Eigentheory of $\mathbb{R}$-Symmetric Matrices from a Graph and Order Theory point of view

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Chapter 1

Introduction with Linear Algebra preamble

1.1 The eigenequations

1.1.1 Eigenvalue equation

Remark 1 Throughout, let $\boldsymbol{M}$ be a $\mathbb{R}$-valued square of some finite size $N \in \mathbb{N}$.

Definition 1 An *eigenvalue* [246, 59, 213, 257, 238, 39, 249, 247, 279, 216, 28, 253] of $\boldsymbol{M}$ is any number λ solving

$$\det(\boldsymbol{M} - \lambda \mathbb{I}) = 0 \,. \tag{1.1}$$

Naming Remark 3 Eigenvalues are alias *proper values* and *characteristic values*. (1.1) is often called the *characteristic equation*. Though, to match, it can also be called the *eigenvalue equation*. It is also alias *secular equation* in some older texts typically on Physics.[1]

Remark 2 Eigenvalues are indeed basis-independent, via

$$\begin{aligned}
\det\left(\widetilde{\underline{\boldsymbol{M}}} - \lambda \underline{\mathbb{I}}\right) &= \det\left(\underline{\boldsymbol{P}} \cdot \underline{\boldsymbol{M}} \cdot \underline{\boldsymbol{P}}^{-1} - \lambda \underline{\mathbb{I}}\right) = \det\left(\underline{\boldsymbol{P}} \cdot \underline{\boldsymbol{M}} \cdot \underline{\boldsymbol{P}}^{-1} - \lambda \underline{\boldsymbol{P}} \cdot \underline{\boldsymbol{P}}^{-1}\right) \\
&= \det\left(\underline{\boldsymbol{P}} \cdot \left[\underline{\boldsymbol{M}} - \lambda \underline{\mathbb{I}}\right] \cdot \underline{\boldsymbol{P}}^{-1}\right) = \det\left(\underline{\boldsymbol{P}}\right) \det\left(\underline{\boldsymbol{M}} - \lambda \underline{\mathbb{I}}\right) \det\left(\underline{\boldsymbol{P}}^{-1}\right) \\
&= \det\left(\underline{\boldsymbol{P}}\right) \det\left(\underline{\boldsymbol{M}} - \lambda \underline{\mathbb{I}}\right) \det\left(\underline{\boldsymbol{P}}\right)^{-1} = \det\left(\underline{\boldsymbol{M}} - \lambda \underline{\mathbb{I}}\right)
\end{aligned} \tag{1.2}$$

Where the first step introduces an arbitrary change of basis. The second uses the identity property. The third factorizes. The fourth uses the multiplicative property of the determinant. The fifth uses the multiplicative property again, now on the identity property. So as to establish that the determinant of an inverse matrix is the reciprocal of the determinant of the matrix itself. The last step then cancels. Revealing overall that the eigenvalue equation with respect to the target basis transforms under change of basis to the eigenvalue equation with respect to the source basis.

Structure 1 Let us celebrate by casting the eigenvalue eqution in coordinate-free terms:

$$\det(E - \lambda I) = 0 \,. \tag{1.3}$$

For E the endomorphism that is represented by the previously considered square matrix $\boldsymbol{M}$ with respect to some particular basis. And I the identity endomorphism.

[1] This refers to its application to modelling slow Astronomical processes: on the timescale of centuries: saecula in Latin.

1.1.2 Eigenvector equation

Definition 2 Having found a particular eigenvalue λ for a matrix $\boldsymbol{M}$, the corresponding *eigenvectors* are any nonzero vector solution $\boldsymbol{v}$ of the following *eigenvector equation*.

$$\underline{\overline{\boldsymbol{M}}} \cdot \overline{\boldsymbol{v}} \;=\; \lambda\, \overline{\boldsymbol{v}}\;. \tag{1.4}$$

This is also basis independent:

$$\begin{aligned} \widetilde{\underline{\overline{\boldsymbol{M}}}} \cdot \widetilde{\overline{\boldsymbol{v}}} \;&=\; \left(\underline{\overline{\boldsymbol{P}}} \cdot \underline{\overline{\boldsymbol{M}}} \cdot \underline{\overline{\boldsymbol{P}}}^{-1}\right) \cdot \left(\underline{\overline{\boldsymbol{P}}} \cdot \overline{\boldsymbol{v}}\right) \;=\; \underline{\overline{\boldsymbol{P}}} \cdot \underline{\overline{\boldsymbol{M}}} \cdot \left(\underline{\overline{\boldsymbol{P}}}^{-1} \cdot \underline{\overline{\boldsymbol{P}}}\right) \cdot \overline{\boldsymbol{v}} \\ =\; \underline{\overline{\boldsymbol{P}}} \cdot \underline{\overline{\boldsymbol{M}}} \cdot \underline{\overline{\mathbb{I}}} \cdot \overline{\boldsymbol{v}} \;&=\; \underline{\overline{\boldsymbol{P}}} \cdot \underline{\overline{\boldsymbol{M}}} \cdot \overline{\boldsymbol{v}} \;=\; \underline{\overline{\boldsymbol{P}}} \cdot (\lambda\, \overline{\boldsymbol{v}}) \;=\; \lambda \left(\underline{\overline{\boldsymbol{P}}} \cdot \overline{\boldsymbol{v}}\right) \;=\; \lambda \widetilde{\overline{\boldsymbol{v}}}\;. \end{aligned} \tag{1.5}$$

Where in the first step we use both the matrix's and the vector's basis changes. In the second step, we refactorize. In the third step, we use the inverse property, and in the fourth, the identity property. In the fifth, we use the eigenvector equation in the original basis. In the sixth step we refactorize, and in the seventh we apply the vector's basis change in reverse.

Structure 2 So let us also cast this in basis-independent form.

$$E(\boldsymbol{v}) \;=\; \lambda\, \boldsymbol{v}\;. \tag{1.6}$$

Chapter 2

Placing eigentheory within Mathematics

2.1 Conceptualizing about the eigenvector equation

2.1.1 The linear systems point of view

Remark 1 The eigenvector equation is a Linear-Algebraic system (LAS). Whose corresponding singularness condition is the eigenvalue equation. Furthermore, the eigenvalue variable λ is the one to solve for so as to enforce singularness.

Remark 2 We can either entertain this from a matrix or from a linear map point of view.

2.1.2 The invariant subspace formulation

Definition 1 Let $\mathfrak{X}$ be a Mathematical space, $End : \mathfrak{X} \longrightarrow \mathfrak{X}$ an endomorphism and $\mathfrak{S} \leq \mathfrak{X}$ a subspace. Then $\mathfrak{S}$ is *invariant under* End if

$$End(\mathfrak{S}) \subseteq \mathfrak{S} .$$

Example 1 Let us now specialize to the case in which $\mathfrak{X} = \mathfrak{V}$ is a v.s. $\mathfrak{S} = \mathfrak{W}$ is a vector subspace. And $End = E : \mathfrak{V} \longrightarrow \mathfrak{V}$ is a linear endomorphism. Then the image (of the restriction of E to $\mathfrak{W}$)

$$\mathfrak{Im}_{E|_{\mathfrak{W}}}(\mathfrak{W}) = E(\mathfrak{W})$$

is itself a vector subspace of $\mathfrak{W}$ and thus of $\mathfrak{V}$. I.e.

$$E(\mathfrak{W}) \leq \mathfrak{W} \leq \mathfrak{V} . \tag{2.1}$$

Example 1.1 The map version of the eigenvector equation (1.4) is a subcase of the specification of an invariant subspace of a v.s.

Here solving for eigenvalues λ amounts to picking values such that

$$E_\lambda := E - \lambda Id$$

has nontrivial kernel,

$$Ker_{E_\lambda}(\mathfrak{V}) > 0 .$$

The possible eigenvectors are then the nontrivial elements of this kernel. For in Linear Algebra, nontrivial kernels rest upon noninvertibility. A diagnostic for which is a zero determinant, And in the current context, this is the eigenvalue equation:

$$0 = \det E_\lambda = \det (E - \lambda Id) . \tag{2.2}$$

More fully, the theory of LASs gives that this kernel consists of all vectors obeying the eigenvector equation [257]. Whether or not one makes use of the above diagnostic, and thus of determinants, along the way.

Remark 3 The invariant subspace formulation moreover itself constitutes a determinant-free formulation of Linear Algebra's eigentheory [246, 257].

Remark 4 Finally both the previous and the current subsections have provided reasons to *conceptualize first about the eigenvector equation*. For all that the mainstream approach is to *calculate first using the eigenvalue equation*, to whose conceptual type we next turn.

2.2 Technical properties of the eigenvalue equation

2.2.1 The eigenvalue equation's Algebraic profile

Remark 1 Since the v.s. dimension N is finite, this determinant equation (2.2) is of polynomial form

$$Eig(\, x;\, \boldsymbol{M}\,) \;=\; 0\;. \tag{2.3}$$

Hence the further matching aliases *characteristic polynomial*, *proper-value polynomial*, *eigenvalue polynomial* and *secular polynomial* for the object on its nonzero side.

Remark 2 This polynomial is furthermore *monic*, meaning that its largest power's coefficient is 1. [This largest power is indeed the Nth power.] I.e.

$$Eig(\, x;\, \boldsymbol{M}\,) \;=\; x^N \;+\; \sum_{k\,=\,n}^{0} a_k\, x^k\;. \tag{2.4}$$

Where also

$$n \;:=\; N \;-\; 1\;. \tag{2.5}$$

Remark 3 Thus specializing our matrix to have real entries does not guarantee Algebraic closure of its eigenvalue equation. This limits how useful it is to study the matrices with real coefficients. To fully get around this, one would need to replace $\mathbb{R}$ with its Algebraic closure, $\mathbb{R}_{\text{clos}} \;=\; \mathbb{C}$.

Remark 4 The eigenvalue equation is a degree-N polynomial equation. Consequently $N \;=\; 5$ is minimum for the eigenvalues to generically not be analytically-expressible in terms of sums, products and powers. This follows from what is usually the main result in a first course in Galois Theory [13, 30]. As a standard feature of degree-5 polynomial equations, alias *quintics*. Which persists for all subsequent N.

Remark 5 Eigentheory is consequently generically forced to be [249] an approximate subject [216, 28, 253].

Only the following subcases can be fully treated exactly.

1) $N \;\leq\; 4$.

2) Those nongeneric cases for $N \;\geq\; 5$ that factor entirely into products of $K \;\leq\; 4$ – at-most quartic – factors.

$$\det E_\lambda \;=\; \prod_{i\,=\,1}^{q} Q_i(\lambda) \;=\; 0\;. \tag{2.6}$$

Additionally the following case admits a partial exact treatment.

3) Eigenequations that partly decompose into products of $K \leq 4$ – at-most quartic – factors. I.e.

$$\det E_\lambda \;=\; R(\lambda) \prod_{i\,=\,1}^{q} Q_i(\lambda) \;=\; 0 \;. \tag{2.7}$$

for $\deg R(\lambda) \geq 5$ – at least quintic – can have all of its other factors treated exactly.

Observe that 1), 3) and 2) form an increasingly general nest of conditions.

Remark 6 Cases with purely $K \leq 2$ – at-most quadratic – factors in place of just $K \leq 4$ – at-most quartic – ones are furthermore far more straightforward to handle exactly. This of course reflects the straightforwardness of linear factors, and the quadratic equation, relative to its cubic and quartic relatives [23].

2.2.2 The eigenspectrum is the eigenvalue equation's solution set

Structure 1 Let us denote by E the total number of distinct eigenvalues possessed by a matrix, or a linear map. This is our first simple example of a *spectral cardinal*. I.e. a well-defined number that one can extract from a matrix's (or tensor's or operator's...) spectral information. We shall encounter many further spectral cardinals over the course of this Review.

Definition 1 The *eigenspectrum* [279, 20, 41, 247] of $\boldsymbol{M}$ is the set of distinct values realized as eigenvalues of $\boldsymbol{M}$. I.e. in the finite v.s. case,

$$\mathfrak{spec}(\boldsymbol{M}) \;=\; \{\, \lambda_e \;,\;\; e \;=\; 1 \;\text{ to }\; E \,\} \;=\; \lambda_1 \quad \ldots \quad \lambda_E \;. \tag{2.8}$$

Notational Remark 1 The first symbol denotes the eigenspectrum formulated as an arena. The second notation is for a set with E elements. Finally the third is for an array symbol, simpler, cleaner, and the smallest of a family of such that we shall build up below.



2.2.3 A first interrelation with the Cayley–Hamilton theorem

Theorem 1 (a weak form of the Cayley–Hamilton Theorem) [257, 252, 247, 249]. Any real-valued square matrix satisfies its own eigenvalue equation.

Remark 8 The eigenvalue equation can consequently also be viewed as a matrix-valued polynomial equation. This is not however an equation to be solved for the matrix, since we know the matrix ab initio. A distinct interpretation of this matrix version of the eigenvalue is teased out in the next section.

2.2.4 First factorized form for the eigenvalue equation

Structure 2 In general,

$$Eig(\,x;\,\boldsymbol{M}\,) \;=\; \prod_{p\,=\,1}^{P} P_p^{\delta_p}(x) \;. \tag{2.9}$$

For *monic prime polynomial factors*

$$P_p(x) \;=\; P_p(\,x;\,\boldsymbol{M}\,) \;.$$

2.2.5 The Algebraically closed case's simplification

Structure 3 In this context, the above splits into the following purely-linear factor form.

$$Eig(\, x; \boldsymbol{M} \,) \;\; = \;\; \prod_{e \,=\, 1}^{E} \, (\, x \;-\; \lambda_e \,)^{\alpha_e} \; . \tag{2.10}$$

Remark 9 Observe that $D \;\leq\; E$ And that this constitutes a fine-graining of (*).

Remark 10 One can also get to this form if one accepts to pass to a corresponding $\mathbb{F}_{\text{clos}}$.

Remark 11 Or by the good fortune that, while one's $\mathbb{F}$ is not Algebraically closed, all of the roots so happen to lie within $\mathbb{F}$.

Remark 12 Or by having some protective theorem that guarantees this outcome, which the next subsection provides for the current Article's central case. Let us subsequently jointly refer to this Subsection's 4 cases as condition (*).

2.2.6 Our central case is secure

Lemma 2 (Cauchy) For real symmetric matrices, all eigenvalues are real.

Remark 13 On the one hand, real-valued matrices with fully real eigenspectra is in general rendered flimsy by $\mathbb{R}$'s lack of Algebraic closure. On the other hand, our central case of real-symmetric matrices grants us a secure subcase.

2.2.7 The eigenvalue polynomial's coefficients are invariant

Motivation 1 The top non-monic coefficient is the trace.

$$\text{tr}\, \boldsymbol{M} \;=\; S_1 \;:=\; \sum_{e \,=\, 1}^{E} \lambda_e \;=\; \deg\, A \; . \tag{2.11}$$

The bottom one is the determinant,

$$\det \boldsymbol{M} \;=\; P \;:=\; \prod_{e \,=\, 1}^{E} \lambda_e \; , \tag{2.12}$$

Both are invariants; this continues to hold for the middling coefficients. We can view the whole lot as [**?**] symmetric polynomials [**?**] in the eigenvalues.

$\mathbf{N \;=\; 3}$ is minimum to possess such a middling invariant,

$$a_1 \;=\; \sum_{e \,\neq\, f \,=\, 1}^{3} \lambda_e \, \lambda_f \;\; = \;\; \frac{1}{2} \, \left({S_1}^2 \;-\; S_2 \right) \; . \tag{2.13}$$

Where

$$S_2 \;:=\; \sum_{e \,=\, 1}^{E} {\lambda_e}^2 \; . \tag{2.14}$$

Remark 14 These invariants form a chain of length N ; see Fig **??** for the smallest few.

♣ ♡ ♠



Figure 2.1:

2.2.8 Algebraic multiplicity

Definition 1 The *Algebraic multiplicity* [213, 20, 253, 238, 249, 279] α_e of λ_e is the number of times that it occurs as a root of the eigenvalue polynomial.

Remark 15 If form (2.10) applies, then just read the Algebraic multiplicities off it.

Remark 16 The definition however transcends to the more general case, via the following rearrangement to the mixed form.

$$Eig(\, x; \boldsymbol{M} \,) \quad = \quad \prod_{q \, = \, 1}^{Q} \, Q_q(x)^{\iota_q} \, \prod_{f \, = \, 1}^{F} \, (\, x \, - \, \lambda_f \,)^{\alpha_f} \, . \tag{2.15}$$

Where the λ_f are the roots in $\mathbb{F}$. And the Q are monic at-least-quadratic prime polynomials. This contains (2.10) and the purely at-least quadratic version of (2.9) as obvious subcases.

Remark 16 We can now furthermore distinguish between the *eigenspectrum-in-*$\mathbb{F}$,

$$\mathfrak{espec}_{\mathbb{F}}(\boldsymbol{A}) \quad = \quad \lambda_1 \, ... \, \lambda_F \, . \tag{2.16}$$

And the eigenspectrum-in-some-$\mathbb{F}_{\text{clos}}$,

$$\mathfrak{espec}_{\mathbb{F}_{\text{clos}}}(\boldsymbol{A}) \quad = \quad \lambda_1 \, ... \, \lambda_E \, , \quad \text{for} \quad E \, \geq \, F \, . \tag{2.17}$$

I.e. it is in general an extension of the previous eigenspectrum's list.

2.2.9 Multi-eigenspectra

Notational Remark 2 The totality of eigenvalues for our matrix form the (*Algebraic*) *multi-eigenspectrum*, now with algebraic multiplicities included. I.e.

$$\mathfrak{ae}\mathfrak{espec}(\boldsymbol{M}) \quad := \quad \begin{matrix} \lambda_1 & \dots & \lambda_E \\ \alpha_1 & \dots & \alpha_E \end{matrix} \, . \tag{2.18}$$

This notation is closely related to that in West [240] in the context of Spectral Graph Theory. This gives some precedent to our using arrays for types of eigenspectra.

Remark 17 If $\mathbb{F}$ is not Algebraically closed, there will once again be general distinct in-$\mathbb{F}$ and in-some-$\mathbb{F}_{\text{clos}}$ versions. These coincide for our main real-symmetric matrix case. Obeying

$$\sum_{\mathfrak{espec}(\boldsymbol{M})} \alpha_e \quad = \quad \sum_{e \, = \, 1}^{E} \alpha_e \quad = \quad N \, . \tag{2.19}$$

Where the first version is a conceptual Arena sum, In contrast the second is a computational sum. Many subsequent equations exhibit such 'conceptual–computational' and 'Arena–counting numbers' doublings.

Naming Remark 2 We take 'multi-eigenspectrum' to be a shorter form of 'multi-set eigenspectrum'. Where a multi-set is a point set (Fig 2.2.a) whose points can each occur in multiple copies (Subfig b). This is how e.g. Roman [249] conceptualizes about multi-eigenspectra (Subfig d), as eigenspectra (Subfig c) whose eigenvalues can each occur in multiple copies, for all that he neither gives this a name nor a notation. In a work containing many specific examples, each significant to the Theory of some part of STEM, however, it is useful to take these two further steps.

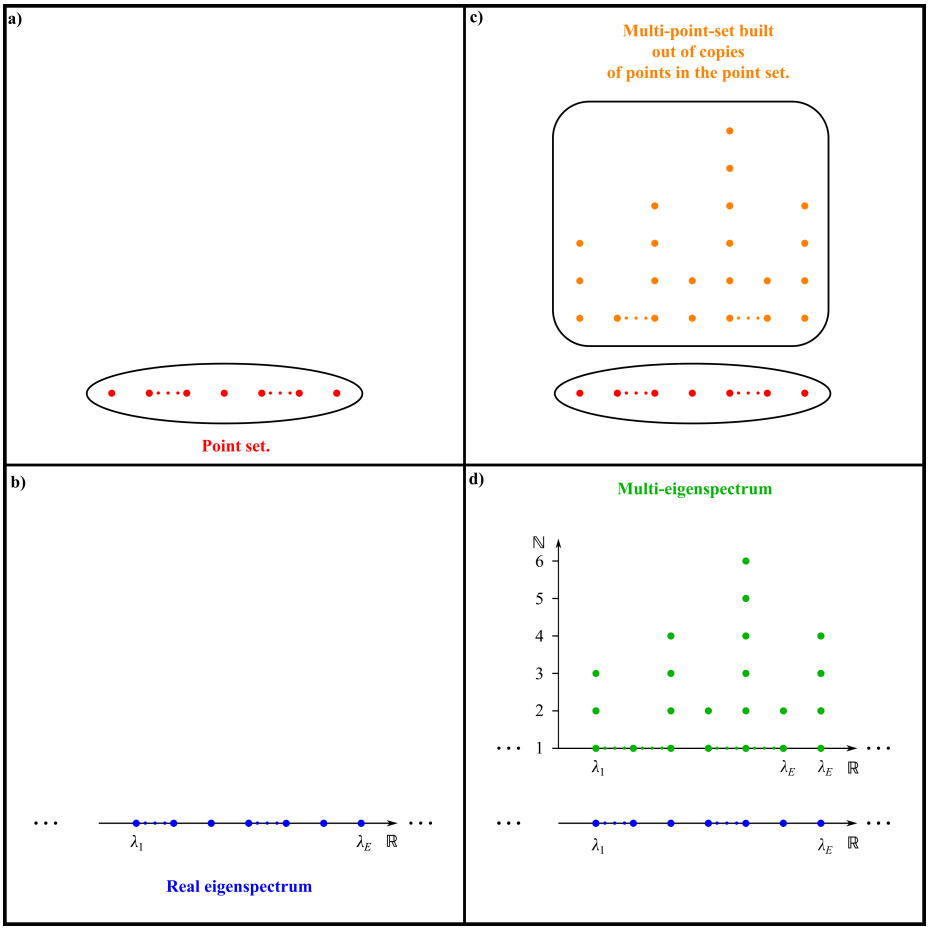


Figure 2.2:

2.3 Examples

Example 1 If $\mathbb{F} = \mathbb{R}$, then we take $\mathbb{R}_{\text{clos}} = \mathbb{C}$. Here non-real roots always form complex pairs, so at-most quadratic collapses to quadratic. Working intrinsically in $\mathbb{R}$ or extrinsically in $\mathbb{C}$ are both probably familiar to the Reader. And our above scheme has begun to lay out how these studies start to diverge from each other in the presence of ≥ 1 pairs of complex roots.

Subexample 1.1 Consider

$$\boldsymbol{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

over $\mathbb{R}$. Then

$$Eig(x; \boldsymbol{M}) = x^2 + 1 .$$

So

$$P = 1 \text{ and } P_1 = x^2 + 1 ,$$

returning no real eigenvalues. Thus

$$\mathfrak{e}\text{spec}_{\mathbb{R}}(\boldsymbol{M}) = \emptyset = \mathfrak{ae}\text{spec}_{\mathbb{R}}(\boldsymbol{M}) .$$

But over $\mathbb{C}$,

$$Eig(x; \boldsymbol{M}) = (x - i)(x + i) = (x - i)(x - (-i)) .$$

For which $E = 2$, $\lambda_1 = i$ and $\lambda_2 = -i$. And $\alpha_1 = 1 = \alpha_2$. So

$$\mathfrak{espec}_{\mathbb{C}}(\boldsymbol{M}) = \begin{matrix} i & -i \end{matrix} .$$

And

$$\mathfrak{aespec}_{\mathbb{C}}(\boldsymbol{M}) = \begin{matrix} i & -i \\ 1 & 1 \end{matrix} .$$

Subexample 1.2 The following, viewed intrinsically within $\mathbb{R}$, is a minimum example for which the mixed form (2.15) is necessary. Consider

$$\boldsymbol{M} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then

$$Eig(x; \boldsymbol{M}) = (x^2 + 1)(x - 1) .$$

So

$$Q = 1 = F \text{ and } Q_1 = x^2 + 1 .$$

While

$$\lambda_1 = 1 \text{ with } \alpha_1 = 1 .$$

So

$$\mathfrak{espec}_{\mathbb{R}}(\boldsymbol{M}) = 1 .$$

And

$$\mathfrak{aespec}_{\mathbb{R}}(\boldsymbol{M}) = \begin{matrix} 1 \\ 1 \end{matrix} .$$

Remark 1 Let us end with a very familiar supporting result, albeit slightly in disguise.

Lemma 3 Any real-valued matrix of size N odd possesses ≥ 1 eigenvalue as modelled intrinsically within $\mathbb{R}$.

2.4 Equal-eigenvalues degeneracy

2.4.1 Degeneracy is an over-used word

Remark 1 Degeneracy is an over-used word, both in Eigentheory and in Mathematics more generally. Consequently, on each use we specify which kind of degeneracy we are referring to...

2.4.2 Equal-eigenvalues degeneracy

Structure 4 A first sense in which a matrix's eigenspectrum can be degenerate is if some of its eigenvalues are equal.

Each Algebraic multiplicity $\alpha_e > 1$ quantifies the extent of this type of degeneracy.

This can be more sharply quantified by the *excess Algebraic multiplicity*[1]

$$\delta\,\alpha_e := \alpha_e - \alpha_{\min} = \alpha_e - 1 . \tag{2.20}$$

[1] In the current Review, *excess* means a relative-difference variable [261, 264, 279, 277]. As formed by subtracting off the minimum nontrivial amount of the corresponding 'absolute' quantity.

For the whole matrix, the following excess sum provides a coarse quantification of this type of degeneracy. I.e.

$$DA \;:=\; \sum_{e \,\in\, \mathfrak{C}\mathrm{spec}(\boldsymbol{M})} \delta\,\alpha_e \;=\; \sum_{e\,=\,1}^{E} \delta\,\alpha_e \overset{(2.20.1)}{=} \sum_{e\,=\,1}^{E} (\,\alpha_e \;-\; \alpha_0\,) \overset{(2.20.2)}{=} \sum_{e\,=\,1}^{E} (\,\alpha_e \;-\; 1\,)$$

$$= \sum_{e\,=\,1}^{E} \alpha_e \;-\; \sum_{e\,=\,1}^{E} 1 \overset{(??,2.28)}{=} N \;-\; E\,. \tag{2.21}$$

Where the penultimate step expands the sum. While the E-tuple

$$(\,\delta\,\alpha_1\,,\quad \ldots\,,\quad \delta\,\alpha_E\,) \tag{2.22}$$

encodes more detailed information. Finally

$$\sum_{e\,=\,1}^{E} \alpha_e \;=\; N\;.?? \tag{2.23}$$

I.e. the algebraic multiplicities partition the dimension N .

2.4.3 Partition interpretation

Remark 2 (**??**) can be reinterpreted as a *partition*. So the Algebraic multiplicities α_e receive the additional interpretation of part sizes, while E is the number of parts. The $\delta\alpha_e$ are then the excess part sizes, supporting a briefer notation.[2]

2.4.4 Examples

Example 2 $N \;=\; 2$ is minimum to realize this type of degeneracy. This corresponds to the *indiscrete partition* 2 : a single piece of size 2 .

Example 0 The least extreme such case has precisely 2 eigenvalues equal. This corresponds to the partition $2\,1^{N\,-\,2}$.

Example 1 The most extreme case supported is has all N eigenvalues equal. This corresponds to the indiscrete partition N .

Example 3 $N \;=\; 3$ is minimum to realize distinct most and least extreme cases. These correspond to the integer partitions $2\,1$ versus $3\,0$.

Example 4 Consider

$$\mathrm{diag}(\,1,\,1,\,2,\,2\,) \;=\; \bigoplus_{i\,=\,1}^{3} i^2 \;\text{ and }\; \mathrm{diag}(\,1,\,1,\,1,\,2\,) \;=\; 1^3 \,\oplus\, 2 \,\oplus\, 3 \,\oplus\, 4\,. \tag{2.24}$$

Then the above coarser quantifier fails to discern between these, returning $\delta\,\alpha \;=\; 2$ in both cases. But the finer quantifier returns $(\,1,\,1\,)$ versus $(\,2,\,0\,)$. These correspond to the partitions $3\,1$ and $2\,2$ respectively.

[2] See Chapter 15 of [264] for an outline of partitions and their denotation. * insert longer Lit-train *

2.5 Annihilating polynomials

2.5.1 The eigenvalue polynomial is an annihilating polynomial

Naming Remark 2 The eigenvalue polynomial, alias proper-values polynomial characteristic(-values) polynomial, and secular polynomial, is furthermore an *annihilating polynomial*:

$$Eig(\boldsymbol{M};\boldsymbol{M}) = 0 . \tag{2.25}$$

For by one of the parts of the Cayley–Hamilton Theorem, a matrix obeys its own eigenequation!

2.5.2 The minimum polynomial

Remark 1 The eigenvalue polynomial has on occasion repeated roots. It is then an interesting question whether striking out some repeated factors preserves the annihilation property. In particular, what is the largest extent to which these that can be struck out while still leaving us an annihilating polynomial?

This simple extremal consideration translates to asking what the minimum-degree annihilating polynomial is. Which is usually referred to as the *minimum polynomial*. I.e. the monic polynomial

$$Min(x;\boldsymbol{M})$$

of minimum degree for which

$$Min(\boldsymbol{M};\boldsymbol{M}) = 0 .$$

We can further unpackage this minimum property as follows. Suppose that

$$Ann(x;\boldsymbol{M})$$

is some other annihilating polynomial. I.e. such that

$$Ann(\boldsymbol{M};\boldsymbol{M}) = 0 .$$

Then

$$Min(x;\boldsymbol{M}) \,\Big|\, Ann(x;\boldsymbol{M}) .$$

Remark 2 In fact, this is not the only conceptualization leading to the minimum polynomial. At least in a v.s.-level study – fields rather than rings – it constitutes a suitable definition [249]. There is furthermore an equivalent definition at the level of modules, which can be found on p. 165 of [249] under the name of *monic order*. Corresponding to reinterpreting the v.s. as a module over the corresponding $\mathbb{F}$-valued polynomial ring. See also [257, 238, 247] for more about the minimum polynomial.

Remark 3 For an arbitrary $\mathbb{F}$,

$$Min(x;\boldsymbol{M}) = \prod_{g=1}^{G} p_g^{\epsilon_g}(x) . \tag{2.26}$$

Where the p_g are the minimum polynomial's own monic prime polynomials.

Remark 4 If $\mathbb{F}$ is closed, then (2.26) takes the further form

$$Min(x;\boldsymbol{M}) = \prod_{e=1}^{E} (x - \lambda_e)^{\mu_e} . \tag{2.27}$$

Remark 1 Furthermore, the degree of the minimum polynomial

$$M := \deg(Min(x;\boldsymbol{M})) = \sum_{\mathfrak{e}\in\mathrm{spec}(\boldsymbol{M})} \mu_e = \sum_{e=1}^{E} \mu_e . \tag{2.28}$$

2.5.3 The maximum annihilating polynomial

Naming Remark 3 *Minimal polynomial* has hitherto been a more common name. Order-Theoretically, however, minimum is a sharper property that this polynomial so happens to possess. Whence our renaming. We shall soon meet the bounded poset for which it serves as bottom element alias greatest lower bound alias in this context minimum.[3]

Naming Remark 4 The eigenvalue polynomial, alias proper-value polynomial, characteristic(-value) polynomial, and secular polynomial, is furthermore the opposing *maximum*, or *top, annihilating polynomial*. Let us denote this by

$$\mathfrak{max}(\, x;\, \boldsymbol{M}\,) \equiv \mathfrak{Eig}(\, x;\, \boldsymbol{M}\,)\ . \tag{2.29}$$

The name *minimal multiplicity* μ is then inherited from the above use of 'minimal'. Thus upon passing to the sharper minimum conceptualization, we correspondingly adjust the inherited name to *minimum multiplicity*. It is true that for any individual eigenvalue, an annihilating polynomial's corresponding multiplicity is chain-valued. And that for chains, the notions of minimal and minimum coincide. However upon considering a matrix as a whole, or its eigenspectrum as a whole, multiplicities become E-tuple-of-numbers-valued. Within these enlarged contexts, then, a poset of annihilating polynomials is in play, giving a reason to favour the name 'minimum multiplicity' over the hitherto more widely used 'minimal multiplicity'.

2.6 Eigenspaces

2.6.1 Piecemeal eigenspaces

Definition 1 Let

$$\mathfrak{Eig}_e(\boldsymbol{M})$$

denote the (*piecemeal*) *eigenspace* [279, 20, 41, 257] spanned by the eigenvectors corresponding to the eigenvalue λ_e .

Lemma 4 Every eigenvalue's corresponding eigenspace is a vector subspace of the underlying v.s. $\mathfrak{V}$.

2.6.2 The total eigenspace

Definition 2 Let us also introduce

$$\mathfrak{Eig}(\boldsymbol{M})$$

to denote the *total eigenspace* of $\boldsymbol{M}$.

Lemma 5 (Eigenspace Decomposition)

$$\mathfrak{Eig}(\boldsymbol{M}) \;=\; \bigoplus_{e\,\in\,\mathfrak{C}\mathrm{spec}} \mathfrak{Eig}_e(\boldsymbol{M}) \;=\; \bigoplus_{e\,=\,1}^{E} \mathfrak{Eig}_e(\boldsymbol{M})\ . \tag{2.30}$$

Remark 1 That some λ_e can have $\alpha_e > 1$ suggests the following. In some subset of cases, there will also be > 1 eigenvectors corresponding to λ_e . Let us form a LI set of these.

Remark 2 These constitute a LB for the corresponding piecemeal eigenspace. In some cases, this is of dimension α_e , while in others, not this many LI eigenvectors can be found. This deficit is measured by the following further multiplicity.

[3] In other contexts, such as Calculus or Optimization, one would need to clarify that this notion is a *unique global* minimum.

2.6.3 Geometrical multiplicity and its rebranding as Arenal multiplicity

Definition 3 An eigenvalue λ_e's *geometrical multiplicity* [213, 20, 257, 247, 238, 249, 279] γ_e is the dimension of the corresponding piecemeal eigenspace,

$$\gamma_e \ := \ \dim \left(\mathfrak{Eig}_e(\boldsymbol{M}) \right) \ . \tag{2.31}$$

Naming Remark 4 This is not however particularly directly tied to Geometry. It is rather a feature of each eigenspace: an ancillary Arena of Linear-Algebra. As such, let us pass to using the name *arenal multiplicity*, which we denote by ρ_e. Standing for 'ancillary Arena: eigenspace', in the context of α_e having already been deployed to denote Algebraic multiplicity.

Remark 3 In (*)'s regime, the arenal multiplicity corresponds to an 'arenal eigenpolynomial'

$$R(x) \ = \ \prod_{e \ = \ 1}^{E} \left(x \ - \ \lambda_e \right)^{\rho_e} \ .$$

As we shall see below, however, this is not necessarily an annihilating polynomial.

2.6.4 The multi^3-eigenspectrum

Remark 1 Let us also introduce the *multi^3-eigenspectrum*

$$\mathfrak{aRMe}\mathrm{spec}(\boldsymbol{M}) \ := \ \begin{matrix} \lambda_1 & \dots & \lambda_E \\ \alpha_1 & \dots & \alpha_E \\ \rho_1 & \dots & \rho_E \\ \mu_1 & \dots & \mu_E \end{matrix} \ . \tag{2.32}$$

This is 'multi' in two further senses: by involving multiple notions of multiplicity! The current Review's example chapters' tables of eigenvalues in effect make use of this notation.

2.7 Diagonal, triangular and band matrices

2.7.1 Diagonal matrices

Remark 1 These solely have nonzero entries along the *leading* diagonal: top-left to bottom-right.

Example 1

$$\boldsymbol{M} \ = \ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \ . \tag{2.33}$$

A standard shorthand notation for which is

$$\boldsymbol{M} \ = \ \mathrm{diag}(\, 2, \, 1, \, 3 \,) \ . \tag{2.34}$$

2.7.2 Diagonalizability

Definition 1 A Linear endomorphism η is *diagonalizable* if its matrix $\boldsymbol{S}$ with respect to some LB is diagonal.

Lemma 6 The elements exhibited on the diagonal of a diagonal matrix are moreover the eigenvalues of the matrix []*ugly misuse of library to look up books I own* .

Lemma 7 The diagonalizing LB-change matrix has the eigenvectors as its columns.

Exercise 6 Redo the first part of Exercise **??**.5 using Exercise **??**.15's line-adapted LB. Generalize this approach to establish that *all* projections have only 2 possible eigenvalues, 1 and 0 (or 'no' and 'yes').

Remark 2 We know that not all matrices admit a LB's worth of eigenvectors from Worked Example 7.1. It follows that not all matrices are diagonalizable, with this Exercise's shear matrix constituting a minimum counterexample. The 2-d (say forwards) shift matrix of Exercise *7.8* also constitutes a minimal counterexample. Such counterexamples are what is meant by 'defective matrices'.

Lemma 8 Eigenspaces corresponding to distinct eigenvalues of a symmetric endomorphism are orthogonal.

Lemma 9 $\mathfrak{V}$ possesses an ONB consisting of eigenvectors of η .

Remark 3 In terms of matrices, this is just recovering the more elementarily-known fact that symmetric matrices are always diagonalizable. And that the diagonalizing matrix $\boldsymbol{P}$ is orthogonal. So in this case, Fig **??** specializes to Fig **??**. And furtherly specializes to

$$\underline{\underline{\boldsymbol{P}}}^{\mathrm{T}} \cdot \underline{\underline{\boldsymbol{M}}} \cdot \underline{\underline{\boldsymbol{P}}} \;=\; \underline{\underline{\boldsymbol{\Lambda}}}$$

for diagonal matrix $\boldsymbol{\Lambda}$.

2.7.3 Eigenvector simplifications for symmetric matrices

Proposition 2 All symmetric matrices $\boldsymbol{S}$ are diagonalizable.

Proposition 3 All symmetric matrices $\boldsymbol{S}$ are orthogonally diagonalizable.

I.e. an $O(N)$ orthogonal matrix $\boldsymbol{P}$ can be found such that

$$\overline{\underline{\boldsymbol{P}}}^{\mathrm{T}} \cdot \overline{\underline{\boldsymbol{S}}} \cdot \overline{\underline{\boldsymbol{P}}} \;=\; \overline{\underline{\boldsymbol{P}}}^{-1} \cdot \overline{\underline{\boldsymbol{S}}} \cdot \overline{\underline{\boldsymbol{P}}} \;=\; \overline{\underline{\boldsymbol{D}}} \,. \tag{2.35}$$

Remark 1 The first and third expressions form a statement of matrix congruence. While the second and third expressions form a statement of matrix similarity. They are united by the basic property of orthogonal matrices that

$$\boldsymbol{P}^{-1} \;=\; \boldsymbol{P}^{\mathrm{T}} \,. \tag{2.36}$$

Proposition 4 The eigenvectors can be taken to form an eigenONBL.

Remark 2 This eigenONBL is furthermore complete.

Remark 3 A matrix's overall eigenbasis carries a partition by the eigenvalues to which each eigenvector belongs. Eigenspaces have a $(\rho_e - 1)$-parameter freedom in choosing eigenvectors up to sign. Which is nontrivially realized whenever the eigenspace is nontrivial. Alongside 2^{ρ_e} discrete sign-choice freedoms.

2.7.4 Triangular matrices

Remark 1 *Upper-triangular matrices* are those with entries all zero below the diagonal. While *lower-triangular matrices* are those with entries all zero above the diagonal. I.e. these two notions are adjointly related.

Example 1

$$\boldsymbol{M} \;=\; \begin{pmatrix} 2 & 4 & 1 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{pmatrix} \,. \tag{2.37}$$

is upper-triangular. One could then write a single larger 0, perhaps as $\mathbb{0}$ as the sole symbol in the lower wedge. So as to indicate, eventually with a lot less ink as matrix size goes up, that this is solely populated by zero entries.

Remark 2 Clearly if a matrix is both upper- and lower-triangular then it is diagonal.

Remark 3 Triangular matrices make for simpler linear systems to solve, by virtue of their being semi-decoupled. I.e. we can solve for 1 variable at a time, albeit without choice of the order of the variables for which solving in this way is enabled. As such, for instance *Gaussian elimination* is an algorithm to send a given matrix to triangular form. Finally, triangular matrices further serve as factors in a number of matrix decompositions [253].

2.7.5 Band matrices

Structure 1 These are zero other than in some subset of the parallels of the leading diagonal. Possibly including the lead diagonal itself. It is often taken to be a contiguous stripe – without gaps – hence the name 'band'. We do not deign to name gapped versions today, out of the current Review not needing any.

What we do need is the intersection of the band matrices with the triangular matrices. Specifically of the 2-string version. I.e.

$$BT_{i\,j} \;=\; a\,\delta_{i\,j} \;+\; b\,\delta_{i,\,j\,-\,1}\;. \tag{2.38}$$

Which is furthermore a simplification as regards each parallel being populated by its own single independent element. This is termed a 'Toeplitz' or 'truncated circulant' simplification. In the sense of managing to form the path part of a circulant's cycle. And yet failing to close up.

2.7.6 Welcome to the Kingdom of Jordan

2.7.7 Complex Jordan blocks

Structure 1 A complex *Jordan block* is a by convention upper-triangular matrix. Whose diagonal contains just copies of the same eigenvalue. And whose first parallel to this diagonal consists solely of 1's. All other elements in it are zero. Revealing this to be our first example of a 2-string quasicirculant in the intersection of the upper-triangular matrices and the band matrices. Specifically, in components,

$$J_{ij} = \lambda \delta_{ij} + \delta_{i,j-1} . \tag{2.39}$$

Notational Remark 1 Let us use the notation $\boldsymbol{J}_s$ for the size s nontrivial Jordan blocks. Nontrivial here means at least 2×2. So picking s to be excess block size, $s \in \mathbb{N}$.

We furthermore make active use of the 4 smallest of these only. So

$$\boldsymbol{J} := \boldsymbol{J}_1 , \quad \boldsymbol{T} := \boldsymbol{J}_2 , \quad \boldsymbol{H} := \boldsymbol{J}_3 , \quad \boldsymbol{F} := \boldsymbol{J}_4 \tag{2.40}$$

is a suitable shorthand. Standing for minimum-Jordan block of excess block size 1. $\boldsymbol{T}$ for Two, $\boldsymbol{H}$ for tHree and $\boldsymbol{F}$ for Four.

2.7.8 Eigenspaces as typically bigger blocks

Structure 1 A fixed eigenvalue can possess a number of separate Jordan blocks, constituting their direct sum. Let us refer to this as our single-eigenvalue's big Jordan block.

The top such is a single Jordan block. While the bottom such consists solely of trivial Jordan blocks of size 1, rendering it diagonal. Other possibilities are supported in between, for sufficiently large big blocks. The totality of these possibilities is indexed by the partitions, whose top and bottom are as above. As interpreted as the 1-piece partition and the partition into pieces of size 1.

Remark 1 Solely the top and bottom single-eigenvalue big Jordan blocks preserve the first upper parallel's constancy of value (**).

2.7.9 Jordan matrices

Structure 1 A *Jordan matrix* is a direct sum of Jordan blocks.

Structure 2 It is possible for an eigenvalue to possess ≥ 1 Jordan block, while not constituting the entirety of a Jordan matrix.

Example 1

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \tag{2.41}$$

contains 2 Jordan blocks corresponding to the eigenvalue 2. 1 trivial and 1 not. While containing also a trivial Jordan block corresponding to the eigenvalue 1.

Observe that distinct eigenvalues break the diagonal away from constancy of value. While solely the top and bottom single-eigenvalue big Jordan blocks preserve the first upper parallel's constancy of value. Either this or (**) suffices to break out of the quasicirculant property. While the weaker 2-string property persists under both big single-eigenvalue and distinct-eigenvalue Jordan block compositions.

Exercise 7 Build the smallest matrix containing i) 2 distinct nontrivial Jordan blocks.

ii) nontrivial Jordan block corresponding to a single eigenvalue.

iii) 2 distinct eigenvalues, each supporting 2 distinct types of nontrivial Jordan blocks.

iv) Consider iii) again but now let none of the blocks be the same size as each other. Describe the 6-fold or 3-fold ambiguity in this answer. Identifying which modelling assumptions give each of these multiplicities. All the while using suitable adapted variables to interpret what you are doing.

2.7.10 $\mathbb{C}$-JNF theorem (Jordan normal form)

Theorem 1 Every $\mathbb{C}$-matrix is equivalent to a $\mathbb{C}$-JNF.

Corollary 1 Every $\mathbb{R}$-matrix is equivalent to a $\mathbb{C}$-JNF.

2.7.11 $\mathbb{R}$-Jordan blocks include a second family

Structure 1 There is a second species of Jordan block in the real rather than complex case. For $\mathbb{R}$ matrices can have nontrivially complex eigenvalues. And since these are not real, one cannot insert them into the diagonal without kicking oneself out of the $\mathbb{R}$-valued matrices... Instead, one uses

$$\boldsymbol{R} := \begin{pmatrix} a & -b \\ b & a \end{pmatrix} . \tag{2.42}$$

For a the real part and b the absolute value of the imaginary part. So as to cover each complex pair of roots in turn.

Then blocking up with a 2-index-values to 1-index-value map, 2×2 identity matrices play the above role of the extra 1's in the above discussion of Jordan blocks.

2.7.12 $\mathbb{R}$-JNF theorem

Theorem 2 Every $\mathbb{R}$-matrix is equivalent to a $\mathbb{R}$-Jordan form.

Corollary 2 Every $\mathbb{R}$-symmetric matrix is equivalent to a $\mathbb{C}$-Jordan form that so happens to be $\mathbb{R}$-valued.

Exercise 8 But which Theorems and Propositions is this a Corollary of?

Remark 1 Because of this, the current Review's primary objects – $\mathbb{R}$-symmetric matrices – enjoy the following somewhat convoluted simplicity. $\mathbb{C}$-JNFs suffice for them, because these get restricted to $\mathbb{R}$-valued $\mathbb{C}$-JNFs. This is a twofold simplicity: firstly it maps $\mathbb{R} \longrightarrow \mathbb{R}$. Where the target space is a $\mathbb{R}$ piece of the $\mathbb{C}$ version. Which then secondly halves the number of constituent block families. Which in turn can be viewed as a benefit of the complex numbers being closed.

In effect, the $\mathbb{R}$-symmetric matrices' eigentheory manages to be closed within the $\mathbb{R}$. For all that this is a fortunate property that does not extend to some other $\mathbb{R}$-matrices...

Counterexample 1 Rotations that are not the identity preserve no lines in $\mathbb{R}^2$ and possess no *Reals*-eigendirections.

For $N = 2$, $\mathbb{R}$ and $\mathbb{C}$ JNFs can be $\mathrm{diag}(a, a)$, $\mathrm{diag}(a, b)$ and

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} .$$

With the $\mathbb{R}$ case also possessing (2.42).

For now ceasing to care about whether eigenvalues are distinct between blocks, the possible $N = 3$ $\mathbb{C}$-JNFs are as follows. $\mathrm{diag}(a, a, a)$,

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} , \quad \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \\ \cdot & & \end{pmatrix}$$

I.e. $1^3, 2\,1, 3$.

The corresponding answer for $N = 4$ is as follows. $\mathrm{diag}(a, a, a, a)$,

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{pmatrix} , \quad \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{pmatrix} , \quad \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{pmatrix} , \quad \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{pmatrix} .$$

I.e. the partitions 1^4, $2\,1^2$, $2^2\,3\,1$ and 4 .

If one does not cease to care, then each eigenspace is assigned a partition into Jordan blocks. So to find all cases, one partitions twice. Once by eigenvalue, which then serves as distinguishing label for each separate piece, i.e. eigenspace. And then a second time within each eigenspace separately, into Jordan blocks.

2.7.13 Generalized eigenvectors

Counterexample 1 (To v.s.s possessing bases of eigenvectors.) For Worked Example **??**.1's 2-d shears, there is only 1 LI eigenvector. Thus there cannot be a LB of eigenvectors in this case.

Matrices exhibiting this kind of behaviour are known as *defective*. We shall relate this below to nondiagonality, and to block structure more generally.

Remark 1 In the event of $\alpha > \mu$ there is however a more general procedure [247] that one can turn to so as to form a sufficient number of 'generalized eigenvectors'.

Definition 1 A vector $\boldsymbol{x}$ is a *generalized eigenvector* of rank k corresponding to the eigenvalue λ of the matrix $\boldsymbol{A}$ if the following holds.

$$\left(\overline{\underline{\boldsymbol{A}}} - \lambda \overline{\underline{\mathbb{I}}}\right)^k \cdot \overline{\boldsymbol{x}} = \overline{\mathbf{0}} \tag{2.43}$$

and yet

$$\left(\overline{\underline{\boldsymbol{A}}} - \lambda \overline{\underline{\mathbb{I}}}\right)^{k-1} \cdot \overline{\boldsymbol{x}} \neq \overline{\mathbf{0}} \, . \tag{2.44}$$

Remark 1 Types of generalized eigenvalue are parametrized by the rank alias index k of the generalization. Clearly $k \in \mathbb{N}$.

Furthermore ordinary eigenvectors correspond to the bottom element, $k = 1$. While the top k is bounded by μ_λ . I.e. the power of the corresponding eigenvalue factor in the minimum polynomial. Thus the arena of possible generalities is the discrete interval of natural numbers $[1, \mu_\lambda]$.

Remark 2 More generally Jordan block matrices possess $< p$ LI eigenvectors [253]. I.e. there is less than a basis amount of them. By which eigenvectors fail to span the eigenspace. *fuse with defective* Clearly likewise matrices similar to nontrivial Jordan form matrices are defective.

Remark 3 A size-k Jordan block fully realizes the above chain, which has come to be called a *Jordan chain*.

By Counterexample 1, $N = 2$ already supports defective matrices. Forming a Jordan chain, conventionally said to be length 2. However, it is more effective to use that the length-1 chain is for a block that is only Jordan trivial. So that we shrink the conventional chains by 1 so as to form 'Jordan nontrivial chains'.

Then the block of size 2 returns the chain that is trivial in the manner of a point. It is however $N = 3$ that is minimum to realize a chain with 2 distinct ends: the $\boldsymbol{T}$ block. While $N = 4$ is minimum for our kind of Jordan chain to have a middle.

Some of an eigenvalue's individual Jordan blocks may however be smaller and yet nontrivial, realizing shorter Jordan chains.

Example 1 $N = 5$ is minimum to exhibit this nontrivially. I.e. for nontrivial Jordan blocks and thus for nontrivial-length chains: ≥ 2 elements. For instance

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.45}$$

realizes a Jordan chain of the maximum length supported – 3 – but also a separate chain of middling length, 2.

Thus μ_λ provides a globally tight bound. But need not be tight for any given Jordan block. Middling nontrivial Jordan blocks included.

2.8 Some organizing principles

End Remark 1 A partial summary of the current subsection is as follows.

$$\overline{\underline{\boldsymbol{P}}}^{-1} \cdot \overline{\underline{\boldsymbol{M}}} \cdot \overline{\underline{\boldsymbol{P}}} = \overline{\underline{\boldsymbol{D}}} \text{ or } \overline{\underline{\boldsymbol{T}}}$$

holds more generally than

$$\overline{\underline{\boldsymbol{P}}}^{\mathrm{T}} \cdot \overline{\underline{\boldsymbol{M}}} \cdot \overline{\underline{\boldsymbol{P}}} = \overline{\underline{\boldsymbol{D}}} \text{ or } \overline{\underline{\boldsymbol{T}}} \, .$$

End Remark 2 Symmetric $\Rightarrow$ diagonalizable leaves no room for nontrivial Jordan blocks. So we have no more to say in the current Article's main topic about Jordan-nontrivial blocks, or ensuing generalized eigenvectors.

Internal Pointer 1 Our approach at this point is to first introduce many eigenadjectives and then to establish that $\mathbb{R}$-valued symmetric matrices discard almost all of these. In the process, $\mathbb{R}$-valued symmetric matrix specific simplifications appear as regards the following of the current Chapter's topics. Eigenvalue multiplicities, annihilator polynomials, both eigenspace and Jordan normal form structure, and equal-eigenvalue degeneracy. The last of these additionally enjoys an underpinning symmetry principle for $\mathbb{R}$-valued symmetric matrices.

Chapter 3

First layer of eigenadjectives

3.1 Conceptual analysis

Remark 1 See Fig 3.1.a) for a conceptual analysis of the possible pairwise inequalities between the following. An eigenvalue's 3 multiplicities. And the 2 extremal values available to these. I.e. 1 – the minimum nontrivial dimension – and N : the dimension of the v.s. acted upon. This yields 9 primary adjectives, underlined in the Fig. Comprising 2 standard ones and 7 new to the current Review.[1]

The current Review quickly specializes to $\mathbb{R}$-valued matrices whose eigenspectra are $\mathbb{R}$-valued as well. Which is in general Mathematically awkward, due to Algebraic closure. But which supports a widely applicable and Mathematically well-defined subcase: the $\mathbb{R}$-valued symmetric matrices. See Subfig b) for a start on how our system of eigenadjectives simplifies in this case. Fig 3.2.a) generalizes instead to the whole-matrix counterpart, via the whole eigenspectrum. Finally Subfig b) 'completes the square' by restricting this generalization to the $\mathbb{R}$-valued symmetric matrices.

[1] Though the solitary concept is certainly not new.

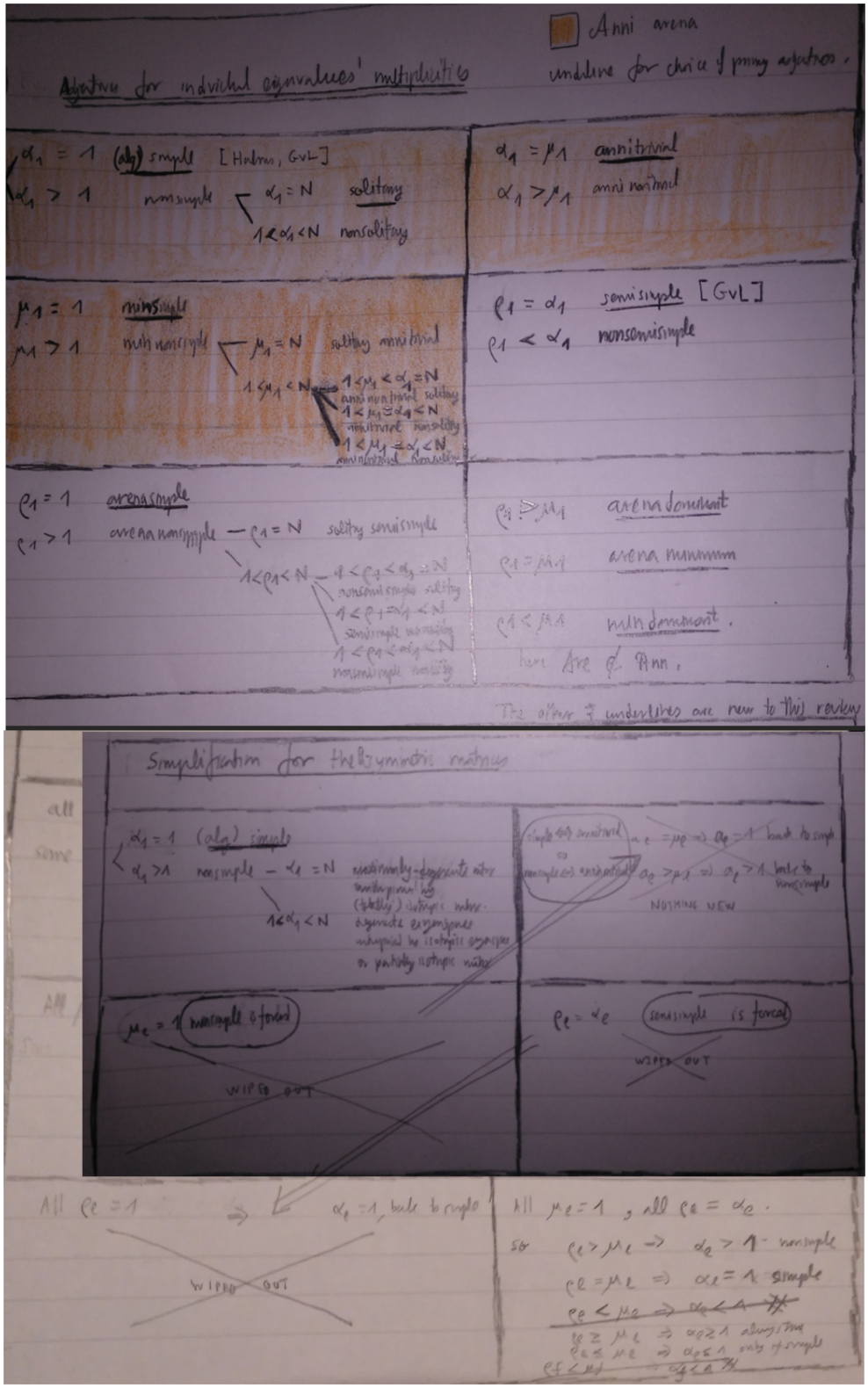

Adjectives for individual eigenvalues' multiplicities
Anni arena
undecided for choice of proving adjectives.
α1 = 1 (alg) simple [Holmes, GvL]
α1 > 1 nonsimple
α1 = N solitary
1 < α1 < N nonsolitary
α1 = μ1 annitrivial
α1 > μ1 annirational
μ1 = 1 minsimple
μ1 > 1 minnonsimple
μ1 = N solitary annitrivial
1 < μ1 < N
ρ1 = α1 semisimple [GvL]
ρ1 < α1 nonsemisimple
ρ1 = 1 arena simple
ρ1 > 1 arena nonsimple
ρ1 = N solitary semisimple
1 < ρ1 < N
ρ1 ≥ μ1 arena dominant
ρ1 = μ1 + 1 arena minimum
ρ1 < μ1 + 1 min dominant.
here Are ∉ Ann.
The others underlined are new to this review
Simplification for the symmetric matrix
α1 = 1 (alg) simple
α1 > 1 nonsimple
1 < α1 < N
NOTHING NEW
μe = 1 nonsimple is forced
WIPED OUT
ρe = αe semisimple is forced
WIPED OUT
All ρe = 1
αe = 1, back to simple
WIPED OUT
All μe = 1, all ρe = αe.
so ρe > μe → αe > 1 nonsimple
ρe = μe ⇒ αe = 1 simple
ρe < μe ⇒ αe < 1

Figure 3.1:

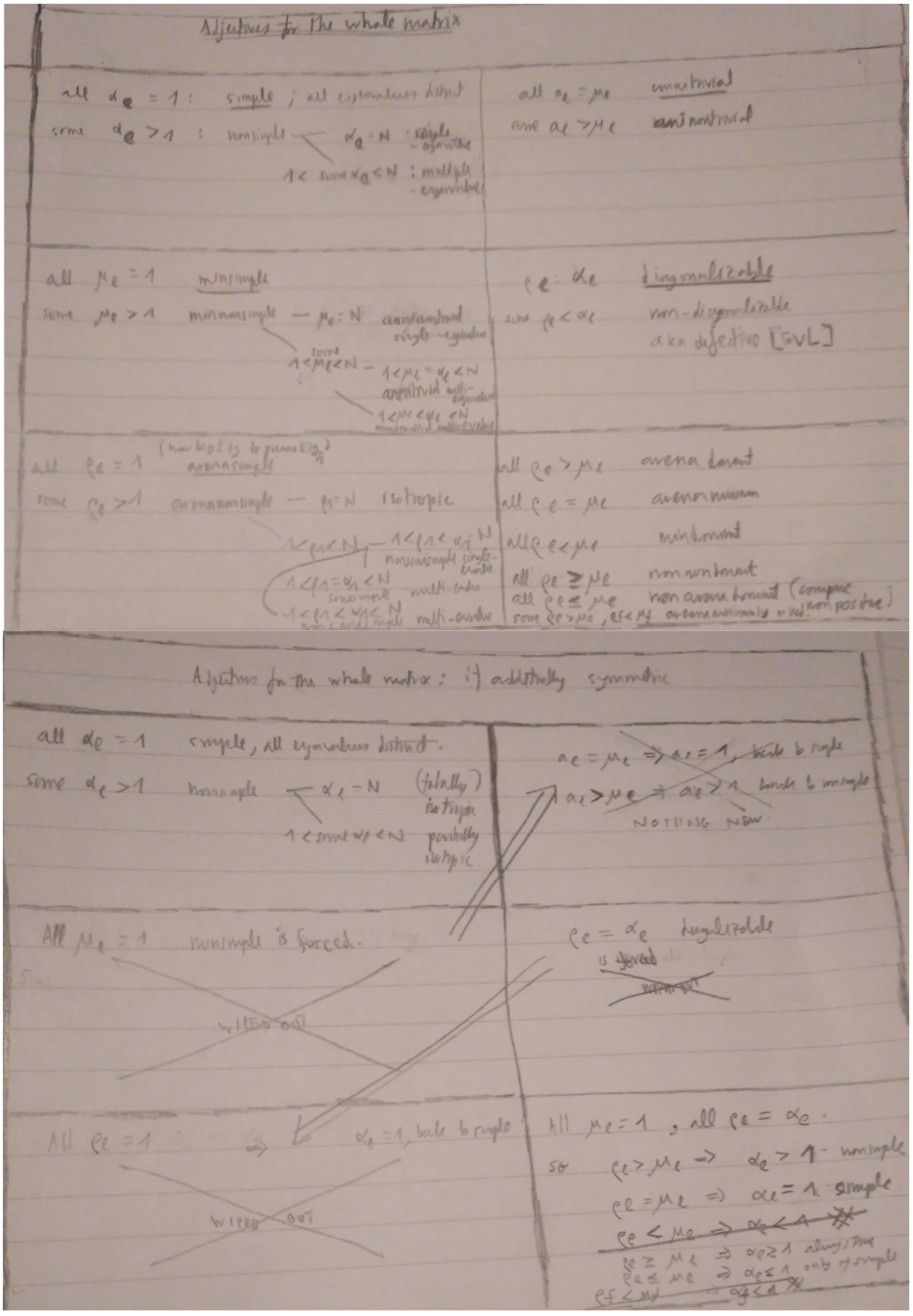

Adjectives for the whole matrix
all $\alpha_e = 1$: simple ; all eigenvalues distinct
some $\alpha_e > 1$: nonsimple — $\alpha_e = N$: totally - algebraic
$1 < \text{some } \alpha_e < N$: multiple - eigenvalue
all $\mu_e = 1$ minsimple
some $\mu_e > 1$ nonminsimple — $\mu_e = N$ antiminimal single - eigenvalue
$1 < \mu_e = \alpha_e < N$ antiminimal multi - eigenvalue
$1 < \mu_e < \alpha_e < N$ antiminimal multi - eigenvalue
all $\rho_e = 1$ (non-bijective to prime Eig.) aronasingle
some $\rho_e > 1$ aronanonsingle — $\rho_e = N$ isotropic
$1 < \rho_1 < N$ — $1 < \rho_1 < \alpha_1 < N$ nonsimple single - eigenvalue
$1 < \rho_1 = \alpha_1 < N$ totally nonsimple multi - eigenvalue
$1 < \rho_1 < \gamma_1 < N$ non-totally nonsimple multi - eigenvalue
all $\rho_e = \alpha_e$ diagonalizable
some $\rho_e < \alpha_e$ non-diagonalizable
aka defective [SVL]
all $\rho_e > \mu_e$ arena transient
all $\rho_e = \mu_e$ arena minimum
all $\rho_e < \mu_e$ minitransient
all $\rho_e \geq \mu_e$ non minitransient
all $\rho_e \leq \mu_e$ non arena transient (compare
some $\rho_e > \mu_e$, $e_f < \mu_f$ arena minimals = rel. non positive)
Adjectives for the whole matrix : if additionally symmetric
all $\alpha_e = 1$ simple, all eigenvalues distinct.
some $\alpha_e > 1$ nonsimple — $\alpha_e = N$ (totally) isotropic
$1 < \text{some } \gamma_e < N$ partially isotropic
$\alpha_e = \mu_e \Rightarrow \alpha_e = 1$, basic to simple
$\alpha_e > \mu_e \Rightarrow \alpha_e > 1$ basic to nonsimple
NOTHING NEW
All $\mu_e = 1$ minsimple is forced.
WIPED OUT
$\rho_e = \alpha_e$ diagonalizable
is forced
WIPED OUT
All $\rho_e = 1$ $\Rightarrow$ $\alpha_e = 1$, basic to simple
WIPED OUT
All $\mu_e = 1$, all $\rho_e = \alpha_e$.
So $\rho_e > \mu_e \rightarrow \alpha_e > 1$ - nonsimple
$\rho_e = \mu_e \Rightarrow \alpha_e = 1$ simple
$\rho_e < \mu_e \Rightarrow \alpha_e < 1$
$\rho_e \geq \mu_e \Rightarrow \alpha_e \geq 1$ always true
$\rho_e \leq \mu_e \Rightarrow \alpha_e \leq 1$ only if simple
$e_f < \mu_f \rightarrow \alpha_f < a$

Figure 3.2:

3.2 Symmetric matrices' eigenadjectives simplicity theorem

Short form of Theorem 1 Within the arena of $\mathbb{R}$-valued matrices, we pass from needing 9 primary adjectives to just 1 : simple.

Remark 1 We observe that this adjective was already in place in Halmos [213].

Long form of Theorem 1 For symmetric matrices,

a) Minsimple and semisimple are forced, while mindominant is prohibited.

b) T.f.a.e.

i) Simple.

ii) Annitrivial.

iii) Arenassimple.

iv) Arenaminimum.

Corollary 1 T.f.a.e. for symmetric matrices.

i) Nonsimple.

ii) Anninontrivial.

iii) Arenanonsimple

iv) Arenadominant

Naming Remark 1 Furthermore, for symmetric matrices, nonsimple is equivalent to *nontrivially partly isotropic*: a symmetry groups criterion. With the *totally isotropic case* included, corresponding to the solitary eigenvalue case. These matrices are proportional to the identity matrix (with the possibility of $\mathbb{0}$ included). Simple is by exclusion *totally anisotropic*, corresponding to just the identity symmetry group. Giving it a new alias within the restricted arena $Sym_{\mathbb{R}}$.

Remark 2 For a symmetric, or more generally, diagonalizable matrix, the ARM-eigenspectrum simplifies down to the AM-eigenspectrum

$$\mathfrak{ame}\text{spec}(\boldsymbol{M}) := \begin{matrix} \lambda_1 & \dots & \lambda_E \\ \alpha_1 & \dots & \alpha_E \\ \mu_1 & \dots & \mu_E \end{matrix}. \tag{3.1}$$

While for arenaminimum matrices, it simplifies down to the AR-eigenspectrum

$$\mathfrak{aR}\text{spec}(\boldsymbol{M}) := \begin{matrix} \lambda_1 & \dots & \lambda_E \\ \alpha_1 & \dots & \alpha_E \\ \rho_1 & \dots & \rho_E \end{matrix}. \tag{3.2}$$

Finally, for Order-Theoretically trivial matrices, we have collapse all the way down to the A-eigenspectrum, i.e. the usual multi-eigenspectrum.

3.3 Possible orders on all the multiplicities

Structure 1 A square matrix of size N 's eigenvalue multiplicities obey the poset of inequalities in Fig 3.3.a). the minimum property. For the $\mathbb{R}$-valued symmetric matrices subcase, this collapses to the chain in Subfig b).

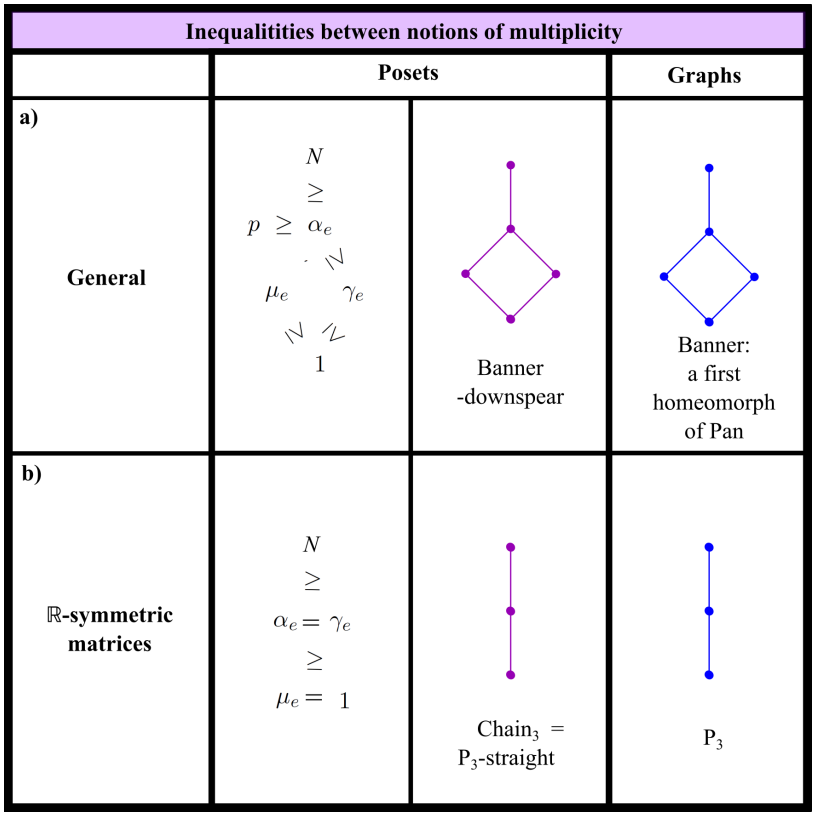


Figure 3.3:

Theorem 2 a) Combinatorially, this poset supports 24 saturation cases.

Due to the Linear Algebraic meaning of the multiplicities' vertices, 12 are prohibited and 12 are realized.

Remark 1 See Fig 3.5 for each prohibition and for a minimum example realizing each allowed case.

Remark 2 See Fig 3.7.a) for these minimum examples classified by size. So size $\leq$ 3 Jordan blocks suffice to realize all cases. Next consult Subfig b) for a structural simplification, based on identifying an inactive 2-factor. Finally Subfig c) cuts down to the 4 cases supported by $\mathbb{R}$-symmetric matrices.

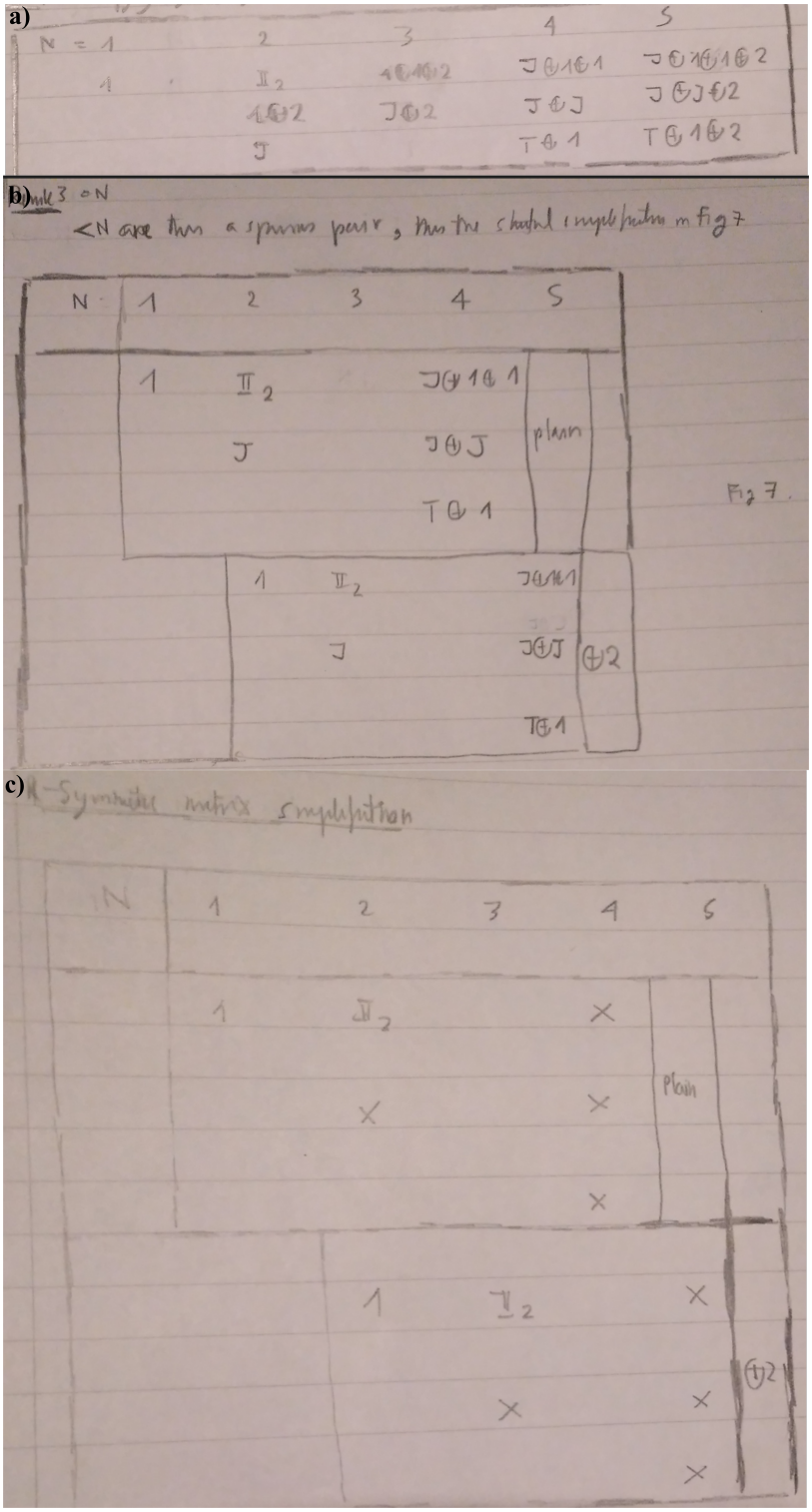


Figure 3.4:

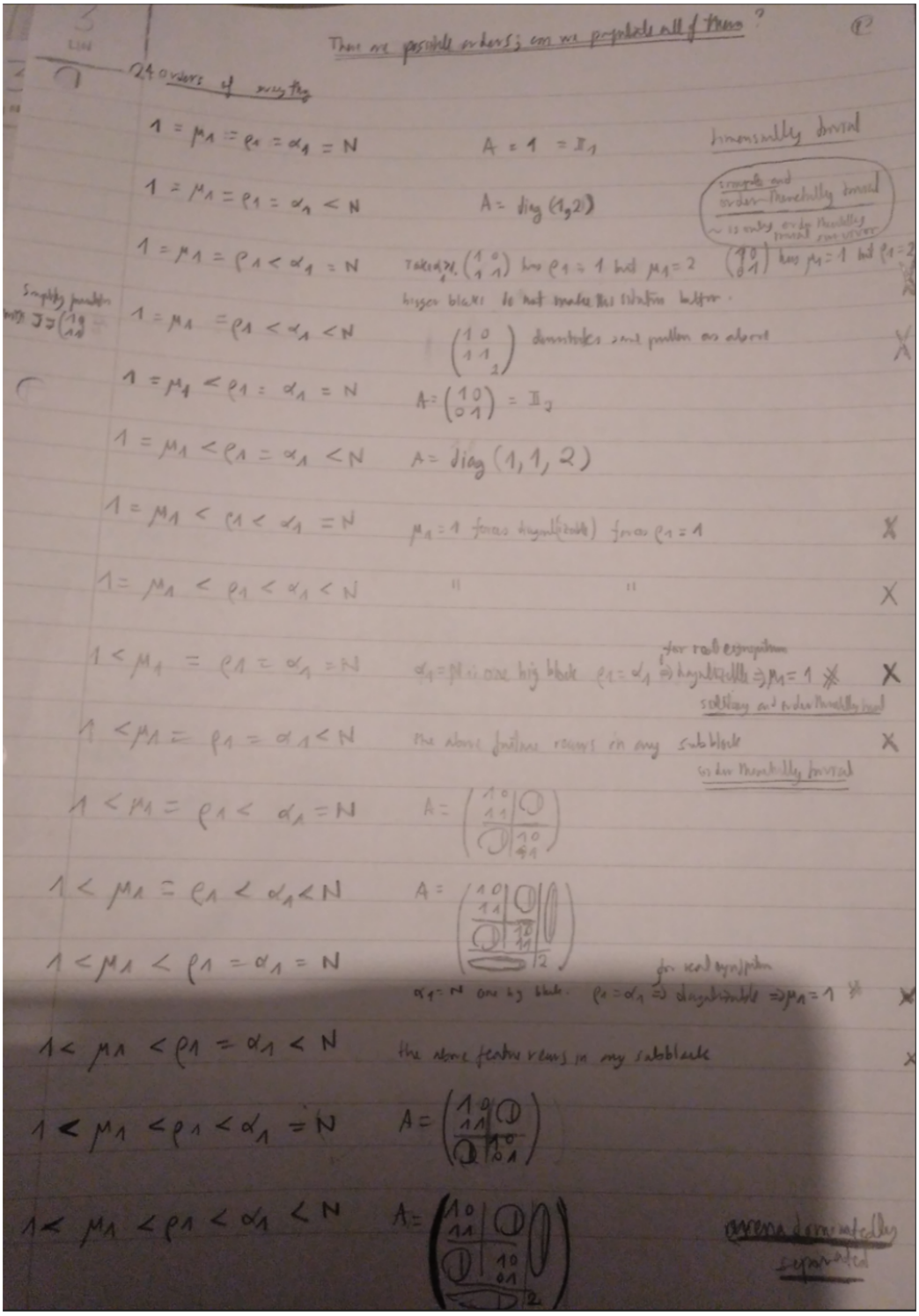


Figure 3.5:

3.4 Corresponding total orders

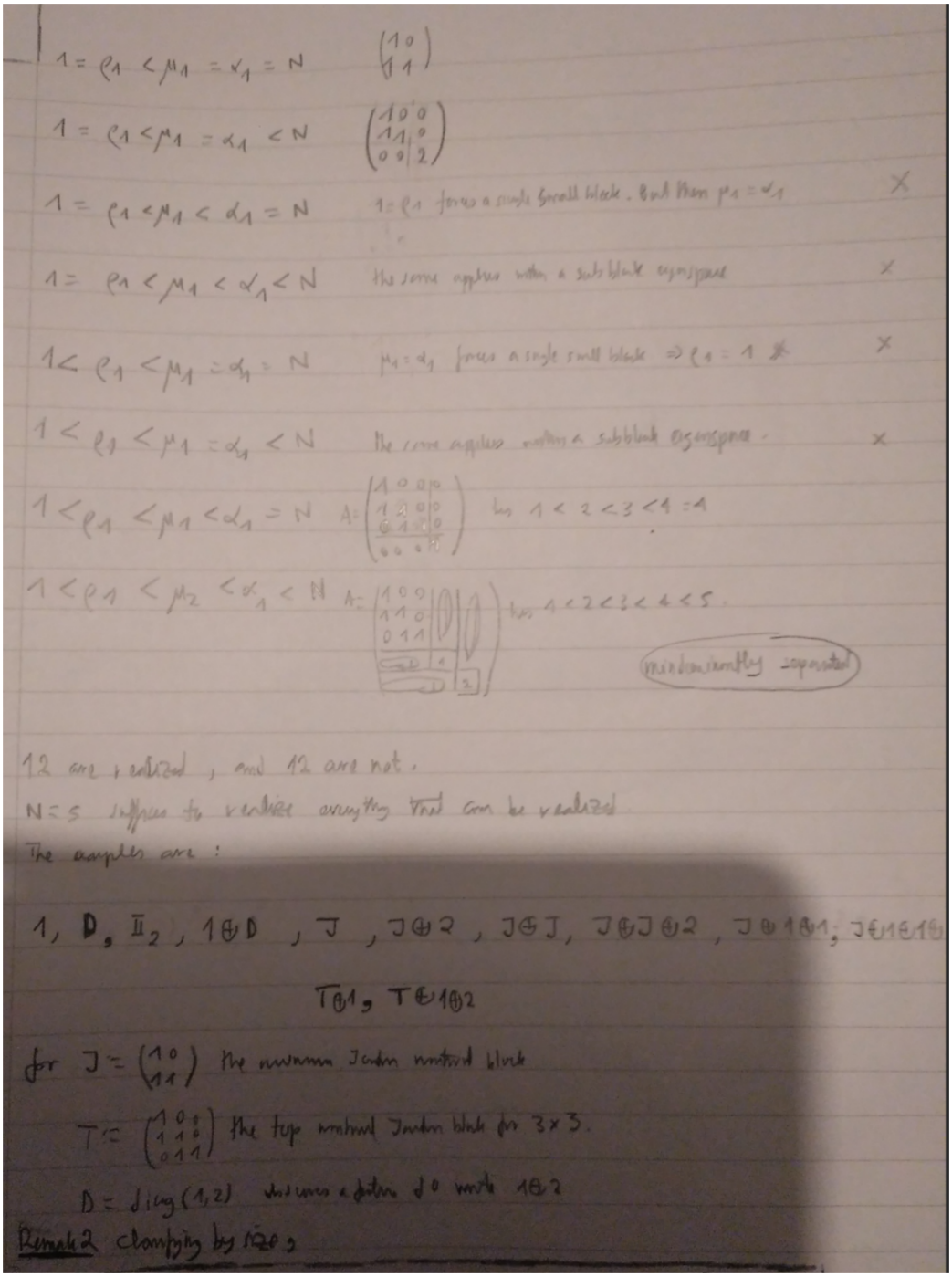


Figure 3.6:

Remark 4 Observe that the fifth case contains a first distinct *Ann* . What minimum examples place an *Ann* into each possible gap in the order? See Subfig b) for identification of the 2 singletons and triplet in question. And Subfig b) for minimum examples of each. Fig 3.8.a) then stacks these onto Fig 3.7's breakdown so as to classify a total of 22 minimum examples. Whose symmetric matrices' subcase of 6 are stacked up in Subfig b).

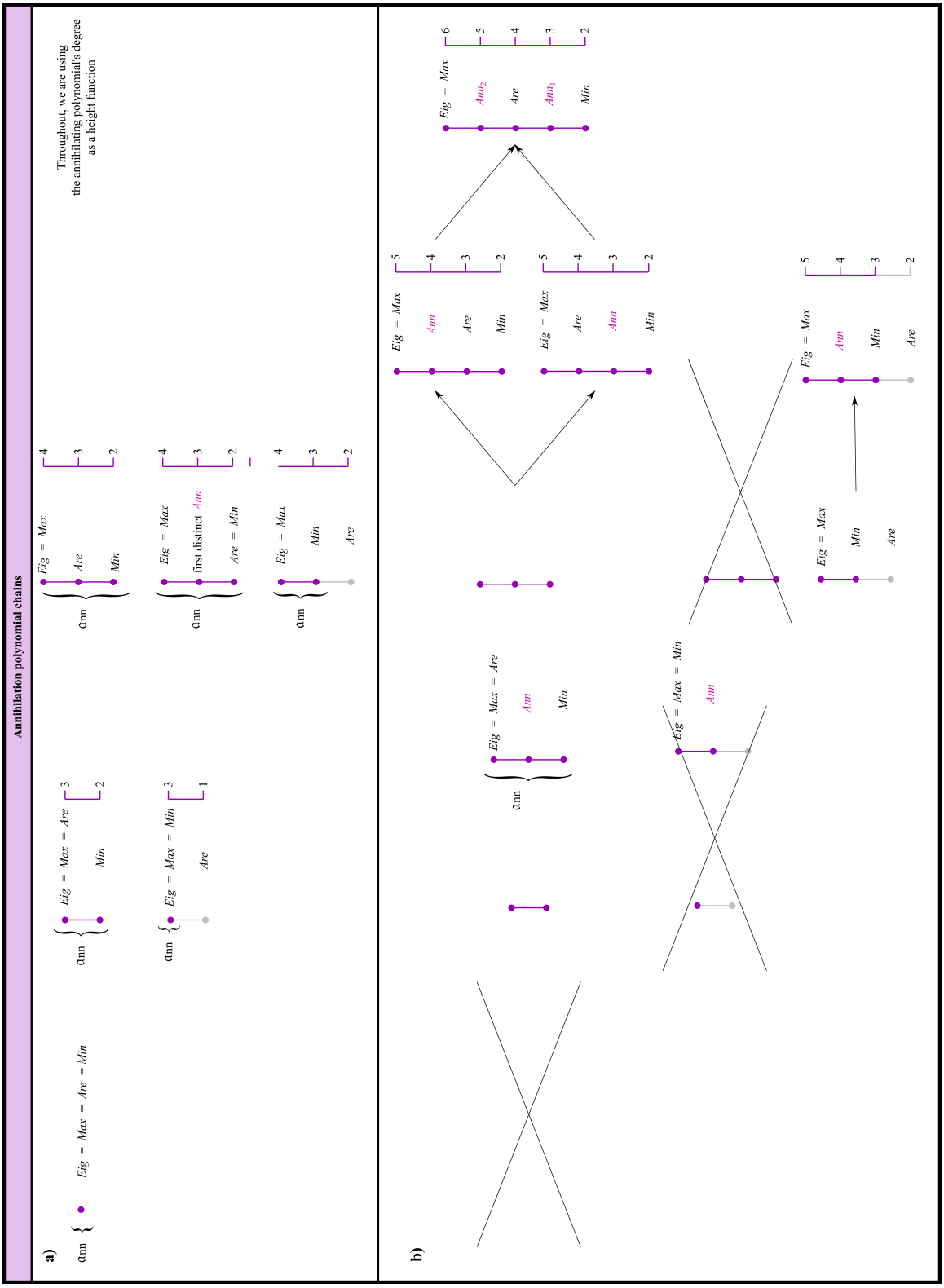

Annihilation polynomial chains
a)
Ann
Eig = Max = Are = Min
Throughout, we are using
the annihilating polynomial's degree
as a height function
Ann
Eig = Max = Are
Min
Ann
Eig = Max = Min
Are
Ann
Eig = Max
Are
Min
Ann
Eig = Max
first distinct Ann
Are = Min
Ann
Eig = Max
Min
Are
b)
Ann
Eig = Max = Are
Min
Ann
Eig = Max = Min
Ann
Eig = Max
Ann
Are
Min
Eig = Max
Are
Ann
Min
Ann
Eig = Max
Ann2
Are
Ann1
Min
Eig = Max
Min
Are
Ann
Eig = Max
Ann
Min
Are

Figure 3.7:

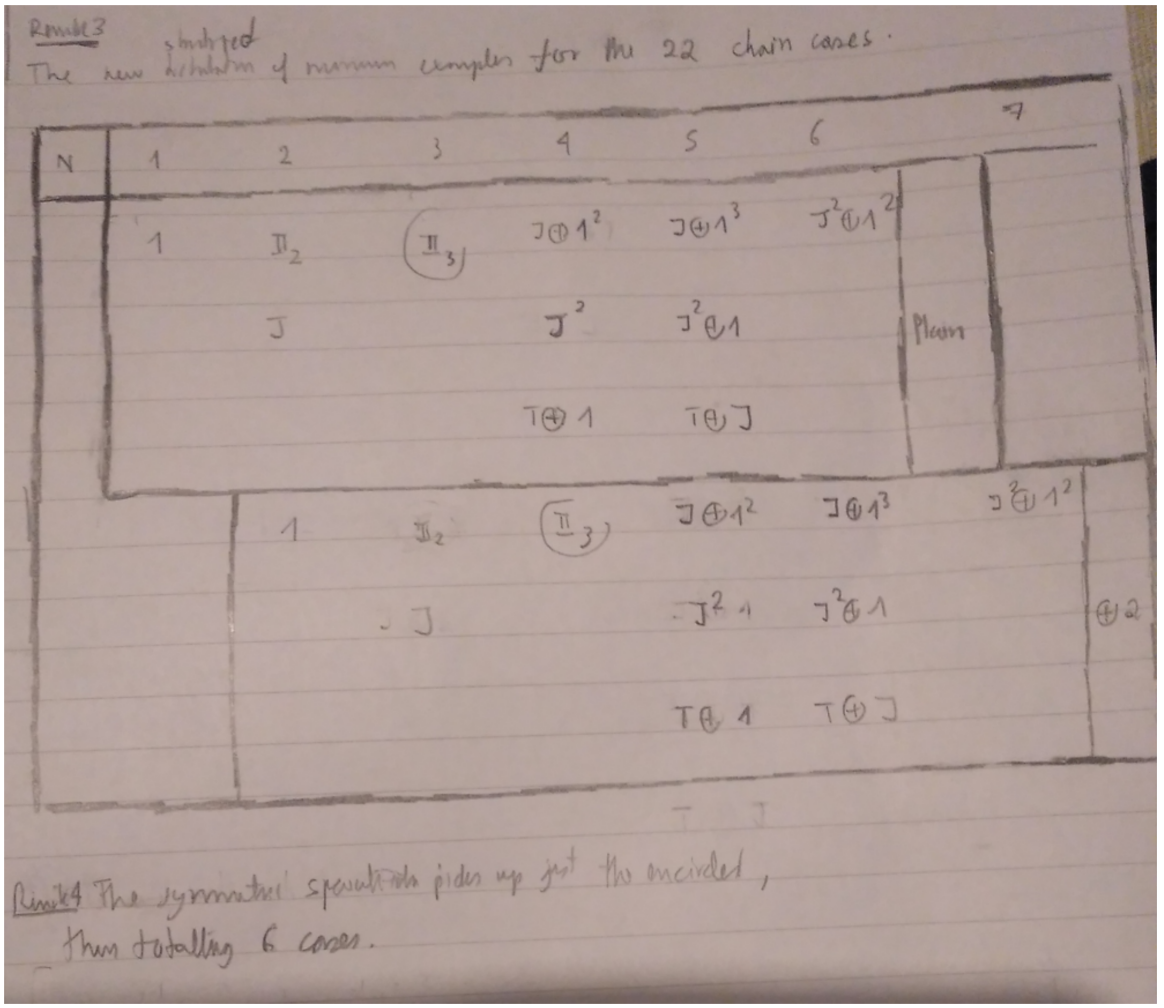

Remark 3
The new distribution of minimum complex for the 22 chain cases.
N
1
2
3
4
5
6
7
Plains
Remark 4 The symmetric specialization picks up just the encircled,
thus totalling 6 cases.

Figure 3.8:

Chapter 4

Nontrivial posets of annihilation polynomials

4.1 Suite of adapted notations

4.1.1 Incipient notation

Example 1

$$\boldsymbol{A} \;=\; 1^2 \,\oplus\, 2^2 \;\;\Rightarrow\;\; \mathfrak{ARMe}\mathrm{spec}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 2 & 2 \\ 2 & 2 \\ 1 & 1\,. \end{matrix} \tag{4.1}$$

This gives the $\mathfrak{Ann}$ poset in Fig 4.1.a).

4.1.2 Excess annihilation polynomial notation

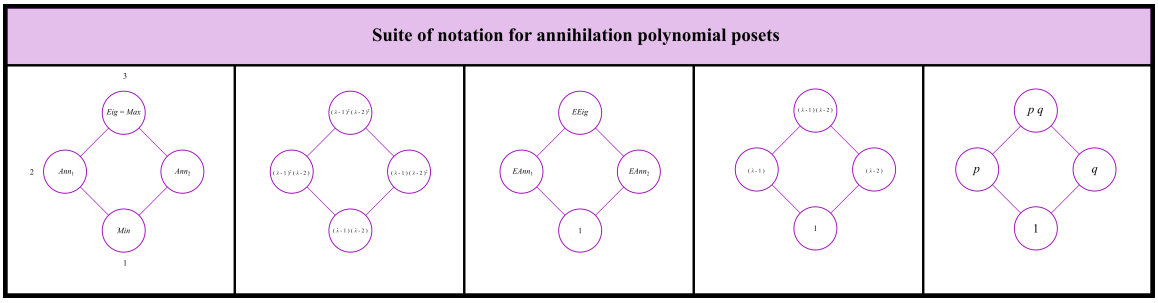


Figure 4.1:

Notational Remark 1 Let us start to streamline the notation for this by introducing the *excess anihilation polynomials* $EAnn$ according to the following.

$$Ann(\,x;\,\boldsymbol{M}\,) \;\;=\;\; Min(\,x;\,\boldsymbol{M}\,)\,EAnn(\,x;\,\boldsymbol{M}\,)\;. \tag{4.2}$$

This both standardizes and simplifies the bottom of each poset to be 1 . See Subfig b) for this simplification in our example.

Examples 2 and 3 The objective here is to pin *Are* onto each of the 2 other available sites.

$$\boldsymbol{A} \;=\; \boldsymbol{J} \,\oplus\, 1 \,\oplus\, 2^2 \;\;\Rightarrow\;\; \mathfrak{ARMe}\mathrm{spec}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 3 & 2 \\ 2 & 2 \\ 2 & 1 \end{matrix} \;.$$

Which pins *Are* into a middling position (Subfig c).

$$\boldsymbol{A} \;=\; \boldsymbol{J} \,\oplus\, 1 \,\oplus\, \boldsymbol{J}_2 \,\oplus\, 2 \;\;\Rightarrow\;\; \mathfrak{ARMe}\mathrm{spec}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 3 & 3 \\ 2 & 2 \\ 2 & 2 \end{matrix} \;.$$

Which now pins *Are* on the bottom position instead (Subfig d).

4.1.3 Arithmetic-divisor poset-isomorphism and ensuing notation

Notational Remark 2 A further notational streamlining is to set

$$p \;:=\; \lambda \,-\, 1 \,, \quad q \;:=\; \lambda \,-\, 2 \,. \tag{4.3}$$

Then the annihilating polynomial posets are labelled isomorphically to arithmetic divisor posets. I.e. a more basic and much more widely known topic and use of posets [242]. This sends Examples 1 to 3 to the simplified presentation in Subfigs 4.1.a)-c). And uses Proposition X below to establish Proposition X' below, for each $X = 1$ to 5 .

4.2 Further examples

Remark 1 Armed with the above suite of notation, we now extend our repertoire the following examples for the domino and the grid.

4.2.1 The domino, alias $p^2\,q$

Example Domino-1

$$\boldsymbol{A} \;=\; \boldsymbol{J}^2 \,\oplus\, \boldsymbol{J}_2 \;\;\Rightarrow\;\; \mathfrak{ARMe}\mathrm{spec}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 4 & 2 \\ 2 & 1 \\ 2 & 1 \end{matrix} \;. \tag{4.4}$$

Example Domino-2

$$\boldsymbol{A} \;=\; \boldsymbol{J}^2 \,\oplus\, 2^2 \;\;\Rightarrow\;\; \mathfrak{ARMe}\mathrm{spec}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 4 & 2 \\ 2 & 2 \\ 2 & 1 \end{matrix} \;. \tag{4.5}$$

Example Domino-3

$$\boldsymbol{A} \;=\; \boldsymbol{J} \,\oplus\, 1^2 \,\oplus\, \boldsymbol{J}_2 \,\oplus\, 2 \;\;\Rightarrow\;\; \mathfrak{ARMe}\mathrm{spec}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 4 & 3 \\ 3 & 2 \\ 2 & 2 \end{matrix} \;. \tag{4.6}$$

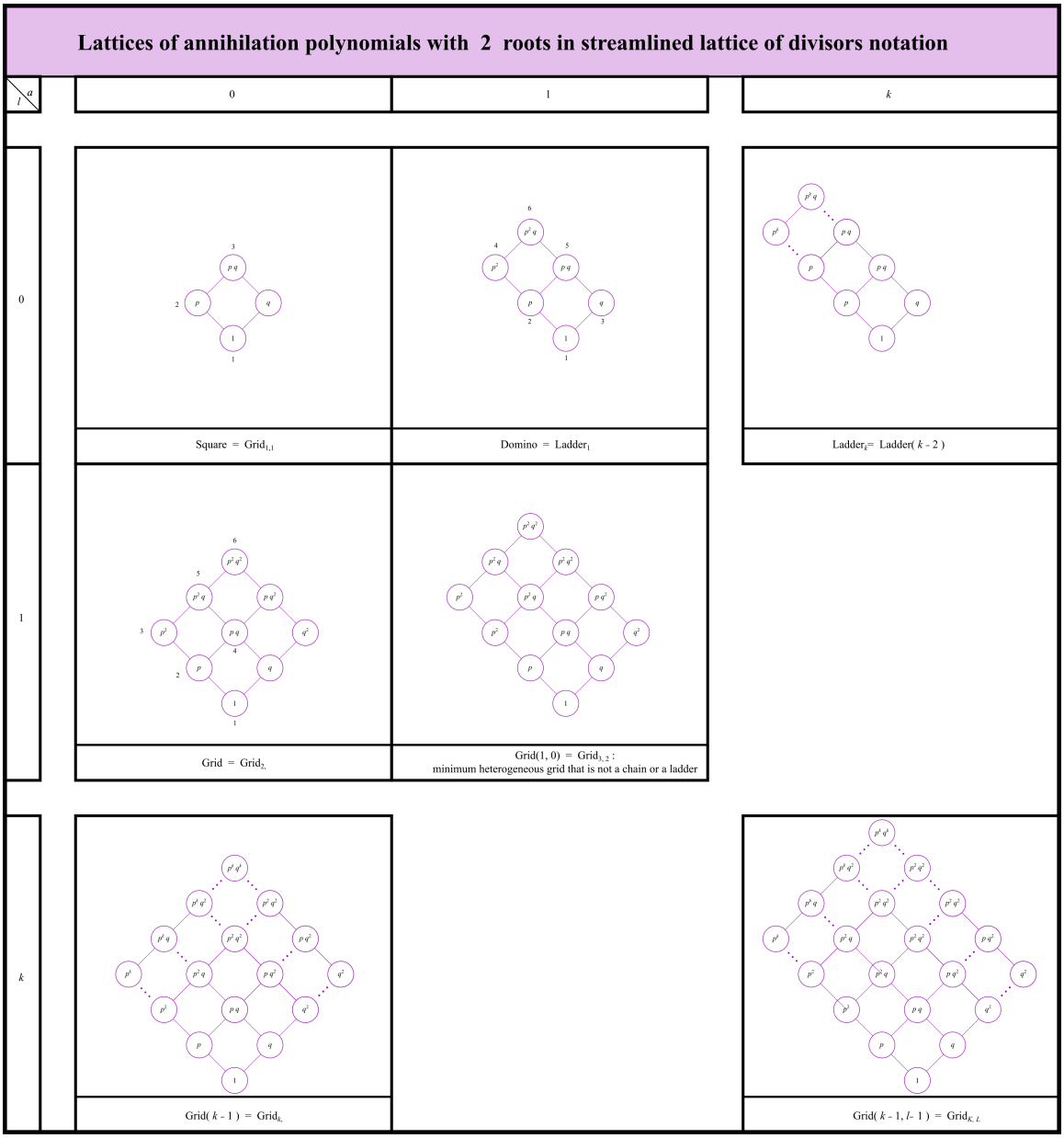


Figure 4.2:

Example Domino-4

$$\boldsymbol{A} \;=\; \boldsymbol{J} \,\oplus\, 1^2 \,\oplus\, 2^2 \;\Rightarrow\; \mathfrak{arme}\mathrm{spec}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 4 & 2 \\ 3 & 2 \\ 2 & 1 \end{matrix} \;. \tag{4.7}$$

Example Domino-5

$$\boldsymbol{A} \;=\; 1^3 \,\oplus\, \boldsymbol{J}_2 \,\oplus\, 2 \;\Rightarrow\; \mathfrak{arme}\mathrm{spec}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 3 & 3 \\ 3 & 2 \\ 1 & 2 \end{matrix} \;. \tag{4.8}$$

Example Domino-6

$$\boldsymbol{A} \;=\; 1^3 \,\oplus\, 2^2 \;\;\Rightarrow\;\; \mathfrak{ARMe}_{\mathrm{spec}}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 3 & 2 \\ 3 & 2 \\ 1 & 1 \end{matrix} \;. \tag{4.9}$$

4.2.2 The grid, alias $p^2\,q^2$

Example Grid-1

$$\boldsymbol{A} \;=\; \boldsymbol{J}^2 \,\oplus\, \boldsymbol{J}_2^2 \;\;\Rightarrow\;\; \mathfrak{ARMe}_{\mathrm{spec}}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 4 & 4 \\ 2 & 2 \\ 2 & 2 \end{matrix} \;. \tag{4.10}$$

Example Grid-2

$$\boldsymbol{A} \;=\; \boldsymbol{J}^2 \,\oplus\, 1^2 \,\oplus\, \boldsymbol{J}_2^2 \;\;\Rightarrow\;\; \mathfrak{ARMe}_{\mathrm{spec}}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 4 & 4 \\ 3 & 2 \\ 2 & 2 \end{matrix} \;. \tag{4.11}$$

Example Grid-3

$$\boldsymbol{A} \;=\; \boldsymbol{J}^2 \,\oplus\, 1 \,\oplus\, \boldsymbol{J}_2^2 \;\;\Rightarrow\;\; \mathfrak{ARMe}_{\mathrm{spec}}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 3 & 4 \\ 3 & 2 \\ 1 & 2 \end{matrix} \;. \tag{4.12}$$

Example Grid-4

$$\boldsymbol{A} \;=\;\; =\; \boldsymbol{J} \,\oplus\, 1^2 \,\oplus\, \boldsymbol{J}_2 \,\oplus\, 2 \;\;\Rightarrow\;\; \mathfrak{ARMe}_{\mathrm{spec}}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 4 & 4 \\ 3 & 3 \\ 2 & 2 \end{matrix} \;. \tag{4.13}$$

Example Grid-5

$$\boldsymbol{A} \;=\; 1^3 \,\oplus\, \boldsymbol{J}_2 \,\oplus\, 2^2 \;\;\Rightarrow\;\; \mathfrak{ARMe}_{\mathrm{spec}}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 3 & 4 \\ 3 & 3 \\ 1 & 2 \end{matrix} \;. \tag{4.14}$$

Example Grid-6

$$\boldsymbol{A} \;=\; 1^3 \,\oplus\, 2^3 \;\;\Rightarrow\;\; \mathfrak{ARMe}_{\mathrm{spec}}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 3 & 3 \\ 3 & 3 \\ 1 & 1 \end{matrix} \;. \tag{4.15}$$

4.2.3 Two further qualitative examples

Examples 4 and 5 Let us end the current Section by finding an example in which *Are* is below $\mathfrak{Ann}$.

$$\boldsymbol{A} \;=\; \boldsymbol{T} \,\oplus\, 1 \,\oplus\, \boldsymbol{J}_2^2 \;\;\Rightarrow\;\; \mathfrak{ARMe}_{\mathrm{spec}}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 4 & 4 \\ 2 & 2 \\ 3 & 2 \end{matrix} \;. \tag{4.16}$$

And another in which it is incomparable with Are's bottom element.

$$\boldsymbol{A} \;=\; 1^3 \;\oplus\; \boldsymbol{J}_2 \;\oplus\; 2 \;\;\Rightarrow\;\; \mathfrak{arme}\mathrm{spec}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 \\ 3 & 3 \\ 3 & 2 \\ 1 & 2 \end{matrix} \quad . \tag{4.17}$$

4.3 Structural study

4.3.1 Graph Theory level study

Proposition 0 As a graph, the poset of divisors of $n = \mathbb{N}$ is connected.

Proposition $0'$ As a graph, the poset of excess annihilation polynomials is connected.

Proposition 1 As a graph, the poset of divisors of $n = \mathbb{N}$ is not planar unless n takes one of the following forms.

0) 1 .

a) p^k .

b) $p^k\, q^l$.

c) $p^k\, q\, r$.

Proof For 0) is a point, a) is a chain and b) is a grid. And c) is a stack of boxes, for which the presentation in Fig 4.3.a) is available. While

d) $p^2\, q^2\, r$ is nonplanar by Subfig b).

e) And $p\, q\, r\, s$ is nonplanar by Subfig c).

Finally, every other poset of arithmetic divisors contains d) or e) as a subposet. □

Proposition $1'$ As a graph, the poset of excess annihilation polynomials is not planar unless the following holds. Its maximum excess annihilation polynomial's multi-eigenspectrum one of the below forms.

$$0)\ \emptyset\ ,\quad \mathrm{a)}\ \begin{matrix} \lambda \\ k \end{matrix}\ ,\quad \mathrm{b)}\ \begin{matrix} \lambda_1 & \lambda_2 \\ k & l \end{matrix}\ ,\quad \mathrm{c)}\begin{matrix} \lambda_1 & \lambda_2 & \lambda_3 \\ k & 1 & 1 \end{matrix} \quad .$$

Remark 1 All graphs corresponding to posets of divisors of $n \in \mathbb{N}$ are cycle systems. Bar those of form 1 or p^k : the prime powers. Similarly, all annihilation polynomial posets are cycle systems bar those with ≤ 1 eigenvalue. See Fig 4.4 for the grids' homeomorph primes. All divisors with ≥ 3 prime factors are already both cycle systems and homeomorph primes, so there is nothing more to draw.

Remark 2 The annihilating polynomials are Graph-Theoretically dull, by being too regular [261, 264], for instance in the following way.

For annihilating polynomials' graph-level Arenas are just Cartesian products of path graphs,

$$G(\,\mathfrak{Ann}(\boldsymbol{e})\,) \;\;=\;\; \bigtimes_{e\,=\,1}^{E} P_{e_i} \;. \tag{4.18}$$

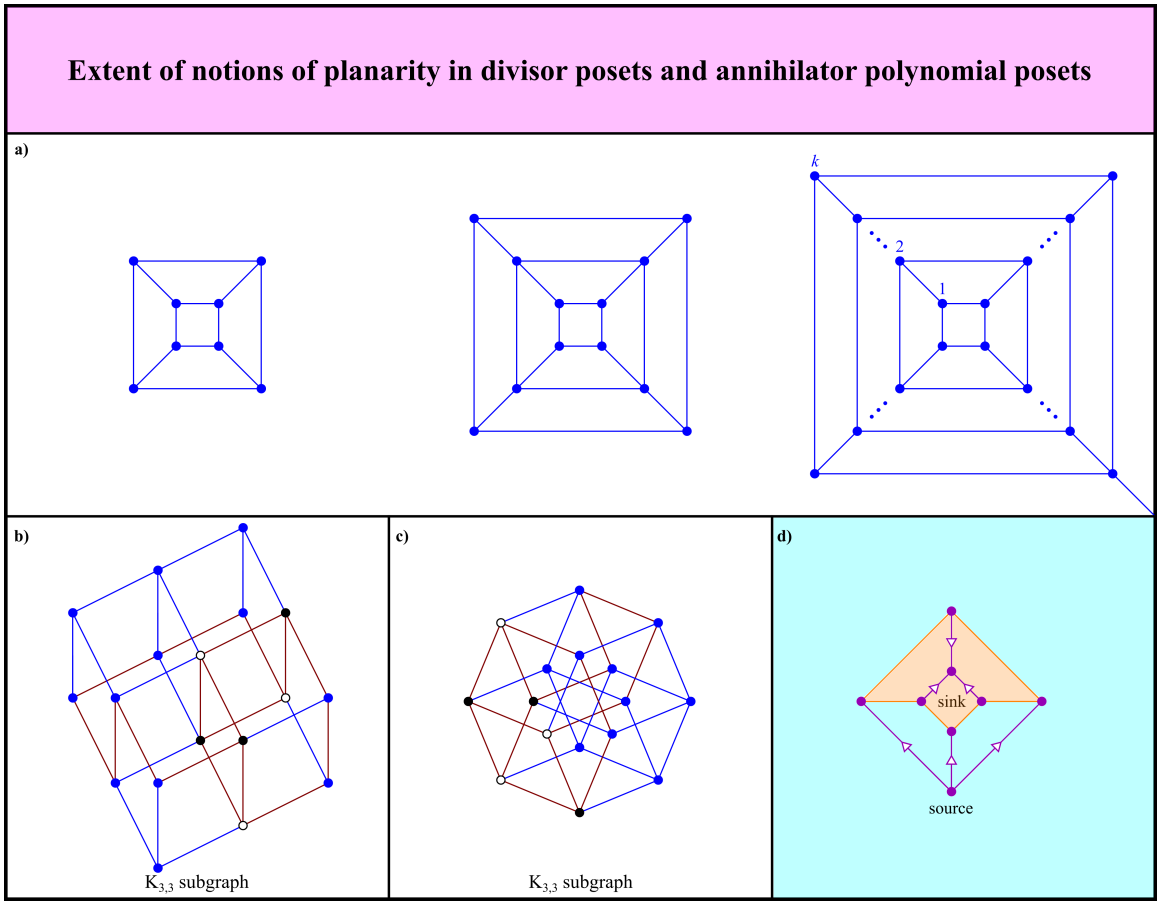


Figure 4.3:

4.3.2 Annihilator chain-lengths

Remark 3 Suppose that our matrix exhibits a nontrivial *annihilator chain-length*.

$$\kappa_e \;:=\; \max_{\mathfrak{a}_{\mathrm{nn}}}\left(\deg(\,Ann\,)\right) \;-\; \min_{\mathfrak{a}_{\mathrm{nn}}}\left(\deg(\,Ann\,)\right) \;=\; \alpha_e \;-\; \mu_e \;>\; 1\;. \tag{4.19}$$

This is a further spectral cardinal. And an excess-length variable corresponding to working with the $EAnn$.

$$K \;:=\; \sum_{e\,\in\,\mathfrak{C}\mathrm{spec}(\boldsymbol{M})} \kappa_e \;=\; \sum_{e\,=\,1}^{E} \kappa_e \;=\; \sum_{e\,=\,1}^{E} (\,\alpha_e \;-\; \mu_e\,) \;=\; N \;-\; M\;. \tag{4.20}$$

Remark 4 Every every natural number $\in\,[\,0,\kappa_e\,]$ is realized by some annihilating polynomial.

Remark 5 Let us now collect some further spectral cardinals. Multiplicities, sums of multiplicities, sums of eigenvalues, or the min and max of whichever kind of real eigenvalue are all simple examples.

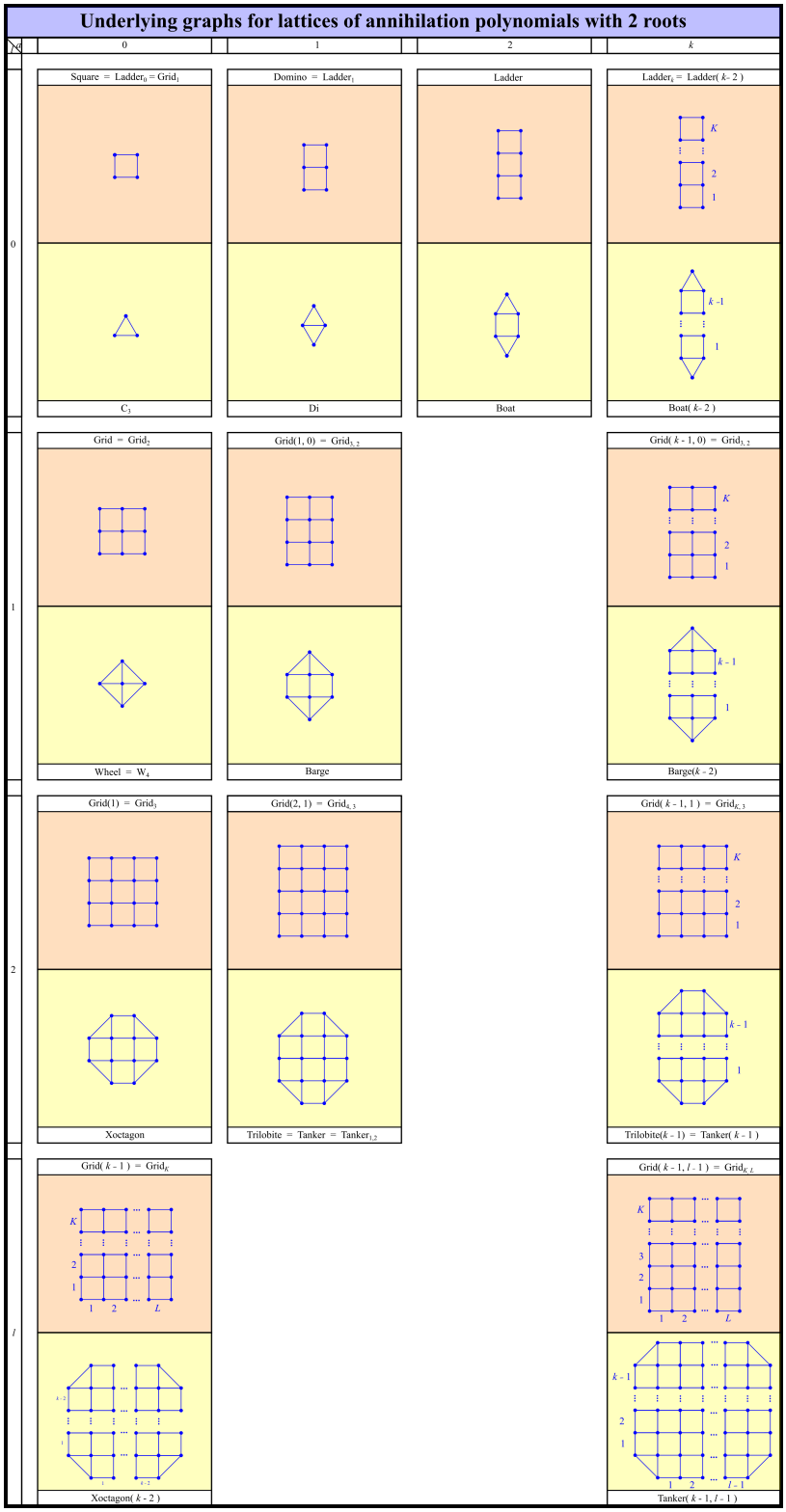

Underlying graphs for lattices of annihilation polynomials with 2 roots
N
0
1
2
k
0
1
2
l
Square = Ladder0 = Grid1
Domino = Ladder1
Ladder
Ladderk = Ladder(k- 2)
C3
Di
Boat
Boat(k- 2)
Grid = Grid2
Grid(1, 0) = Grid3, 2
Grid(k - 1, 0) = Grid3, 2
Wheel = W4
Barge
Barge(k - 2)
Grid(1) = Grid3
Grid(2, 1) = Grid4, 3
Grid(k - 1, 1) = GridK, 3
Xoctagon
Trilobite = Tanker = Tanker1,2
Trilobite(k - 1) = Tanker(k - 1)
Grid(k - 1) = GridK
Grid(k - 1, l - 1) = GridK, L
Xoctagon(k - 2)
Tanker(k - 1, l - 1)

Figure 4.4:

4.3.3 2 to k distinct factors

Remark 5 Order-Theoretically, furthermore each of these paths is implemented straight.[1] By which our Arenas, as posets, are furthermore just Cartesian products of chains,

$$\mathfrak{Ann}(\boldsymbol{e}) \;=\; \bigtimes_{e\,=\,1}^{E} P_{e_i\text{-straight}} \;=\; \bigtimes_{e\,=\,1}^{E} Chain_{e_i} \;. \tag{4.21}$$

Remark 6 If precisely 2 eigenvalues have nontrivial κ_e , then we have a rectangular grid of square-diamond subposets. The above square, domino and grid are 3 of the smallest nontrivial such.

Exercise 1 The missing case is the Ladder – 2×4 – whose graph is a homeomorph of the Boat (Fig 4.4). Find minimum-sized $\mathbb{R}$-valued matrices placing the Arenal polynomial at each symmetry-distinct site within this annihilation polynomial poset. And beneath it. And incomparable to its bottom element. Also find minimum-sized matrices realizing these last two situations for the Domino and the Grid.

Remark 7 If 3 have nontrivial κ_e , then we have a big box stacking together cube-spearpoint subposets.

If 4 have nontrivial κ_e , then we have the same kind of structure built out of tesseract-pointy subposets.

The general case is a ' k-hyperbox' stacking together k-hypercube-pointy subposets.

The most regular such have each eigenvalue's multiplicities have the same gap. The big square, the big cube, the big tesseract ... the big hypercube. The above chain is one example of highly anisotropic k-hyperbox. Others include the ladders stacking whichever number of squares, the cigars of boxes and the pancakes of boxes.

4.3.4 Poset level study

Remark 8 Our posets are bounded, with the discrete and indiscrete partitions as bottom and top.

Definition 1 A poset is *upward planar* [232, 113] if its underlying graph can be cast in planar form without spoiling the upward encoding of the partial ordering. It is *hupward planar* if this can be done furthermore without spoiling a prescribed height function. The Reader is warned at this stage that some sources do not distinguish between these two notions, or use upward planar for the second...

Proposition 2 The poset of divisors of $n \in \mathbb{N}$ is not upward planar, and thus not hupward planar, unless it takes one of the following forms.

0), a) or b).

Proof Non-planar $\Rightarrow$ non-upward planar. Case c) is then eliminated as per Subfig d). □

Proposition 2′ The poset of excess annihilation polynomials is not upward planar, and thus not hupward planar, unless the following holds. Its maximum excess annihilation polynomial's multi-eigenspectrum takes one of the below forms.

0), a) or b).

[1] This is as opposed to their being hockey-stick- or furtherly kinked. See e.g. [264, 121] for further nomenclature for rooting and bending graphs, including into posets.

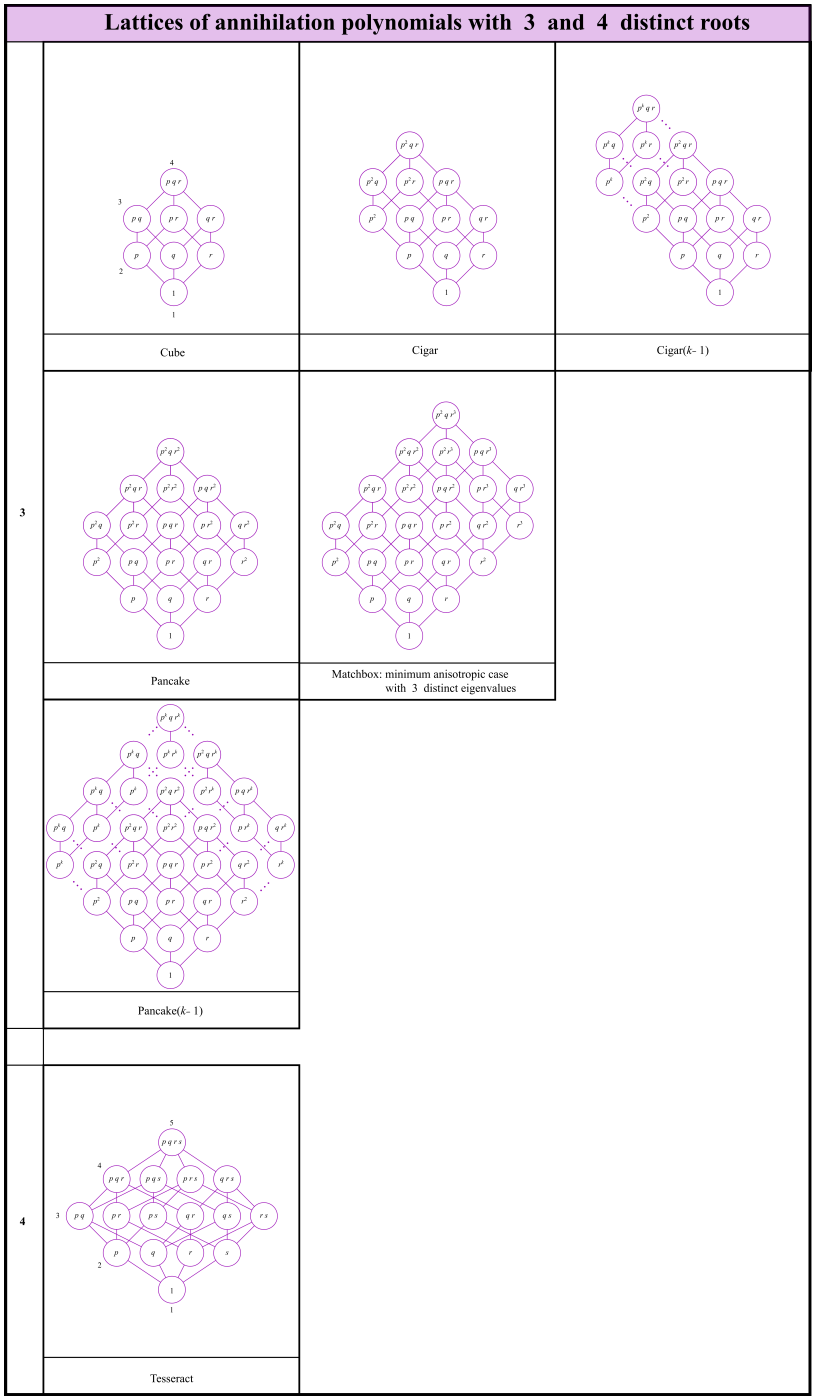


Figure 4.5:

Proposition 3 The poset of divisors of some $n \in \mathbb{N}$ is bounded. With 1 as bottom element and n as top element.

Proposition $3'$ The poset of excess annihilation polynomials is bounded. With 1 as bottom

element and

$$EMax(x) \;=\; \frac{Eig(x)}{Min(x)}$$

as top element.

4.3.5 Lattice level study

Proposition 4 The poset of divisors of some $n \in \mathbb{N}$ is a fortiori

i) a *lattice*, with l.c.m. as *join* and g.c.d. as *meet* .

ii) *Complete*.

iii) *Modular* – diagnosed by there being no C_5 subposets – and a fortiori *distributive*: with no $K_{3,3}$ subposets either [242, 264].

iv) *Atomic*, with the prime numbers as atoms.

Proposition 5 The lattice of divisors of some $n \in \mathbb{N}$ which is *squarefree* – no repeated prime factors – is, furthermore,

v) *complemented*.

vi) *Boolean*.

vii) A *hypercube lattice*, thus furthermore also modelling a power set.

Remark 8 vi) follows by definition from iii) and v):

$$(\,\text{distributive}\,) \;+\; (\,\text{complemented}\,) \;=\; (\,\text{Boolean}\,).$$

4.4 A last few examples

4.4.1 The cube, here alias $p\,q\,r$

Example Cube-1 With *Are* placed at the top. This is realized by the $N = 6$

$$\boldsymbol{A} \;=\; \bigoplus_{i\,=\,1}^{3} i^2 \;\;\Rightarrow\;\; \mathfrak{ARMe}\text{spec}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{matrix}\;. \tag{4.22}$$

Example Cube-4 With *Are* placed at the bottom. This is realized by the $N = 9$

$$\boldsymbol{A} \;=\; \bigoplus_{i\,=\,1}^{3} (\,\boldsymbol{J}_i \,\oplus\, i\,) \;\;\Rightarrow\;\; \mathfrak{ARMe}\text{spec}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{matrix}\;. \tag{4.23}$$

Example Cube-2 With *Are* placed on the second height. This is realized by the $N = 8$

$$\boldsymbol{A} \;=\; \boldsymbol{J} \,\oplus\, 1 \,\oplus\, 2^2 \,\oplus\, \boldsymbol{J}_3 \,\oplus\, 3 \;\;\Rightarrow\;\; \mathfrak{ARMe}\text{spec}(\boldsymbol{A}) \;=\; \begin{matrix} 1 & 2 & 3 \\ 3 & 2 & 3 \\ 2 & 2 & 2 \\ 2 & 1 & 2 \end{matrix}\;. \tag{4.24}$$

Example Cube-3 With *Are* placed on the third height. This is realized by the $N = 7$

$$\boldsymbol{A} = 1^2 \oplus \boldsymbol{J}_2 \oplus 2 \oplus 3^2 \;\Rightarrow\; \mathfrak{arme}\text{spec}(\boldsymbol{A}) = \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 2 & 2 & 2 \\ 1 & 2 & 1 \end{matrix} . \tag{4.25}$$

4.4.2 The tesseract, alias $p\,q\,r\,s$

Remark 1 Reverse-ordering and working systematically down the lattice, we arrive at the following.

Example Tesseract-5 With *Are* placed on the top = 5th floor. This is realized by the $N = 8$

$$\boldsymbol{A} = \bigoplus_{k=1}^{4} i^2 \;\Rightarrow\; \mathfrak{arme}\text{spec}(\boldsymbol{A}) = \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{matrix} . \tag{4.26}$$

Example Tesseract-4 With *Are* placed on the 4th floor. This is realized by the $N = 9$

$$\boldsymbol{A} = \boldsymbol{J} \oplus 1 \oplus \bigoplus_{j=2}^{4} j^2 \;\Rightarrow\; \mathfrak{arme}\text{spec}(\boldsymbol{A}) = \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{matrix} . \tag{4.27}$$

Example Tesseract-3 With *Are* placed on the 3rd floor. This is realized by the $N = 10$

$$\boldsymbol{A} = \bigoplus_{a=1}^{2} (\boldsymbol{J}_a \oplus a) \oplus \bigoplus_{b=3}^{4} b^2 \;\Rightarrow\; \mathfrak{arme}\text{spec}(\boldsymbol{A}) = \begin{matrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 1 & 1 \end{matrix} . \tag{4.28}$$

Example Tesseract-2 With *Are* placed on the 2nd floor This is realized by the $N = 11$

$$\boldsymbol{A} = \;=\; \bigoplus_{k=1}^{3} (\boldsymbol{J}_k \oplus k) \oplus 4^2 \;\Rightarrow\; \mathfrak{arme}\text{spec}(\boldsymbol{A}) = \begin{matrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 \end{matrix} . \tag{4.29}$$

Example Tesseract-1 With *Are* placed on the bottom = 1st floor. This is realized by the $N = 12$

$$\boldsymbol{A} = \;=\; \bigoplus_{i=1}^{4} (\boldsymbol{J}_i \oplus i) \;\Rightarrow\; \mathfrak{arme}\text{spec}(\boldsymbol{A}) = \begin{matrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{matrix} . \tag{4.30}$$

4.4.3 The k-hypercube, alias $p_1\,p_2 \ldots p_k$

Example p With *Are* placed on floor $p \in [1, K]$. This is realized by the

$$N = 3p + 2(k - p) = 2k + p$$

$$\boldsymbol{A} = \bigoplus_{o=1}^{p} (\boldsymbol{J}_o \oplus o) \oplus \bigoplus_{q=p+1}^{k} q^2 \;\Rightarrow\; \mathfrak{arme}\text{spec}(\boldsymbol{A}) = \begin{matrix} 1 & \ldots & p & p+1 & \ldots & k \\ 3 & \ldots & 3 & 2 & \ldots & 2 \\ 2 & \ldots & 2 & 2 & \ldots & 2 \\ 2 & \ldots & 2 & 1 & \ldots & 1 \end{matrix} . \tag{4.31}$$

This covers all of Subsec * and the current Sec's examples (on occasion modulo reassigning which block structures are realized by which eigenvalue).

Exercise 2 With reference to Fig 4.4, find minimum-sized $\mathbb{R}$-valued matrices whose annihilating polynomials form each poset whose graph is a homeomorph of one of the following.

a) The smallest Barge.

b) The smallest Xoctagon.

c) The smallest Trilobite.

d) The smallest Tanker that is neither a Xoctagon nor a Trilobite.

In each case, find a minimum example that places the Arenal polynomial at each symmetrically-distinct site within the poset. And beneath the poset. And incomparable with the bottom of the poset.

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