an unexpected constancy arising in the proof to ± 1 . And that this so happens to be equal to a reflection-symmetric difference. On which grounds we interpret this difference as an index that is forced to be binary sign-valued. The proof proceeds by showing that this difference is furthermore zero, generating a contradiction. We then interpret this as deriving a contradiction between 2 toy index theorems. The binary-sign valued one above and a null one.

Via this observation, we truer-name Barnette's theorem. At least in its aspect of being provable in the above manner. We also point out that in the case of interpreting Grinberg's theorem as a null index theorem in the plane, the following happens. Contentful index theorem behaviour arises from extending the result to a graph embedded in a surface. This leads us to pose an analogous question of whether the Barnette toy index theorems become contentful index theorems for graphs embedded in surfaces. And whether these sometimes, or even always, continue to furnish a contradiction. That would extend Barnette's theorem to some, or all, surfaces respectively.

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1 Introduction

Remark 1 We here offer further interpretation to Open-AI's [23] recent proof of the long-standing Barnette conjecture [6, 19, 22]. We point out that adapted variables [21] scale an unexpected constancy arising in the proof to ± 1 . And that this so happens to be equal to a reflection-symmetric difference. On which grounds we interpret this difference as an index [12, 17, 26] that is forced to be $\mathbb{Z}_2$-valued. The proof proceeds by showing that this difference is furthermore zero, generating a contradiction. We then interpret this as deriving a contradiction between 2 toy index theorems. The $\mathbb{Z}_2$-valued one above and a null one.

Aside 1 We consider the current Article to be a commentary on Open-AI's proof. At the level of further organizing and streamlining its written preprint summary. We shall thus be referring to equation numbers and locations in that preprint without reproducing them. With reference to the state it was in when we last checked: October 16 2026. From the conceptually significant point of view of spotting that toy index theorems are playing a major role in this proof.

Remark 2 Via this observation, we truer-name Barnette's theorem. At least in its aspect of being provable as per [23]. We also point out that in the case of interpreting Grinberg's theorem [5] as [21] a null index theorem in the plane, the following happens [25]. Contentful index theorem behaviour arises from extending the result to a graph embedded in a surface. This leads us to pose an analogous question of whether the Barnette toy index theorems become contentful index theorems for graphs embedded in surfaces. And whether these sometimes, or even always, continue to furnish a contradiction. That would extend Barnette's theorem to some, or all, surfaces respectively.

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2 Preamble

2.1 Colour vs. sign for meaningless yet distinguishable labels

Notational Remark 1 Black and white are quite traditional meaningless and yet distinguishable label names in Combinatorics. For instance, these are also used in the literature on and extending Grinberg's theorem [5], a topic that we shall be returning to later. In the present case, black and white are used to form a shading with alternation across edges. This is a useful 'Geometrical' context in which to shade. Relating such shadings to (say Combinatorial) states is also standard.

Notational Remark 2 = Foundational Remark 1 Something conceptual and technical is however gained at the initially-Algebraic level by the following. Using not colours but signs for pairs of meaningless labels. In the common sense of 2-valued signs $+ -$. As opposed to the also common 3-valued signs $+ - 0$.

We elect to use not b and w for black and white counts but $n_{\pm}$ sign counts. For each of the three fonts they use: lower-case primed and unprimed, and upper case.

2.2 Sum and reflective difference = proto-index

Definition 1 The *sign sum* and *sign difference* are as follows.

$$\Sigma n \ := \ n_+ \ + \ n_- \ . \tag{1}$$

$$\Delta n \ := \ n_+ \ - \ n_- \ . \tag{2}$$

Remark 1 Here are two ways in which differences are more distinguished than the corresponding sums.

Remark 2 This difference is furthermore 'reflective': between like objects. Since the sign values are meaningless yet distinguishable, one could also use the following.

$$n_- \ - \ n_+ \ = \ - \, (\, n_+ \ - \ n_- \,) \ = \ - \, \Delta n \ . \tag{3}$$

I.e. our difference switches sign under meaningless label exchange.

Most other differences are not the differences of like quantities to this extent. One convention is to use δ for these other differences, reserving Δ for reflective differences.

Remark 3 'Proto-indices' is a further alias, since a subset of reflective differences are *indices*. A further property in this case is $\mathbb{Z}$-valuedess. In some cases, there is an obvious reason for this property, while in others there is not. It can come to pass that an index from one branch of Mathematics can be equal to that from another. This is an incipient recipe for an index theorem; more specifically, one is to equate Analytic information to Topological information. Then via an index theorem a non-obviously $\mathbb{Z}$-valued quantity can be indirectly established to in fact be $\mathbb{Z}$-valued.

Using sign labels in place of colour labels places one in the framework in which looking for indices and index theorems is specifically on the cards.

Structure 1 Together, the adapted variables Δ and Σ amount to using flat 2-d characteristic variables [3]. I.e. those for the wavespeed $= 1$ 2-d wave equation in 2-d flat space. Or those picking out the null cone – exceptionally as a null wedge – in 2-d Minkowski spacetime [15]. Now under the fundamental choice of units of Theoretical Physics, in which the speed of light is 1 .

3 A new perspective on the world's first proof of Barnette's conjecture

3.1 Incipient reformulation in terms of a proto-index Δ

Remark 1 Below (3.2), their equation can be recast as follows in proto-index form.

$$3\,\Delta n' \;=\; \sum_{j\,=\,1}^{d} \sigma_j \;. \tag{4}$$

Where the σ_j are themselves already signs in the original account.

Remark 2 (3.3) is then

$$|\,\Delta n'\,| \;\leq\; d \,-\, 4 \;. \tag{5}$$

Leading to the summed version

$$\Delta n \;:=\; \sum_{non-\triangle\, s} \Delta n' \;\leq\; \sum_{non-\triangle\, faces} (\,d \,-\, 4\,) \;. \tag{6}$$

3.2 First instance of a sum Σ

Structure 1 The bottom of their page 4 marks the first appearance of a Σ . So the above widely known characteristic variables recur as adapted variables for this affirmative proof of Barnette's conjecture. In the form of a Σ, Δ pair.

Specifically, from our perspective,

$$2\,m \;=\; 4\,f \,-\, \Sigma n \,+\, \sum_{non-\triangle\, faces} (\,d \,-\, 4\,) \;\geq\; 2\,(\,2\,f \,-\, n_+\,) \;. \tag{7}$$

Which amounts to a sign-asymmetric solving step. Furthermore, division by 2 would save a bit of ink here.

3.3 A surprising integer solution equal to a proto-index...

Structure 1 (4.1) then comprises the characteristic doublet

$$\Sigma N \;=\; 2\,I \,+\, h \,-\, 2\;, \tag{8}$$

$$\Delta N \;=\; \frac{2\,l \,-\, h}{3} \;. \tag{9}$$

Together with (4.2)

$$N_+ \;=\; I \,+\, l\;, \tag{10}$$

these form a system of simultaneous linear equations.

Remark 1 If presented with no further context, it is under-determined: 3 equations in 5 unknowns. Generically, one can eliminate 2 variables to leave the equation of a flat.

Remark 2 Let us look however at what happens for this particular system. Firstly treat the third equation as a solution for N_+ . Secondly, use this to eliminate N_+ from the first 2 equations. Thirdly, make the following judicious choice of adapted variable

$$S \;:=\; \frac{2\,l \,-\, h}{3} \;. \tag{11}$$

Fourthly, eliminate N_- :

$$h + 2(I - 1) - (I + l) = (I + l) - S . \tag{12}$$

$$\Rightarrow \quad -2 = 2l - h - S = 3S - S = 2S \tag{13}$$

$$\Rightarrow \quad S = -1 : \tag{14}$$

a constant and an integer.

They do observe the presence of this linear combination returning a specific constant value. They do not however remark on this being furthermore $\mathbb{Z}$-valued. Nor do they adhere to presentational homogeneity so as to exhibit the third piece of the solution:

$$N_- = I + l - S = I + l - 1 . \tag{15}$$

Consequently, we write the following more lucid line.

$$\Delta N = I + l - (I + l + 1) = -1 . \tag{16}$$

So the unexpected constant-integer portion of the solution is furthermore equal to a proto-index. Fixing the value of this to be a constant integer. I.e. one is dealing with an actual index that is participating in a toy index theorem.

They did not reach this interpretation in their article. Nor did they rescale their linear combination to standardize its magnitude to 1 . I.e. our perspective's third adapted variable is specifically the scale-factor $\frac{1}{3}$-rescaling of their linear combination.

Our perspective's key initial equation involving the index is as follows.

$$\Delta N = S = -1 . \tag{17}$$

Remark 3 The other part of their proof in Sec 4 returns the $+$ sign counterpart as its constant. So a fortiori, our rescaling renders the constant integers in question sign-valued: ± 1 Signifying that we have more specifically hit upon a $\mathbb{Z}_2$-valued index: the smallest non-null kind. This underpins our notation S for this adapted variable, standing for 'sign-valued'...

3.4 The closed-spoke-curve proto-index

Structure 1 One can interpret their Sec 5 in terms of a further proto-index

$$\Delta L(C) . \tag{18}$$

Which, according to (5.3) obeys the strict null inequality

$$\Delta L(C) > 0 . \tag{19}$$

While (5.6) can be recast as the expression

$$-\prod_{C \subset D} \Delta (i \exp(x L(C))) . \tag{20}$$

With first nonvanishing Taylor coefficient

$$(-i)^{C(D)} \prod_{C \subset D} \Delta L(C) . \tag{21}$$

I.e. now with a power-of-i fine-graining in $\mathbb{C}$ of the notion of sign factor: periodicity 4 .

3.5 The contradiction step

Remark 1 Since so far we have an under-determined set of linear equations, it is desirable to increase the number of equations. A common hope in such a search is to eventually reach a well-determined system of linear equations. This is not however the only possible outcome. For instance one might find a coupled nonlinear equation, changing the character of the problem. Or one might eventually over-determine, generically leading to inconsistency. Or, as per the below, one might even find inconsistency before over-determining. See the next Section for some context and basic references about the role of inconsistency.

Structure 1 In our conceptualization and notation, (6.1) is

$$\Delta N \;=\; 3\,S \tag{22}$$

While also

$$\Delta N \;=\; S \tag{23}$$

holds. Consequently

$$S \;=\; 0\;. \tag{24}$$

Contradicting its previously established possible fixed values of ± 1.

Naming Remark 1 Hence we call (22, 23) the *zero-weight-fixing* pair of equations

Remark 2 A missing line of note according to our current perspective is as follows. Back-substitute to obtain

$$\Delta N \;=\; 0\;. \tag{25}$$

Which, since its RHS is $0 \in \mathbb{Z}$, we can interpret as a toy index theorem. Of the simplest kind – *null* – since a sole integer – 0 – solves.

3.6 Inconsistencies in general versus inconsistencies between toy index theorems

Remark 1 It is very widely known that systems of equations can include inconsistencies. Linear Mathematics courses quite often illustrate this [18]. While Dirac [4] emphasized this in a Physical context. An inconsistent outcome is thus not particularly remarkable from the point of view of a simultaneous linear system of equations.

Remark 2 What is more unusual is for the inconsistent equations in question to be toy index theorems. Let us end by packaging up this inconsistency into the conceptual form

$$\underbrace{\pm 1}_{\text{sign value}} \quad = \quad \underbrace{\Delta N}_{\text{sign-count index}} \quad = \quad \underbrace{0}_{\text{null value}}\;. \tag{26}$$

4 Discussion

4.1 Versus the past: Tutte and Tait

Remark 1 To Tait's version of the conjecture, without bipartiteness, Tutte produced a counterexample. Which was subsequently massaged into minimum counterexample form by Barnette himself, among others.

Remark 2 Tutte [7] then tried to use bipartiteness instead of planarity. But the Horton graph is a counterexample to this [8]. In this case, the world has not yet finished minimizing our known counterexamples...

Remark 3 In contrast, consider including both of these concepts, as Barnette did. I.e. the top element of the simple Wille concept lattice drawn out in [24]. Then enough interlocking structure turns out to be present for a theorem rather than a counterexample. And, at least looking at the one available claimed proof of this, some of the structure present takes the form of toy index theorems. Far from any pair of adjectives like 'bipartite' and 'planar' would play along so nicely.

Tutte must therefore be credited for bringing in bipartiteness. And Barnette as well, for considering the combination of these two adjectives. As opposed to Tutte's replacement of Tait's 'planar' with his 'bipartite'.

4.2 Toward the future: versus Grinberg and its extension to surfaces

Remark 1 In the plane, Grinberg's theorem [5, 13, 14] can be massaged into the form

$$\Delta T \ = \ 0 \ . \tag{27}$$

Where T is the triangulation count of each of the 2 outerplanar strips. That the graph is partitioned into by the Hamiltonian cycle. As resulting from triangulating up whichever higher-polygon elementary faces are present. I.e. a conceptually clearer reckoning of what used to be called face strength. Or more eventually excess face strength [21].

It is thereby a null index theorem. In contrast, the known claimed proof of Barnette's conjecture amounts to forming a contradiction. By deriving a null index theorem Alongside a signed index theorem: equal to a constant $\mathbb{Z}_2$-valued sign . The $0 \ = \ 1$ ensues, which is a base-level contradiction.

Naming Remark 2 This led to a truer-naming of Grinberg's theorem as the *ZIPHoN* theorem [21, 25]. I.e. the zero-index planar Hamiltonian Necessity theorem... Pronounced 'siphon!'

Leading me to now pin the truer name *SIPBHaN* theorem on Barnette's theorem [To the extent that an affirmative proof has been claimed for this.] I.e. the Signed-index planar-bipartite Hamiltonian-necessity theorem. Pronounced, to be contrarian about one's sibilences, 'zip-ban!' This is with further reference to the above claim. Only Tolstoy's two strongest warriors will tell if other proofs totally bypass constructing what I have here identified to be *toy-index-theoretic* contradictions...

Grinberg's theorem is moreover well-known to [9, 10, 20] more intricately extend away from the planar case to that of graphs embedded in surfaces. We furthermore interpreted this departure as a contentful (rather than null) index theorem [25, 26]. To the extent that we were already in the process of organizing a toy index theorems Summer School for 2027 even before the candidate proof outlined in [23] appeared. We shall now do so with increased vigour. Posing for today a question that is well-motivated by the conceptualization we unveiled in the current Article.

Open Question 1 Does the toy-index-theoretic contradiction in proving Barnette's theorem in the plane extend to the following. A more contentful index-theoretic contradiction in extending Barnette to graphs embedded in some other, or all, surfaces? In two parts: firstly do the toy index theorems picked out in the current Article extend to graphs embedded on surfaces? And, if so, do they continue to conspire to engineer a contradiction? Or are there cases in which a contradiction is avoided?

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