Tensor Networks throughout the Basic Theory of STEM. IV. Cartesian Tensors

E. Anderson*

Abstract

We continue our program of applying tensor networks throughout the basic theory of STEM, today with a Review on Cartesian tensors. While basic accounts often treat these 1) plainly rather than 2) with upstairs-downstairs distinction, we entertain and compare both. 2) extends to vastly more general notions of tensor. We cover tensor transformation laws, Tensor Algebra, tensor symmetries, and isotropic tensors, all in tensor networks notation.

We also cover arenas of tensors, i.e. the totalities of various type of tensor. Not only A) tensors' (co)vector space products of fixed rank and dimension and their tensor field generalizations in fibre bundle form. But also the arenas of B) ranks, C) dimensions, and D) both together. And then the bigger arenas placing A) over B-D), now as general bundles. In adapted variables, B) can be viewed in terms of Pascal's triangle and 2-part partitions, while D) is somewhat more intricate. Various meaningful quotients of these arenas are also considered. As is the simple Tensor Calculus afforded by Cartesian tensor fields. Such extensive focus on arenas is part of Modern Applied Topology, with the current Article benefiting from simpler Combinatorial and Linear instances.

Arenizing tensor symmetries returns the full Young lattice arena of partitions. While arenizing isotropic tensors gives a rather interesting application of the joint arena of dimensions and ranks. Aside from the current Review's overall distinctively Combinatorial point of view arising standardly from symmetries and novelly from arenization, more pseudotensor, quotient-theorem and isotropic-tensor variants than usual are included.

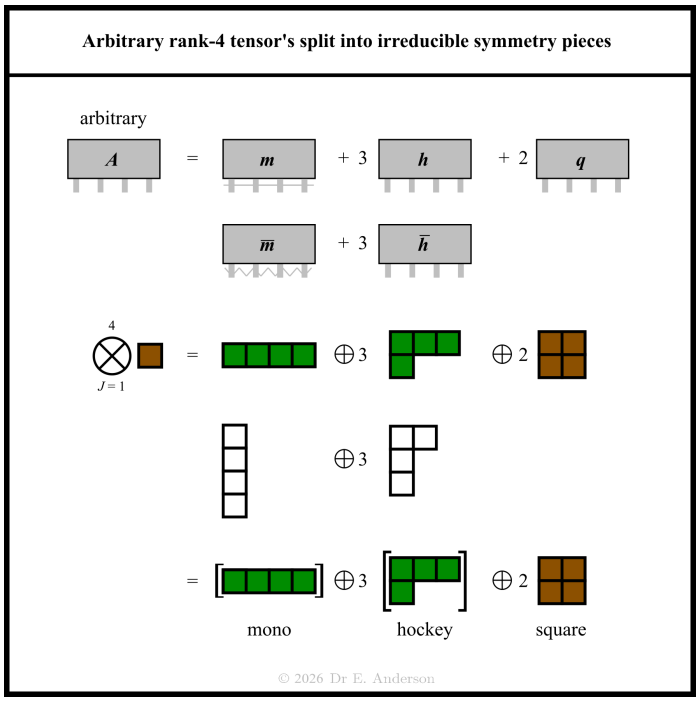


Figure 1:

This Review is ⟨3⟩: accessible to third-year undergraduates.

Cite as E. Anderson, "Tensor Networks throughout the Basic Theory of STEM. IV. Cartesian Tensors", Online Encyclopaedia of Tensors and other Multi-Linear Arrays, institute-theory-stem.org/oetoma-tensor-networks-for-Cartesian-Tensors/ (2026).

Date-stamp: v0 18-09-2026, v1. 27-09-2026. Copyright of Dr E. Anderson.

* Dr.E.Anderson.Maths.Physics *at* protonmail.com . Institute for the Theory of STEM

Contents

1 Introduction **5**
- 1.1 The underlying transformations . . . 5
- 1.2 Aside: coordinate transformations more generally, leading to further kinds of tensor 7
  - 1.2.1 The other linear transformations . . . 7
  - 1.2.2 Special Relativity's inertial frames . . . 8
  - 1.2.3 General curvilinear transformations . . . 8
  - 1.2.4 Differentiable manifolds and General Relativity . . . 9
  - 1.2.5 The upstairs–downstairs distinction (u.d.d.) . . . 11
- 1.3 Outline of contents . . . 13
  - 1.3.1 The main body . . . 13
  - 1.3.2 The Appendices . . . 13
  - 1.3.3 Conceptual underpinning for the current Review's range of Mathematics . . . 14

2 The Cartesian tensor transformation laws **15**
- 2.1 $O(D)$ alias Cartesian tensors . . . 15
- 2.2 $SO(D)$-tensors alias Cartesian pseudotensors . . . 17

3 Tensor arenas **18**
- 3.1 Arenas of tensors for fixed dimension and rank . . . 18
- 3.2 Combinatorial tracking arenas (CTAs) . . . 19
  - 3.2.1 Introduction . . . 19
  - 3.2.2 The arena of dimensions . . . 19
  - 3.2.3 Plain-rank arena . . . 20
  - 3.2.4 U.d.d. rank arena . . . 21
  - 3.2.5 Adapted variables . . . 22
  - 3.2.6 0- and 1-D collapses . . . 24
  - 3.2.7 The u.d. distinguishable and yet meaningless quotient arena . . . 24
  - 3.2.8 Aside: fate of CTAs for manifolds . . . 26
- 3.3 Order-Theoretic and Graph-Theoretic analysis of dimension and rank separately . . 27
  - 3.3.1 The trivial cases . . . 27
  - 3.3.2 Full u.d.d. version . . . 28
  - 3.3.3 Motivational aside on arenas . . . 30
  - 3.3.4 A first two Mathematical analogues of u.d.d. tensor rank arenas . . . 31
  - 3.3.5 Corresponding quotient arenas . . . 31
  - 3.3.6 Analogy 3: arena of partitions with $\leq$ 2 parts . . . 31
- 3.4 Dimension-and-rank arenas, including Order- and Graph- Theoretic analysis . . . 32
  - 3.4.1 Full plain version . . . 32
  - 3.4.2 Plain quotients . . . 35
  - 3.4.3 Full u.d.d. version . . . 35
  - 3.4.4 Analogues of the preceding quotients . . . 36
  - 3.4.5 Interplay with $\updownarrow$ quotient . . . 37

4 Bigger arenas of tensors **39**
- 4.1 Definitions . . . 39
- 4.2 Interlude on Bundle Theory . . . 40
  - 4.2.1 General bundles . . . 40
  - 4.2.2 Fibre bundles . . . 42
  - 4.2.3 Sections of a bundle . . . 42
  - 4.2.4 Discretized base spaces . . . 42
- 4.3 Some plain examples of bigger tensor arenas . . . 43
  - 4.3.1 The full plain $\mathfrak{Tensor}(D)$ . . . 43
  - 4.3.2 The full plain $\mathfrak{Tensor}(R)$ . . . 43
  - 4.3.3 Quotients so far . . . 44
- 4.4 Motivation for CTAs . . . 44
  - 4.4.1 $\mathfrak{Tensor}(R)$ . . . 44
  - 4.4.2 $\mathfrak{Tensor}(D)$ . . . 44

4.4.3 𝔗ensor . . . 45

5 Cartesian tensor algebra **46**
5.1 The full Cartesian case: $O(D)$ . . . 46
5.1.1 Linearity . . . 46
5.1.2 The outer product . . . 47
5.1.3 Contraction and inner product. i. Definitions and notation . . . 48
5.1.4 ii. Inner-product tensoriality . . . 49
5.1.5 iii. Self-contraction tensoriality . . . 50
5.1.6 The quotient theorem . . . 52
5.2 $SO(D)$ counterparts . . . 54
5.2.1 Linearity . . . 54
5.2.2 Outer product . . . 54
5.2.3 Self-contraction . . . 55
5.2.4 Inner products . . . 55
5.2.5 Quotient theorem . . . 55
5.3 Parity unification . . . 55
5.3.1 Parity conservation . . . 55
5.3.2 Parity multiplication. i. Binary . . . 56
5.3.3 ii. n-ary . . . 56

6 Tensor symmetries **57**
6.1 The minimum setting: rank 2 . . . 57
6.1.1 Pairwise (anti)symmetry . . . 57
6.1.2 Further notations . . . 58
6.1.3 The trace– tracefree and combined splits . . . 59
6.2 A simpler extension . . . 60
6.2.1 Minimum, total and partial (anti)symmetry . . . 60
6.2.2 Partition notation for symmetry type . . . 61
6.3 A harder extension: including mixed symmetries . . . 62
6.3.1 Existence, minimum occurrence and persistence . . . 62
6.3.2 A start on determining the nature of the minimum example . . . 63
6.3.3 Applying a subcase of a subcase of how to tensor Young diagrams . . . 64
6.3.4 Determining the nature of the minimum example continued . . . 64
6.3.5 Rank 4 . . . 65
6.3.6 Exercises . . . 66
6.3.7 More advanced Exercises . . . 67
6.4 Symmetries of u.d.d. tensors . . . 67
6.5 Interplay with quotient rule . . . 68
6.5.1 Contractions with symmetric pieces . . . 68
6.5.2 Interplay with trace–tracefree split . . . 69
6.5.3 Interplay with pseudotensors . . . 69
6.5.4 Parity reformulation of four quartets of quotient theorems and corollaries . . . 71

7 Tensor fields and Cartesian Tensor Calculus **72**
7.1 Tensor fields and tensor bundles . . . 72
7.1.1 Introduction . . . 72
7.1.2 Scalar and vector field arenas . . . 72
7.1.3 Continuing in tensor field arena notation . . . 73
7.1.4 Arenas of tensor fields of arbitrary fixed rank . . . 73
7.1.5 (Co)tangent bundle approach to the above arenas . . . 73
7.1.6 With sections recovering individual tensor fields . . . 74
7.1.7 Route 2 to Cartesian tensor fields and their arenas . . . 74
7.2 Interplay with CTAs . . . 74
7.3 Cartesian Tensor Calculus . . . 75
7.3.1 Grad is a covector . . . 75
7.3.2 Partial derivation is tensorial . . . 76
7.3.3 Cartesian Tensor Calculus counterpart of the quotient theorem . . . 76

7.3.4 Tensor Calculus subtleties . . . 77

8 Plain tensor networks **78**
8.1 Tensor transformation laws . . . 78
8.2 Tensor Algebra . . . 79
8.3 Tensor symmetries . . . 83
8.4 Tensor Calculus . . . 83
8.5 Plain versus u.d.d.'s vertical variant . . . 83

9 Conclusion **85**
9.1 The current Review's Combinatorics and arenas . . . 85
9.2 Extending to other notions of tensor . . . 85

A The world of combinations **86**
A.1 Combinations themselves . . . 86
A.1.1 Definition and notation . . . 86
A.1.2 Various binomial applications . . . 86
A.2 Combination arenas . . . 87
A.2.1 Binomial expansion's chains of combinations . . . 87
A.2.2 Pascal's triangle of combination values . . . 87
A.2.3 Pascal's triangle as the arena of combinations, $= \mathfrak{C}\text{ombi}$ . . . 89
A.2.4 The corresponding count returns the triangular numbers . . . 89
A.2.5 Pascal's triangle as presented is already-centred . . . 89
A.2.6 Reflection-quotient arena . . . 90
A.3 Graph and Order Theory continuation . . . 90
A.3.1 For Pascal's triangle $= \mathfrak{C}\text{ombi}$ . . . 90
A.3.2 Discussion . . . 91
A.3.3 ... and of its in-out quotient . . . 92
A.3.4 Further Order Theory . . . 92
A.4 Aside on naming some of the current Review's graphs . . . 92

B The world of partitions **93**
B.1 Partitions . . . 93
B.1.1 Definition and smallest examples . . . 93
B.1.2 Young diagrams . . . 94
B.1.3 Conjugate partitions . . . 95
B.1.4 k-part partitions . . . 95
B.1.5 Tensoring Young diagrams . . . 96
B.1.6 Hook lengths and box values . . . 97
B.1.7 Exercises . . . 98
B.2 Partition arenas . . . 99
B.2.1 Adapted variables for partitions . . . 99
B.2.2 The Young lattice arena of partitions . . . 99
B.2.3 Runnels and interesting irregularity . . . 101
B.2.4 The Young lattice's quotient by partition conjugation . . . 101
B.2.5 Some other smallest arenas of partitions . . . 101

1 Introduction

1.1 The underlying transformations

Structure -1 The current Article works with frames in

$$\mathbb{R}^D \ .$$

With a shared origin, and a globally-valid set of axes at right angles to each other. These are *Cartesian frames*, corresponding to *Cartesian coordinate systems*. For all that these italicized terms continue to make sense without the shared origin.

Structure 0 The following transformations

$$\boldsymbol{L}$$

[105, 106, 98] preserve these frames, shared origin included. *Rotations* (Fig 2.a) about an axis passing through the origin. And *reflections* (Subfig b) about a hyperplane $\mathbb{R}^d$ though the origin. Where

$$d \ := \ D \ - \ 1 \ . \tag{1}$$

In 2-D , this is just a line through the origin, and in 3-D , just a plane through the origin.

For whichever fixed D , the $\boldsymbol{L}$ form the *orthogonal group*

$$O(D)$$

[105, 107, 100]. These are the homogeneous-linear transformations that furthermore preserve the usual (*Euclidean*) *scalar product*, and thus both *Euclidean lengths* and *Euclidean angles*. Schematically, using the habitual primed and unprimed frame notation,

$$\overline{\boldsymbol{x}} \ \longrightarrow \ \overline{\boldsymbol{x}}' \ = \ \underline{\overline{\boldsymbol{L}}} \cdot \overline{\boldsymbol{x}} \ . \tag{2}$$

Two equivalent phrasings to preserving the Euclidean scalar product are preserving the *Euclidean inner product*, and preserving the *Euclidean metric*

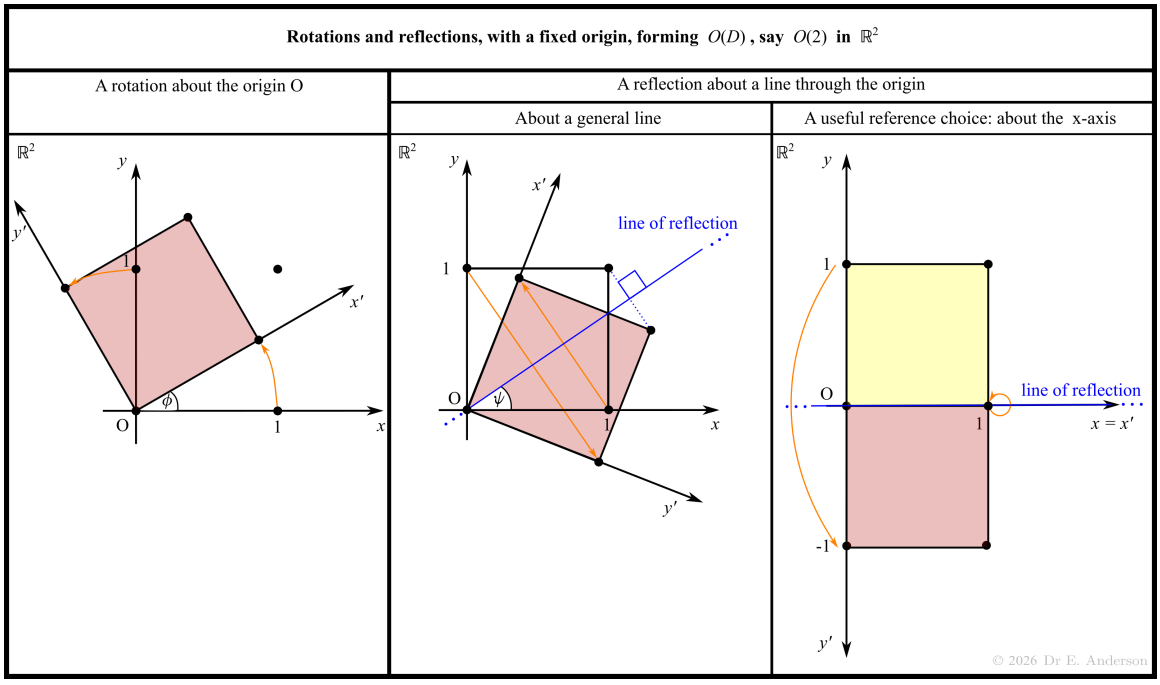


Figure 2:

Structure 0′ We often also drop the discrete requirement of preservation under reflections. The transformations left are then purely rotations, forming the group

$$Rot(D) \quad = \quad SO(D) \ .$$

The ensuing tensors are often referred to as (*Cartesian*) *pseudotensors* in introductory and Physics texts.

Remark 1 First courses on Cartesian tensors [3, 6, 23, 10, 22] often continue at this point by presenting some preliminary identities afforded by orthogonality. The current Review is however a second course, and so benefits from its intended audience having seen this before. Furthermore, the current Review is part of an online Encyclopaedia [27] that concentrates on reformulating the basic theory of STEM in terms of tensor networks. And is the Encyclopaedia's fourth Article. By this point we have already presented tensor networks formulations of Vector Algebra [29], Vector Calculus [30] and Linear Algebra [31]. So our audience is already familiar with what tensor networks are.[1]

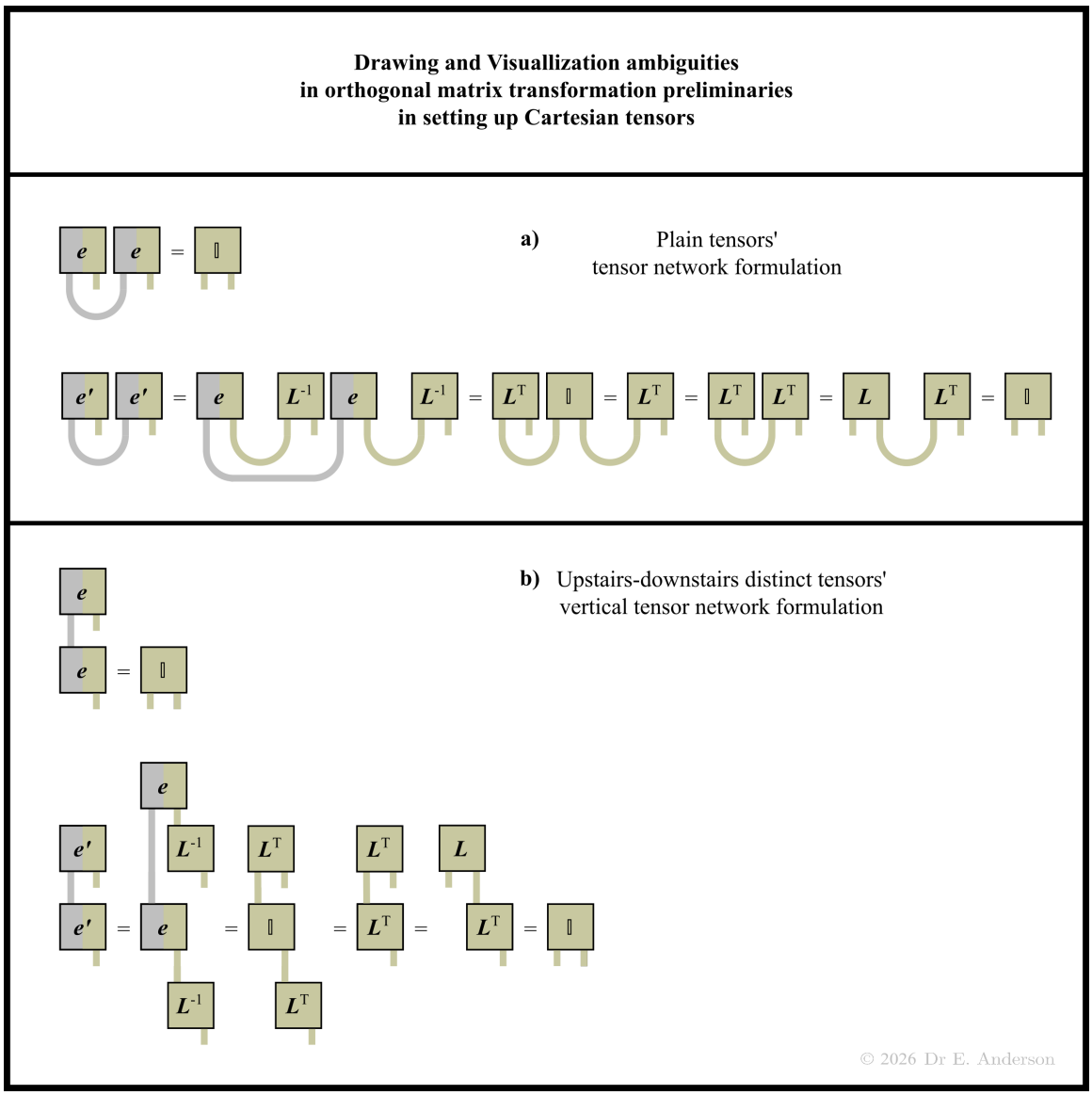


Figure 3:

[1] If not, consult [29]...

Exercise 0 And so we do not present the abovementioned identities in a first course's habitual component notation. Instead, we present them in firstly one tensor network notation, and then another![2] And expect the Reader to figure out what these equations (Fig 3) look like in components. And thus to identify them with equations that featured in your first course on Cartesian tensors... What basic Mathematical move does each step in the figure correspond to? Where $\boldsymbol{e}$ and $\boldsymbol{e}'$ are an unprimed and primed pair of orthonormal linear bases...

1.2 Aside: coordinate transformations more generally, leading to further kinds of tensor

1.2.1 The other linear transformations

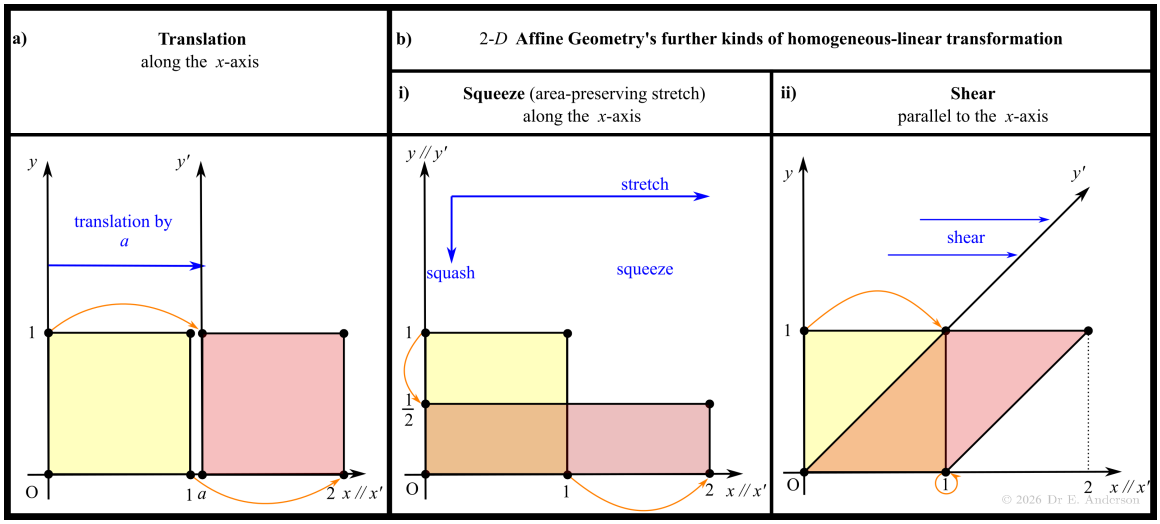


Figure 4:

Remark 1 Since this is a second course, comparison with other coordinate transformations that you have seen elsewhere is reasonable. Leading in various cases to notions of tensor that many of you will have seen in other courses, or shall be studying soon...

Structure I Were the origin to be free to shift under transformations, *translations* (Fig 4.a) would also be included. Then the whole of the *Euclidean group* $Eucl(D)$ [102] of transformations would be in play. Corresponding to Euclidean Geometry. Which is very widely taught in the case of the Euclidean plane $\mathbb{R}^2$ as equipped with the Euclidean metric (or norm $||\,||$ or inner product). Though some of you will have also specifically studied Geometry in Euclidean space $\mathbb{R}^3$. And all of you will have used $\langle \mathbb{R}^3, ||\,|| \rangle$ as an excellent model for Physical 3-space....

In this context, we pass from (2) to its inhomogeneous-linear counterpart,

$$\overline{\boldsymbol{x}} \longrightarrow \overline{\boldsymbol{x}}' = \underline{\overline{\boldsymbol{L}}} \cdot \overline{\boldsymbol{x}} + \overline{\boldsymbol{a}} \, . \tag{3}$$

Structure II An alternative generalization is from Euclidean length preserving homogeneous-linear transformations to merely invertible homogeneous-linear transformations. This enlarges $O(D)$ to the *general linear group*

$$GL(D) \, .$$

At the level of matrices, passing from $GL(D)$ to its $O(D)$ subgroup enforces that the transposed matrices are to serve as the inverses. And consequently that the determinant is ± 1 . With $+1$ characterizing the special orthogonal group. And -1 the unique group coset of other orthogonal transformations. Which can be taken to comprise $SO(D)$ multiplied by whichever reflection.

[2] See [31] for this bicoloured notation, alongside motivation for multicoloured or 'rainbow' tensor networks...

There are however other determinant ± 1 general linear transformations. I.e. the $+1$ determinant special linear transformations

$$SL(D) \ .$$

These come in 2 conceptual types, 2-D representatives of which are drawn in Subfig b). And then their group coset upon multiplying by some reflection, covering the -1 determinant case. The independent further transformation in the general-linear case is the overall *dilation*, alias change of scale.

Structure III Both of the above generalization can be applied at once, giving the inhomogeneous general-linear transformations,

$$\overline{\boldsymbol{x}} \ \longrightarrow \ \overline{\boldsymbol{x}}' \ = \ \underline{\overline{\boldsymbol{G}}} \cdot \overline{\boldsymbol{x}} \ + \ \overline{\boldsymbol{a}} \ . \tag{4}$$

Which form the *Affine group*

$$Aff(D)$$

[102, 97] in the $\mathbb{R}^D$ setting.

Remark 2 Tensors can also be defined for $SL(d)$ and $GL(d)$. With or without reflections. Which indicates part of why '*Cartesian* pseudotensors' is a more precise description of part of what the current Review covers than just 'pseudotensors'.

1.2.2 Special Relativity's inertial frames

Structure IV Those Readers who have already studied Special Relativity will have seen a close parallel of the above discussion there. With in particular the Lorentz group in place of the orthogonal group, and the Poincaré group in place of the Euclidean group. Living on Minkowski spacetime $\mathbb{M}^{D\,+\,1}$ in place of $\mathbb{R}^D$. And leading to a theory of Lorentzian tensors that closely parallels that of Cartesian tensors. Now modelling Physics in Special Relativity's [19] notion of inertial frames.

1.2.3 General curvilinear transformations

Structure V Let us next entertain *general curvilinear coordinate transformations* in $\mathbb{R}^D$. Now with not some constant transformation matrix but some coordinate-dependent matrix

$$\overline{\boldsymbol{x}} \ \longrightarrow \ \overline{\boldsymbol{x}}' \left(\overline{\boldsymbol{x}} \right) \ . \tag{5}$$

With corresponding transformation matrix with components

$$J_i{}^{j'} \ := \ \frac{\partial x'^{\,j'}}{\partial x^i} \ . \tag{6}$$

A first coordinate-free notation for this is

$$\underline{\overline{\boldsymbol{J}}} \ := \ \frac{\partial \overline{\boldsymbol{x}}'}{\partial \overline{\boldsymbol{x}}} \ . \tag{7}$$

This is the *Jacobian matrix*. We leave it to the Reader to cast this definition in tensor networks form.

Remark 1 From here on it will not do to transform just our little yellow square. Since now distinct small approximate squares will in general transform distinctly from each other. Thus we need a *coordinate grid* containing at least a few small approximate squares both to see what is going on and to indicate this variability with location. See Fig 5 for a first simple example.

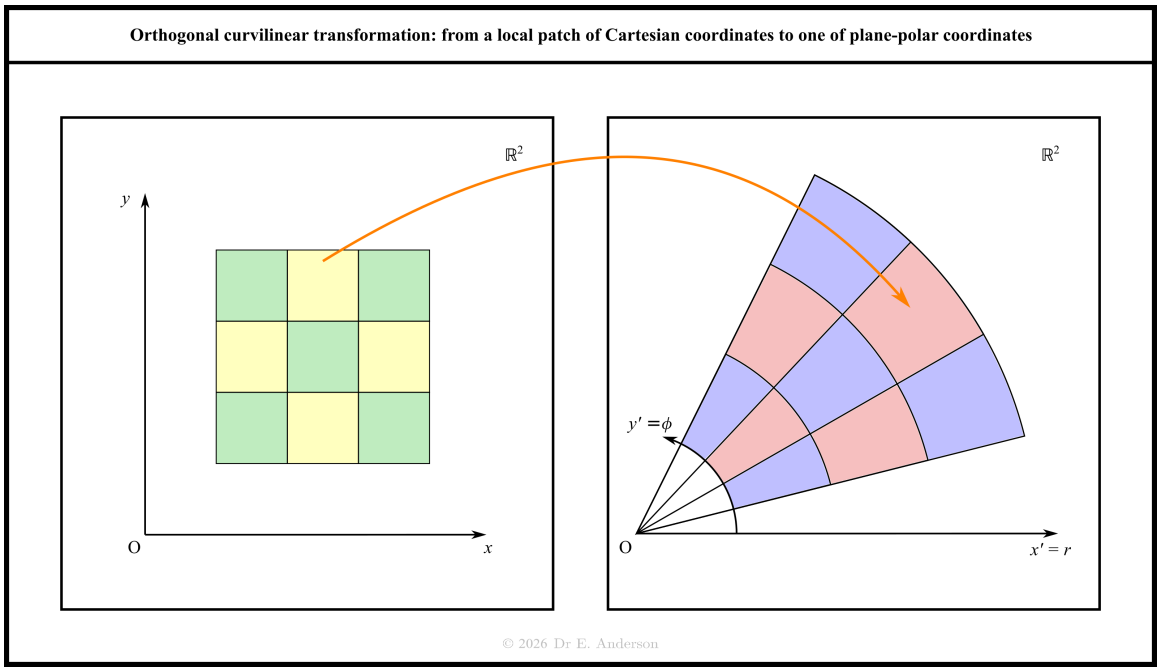


Figure 5:

A theory of tensors is then supported using the general-curvilinear $\boldsymbol{J}$ in place of the Cartesian tensors' orthogonal $\boldsymbol{L}$. The corresponding group of transformations is the *diffeomorphism group*[3]

$$Diff\left(\mathbb{R}^D\right) \ .$$

Everything in the current SSSec can also be carried out in $\mathbb{M}^{D+1}$. Physically corresponding to modelling not just Special-Relativistic inertial frames but also accelerated frames as well. For all that this is not usually included in basic courses.[4]

1.2.4 Differentiable manifolds and General Relativity

Structure 6 Even more generally, Jacobians can be locally defined in differentiable manifolds [108] $\mathfrak{m}^D$. To provide some context, if these are given their own Affine structure [18, 25], then a notion of curvature is supported. And so our modelling scope includes in general curved spaces, as considered in Differential Geometry [20]. If these are furthermore given a metric level of structure, then we arrive at Riemannian Geometry [20] for positive-definite metrics. And at semi-Riemannian Geometry [16] for the $-+...+$ subcase of indefinite-signature metrics. Part of whose modelling scope is the general curvilinear coordinate frames of the in-general curved spacetimes of General Relativity [17, 13].

The way this generalization works is as follows. Differentible manifolds are guaranteed to support *local coordinate charts* (Fig 6.a),

$$\phi \ : \ \mathfrak{U} \longrightarrow \mathbb{R}^D \ .$$

For $\mathfrak{U}$ some open subset within $\mathfrak{m}^D$, amounting to a first use of the above 'local'.

The second use is to consider pairs of coordinate charts ϕ, ϕ' whose respective $\mathfrak{U}, \mathfrak{U}'$ overlap (Subfig b). So then one can form the Jacobian matrix from ϕ 's unprimed local coordinates to ϕ 's primed ones. As enabled by now one locally working with a map

$$\mathbb{R}^D \longrightarrow \mathbb{R}^D \ .$$

[3] There are multiple specific such, depending on which sufficiently tractable function space [91] we wish to work with. And on whether we choose to include 'large diffeomorphisms': an analogue of the above ambiguity as to whether to include rotations.

[4] If interested in this, carefully peruse [19]. Whose text is not only an introduction to the subject but also interwoven with more advanced comments on this particular subtlety.

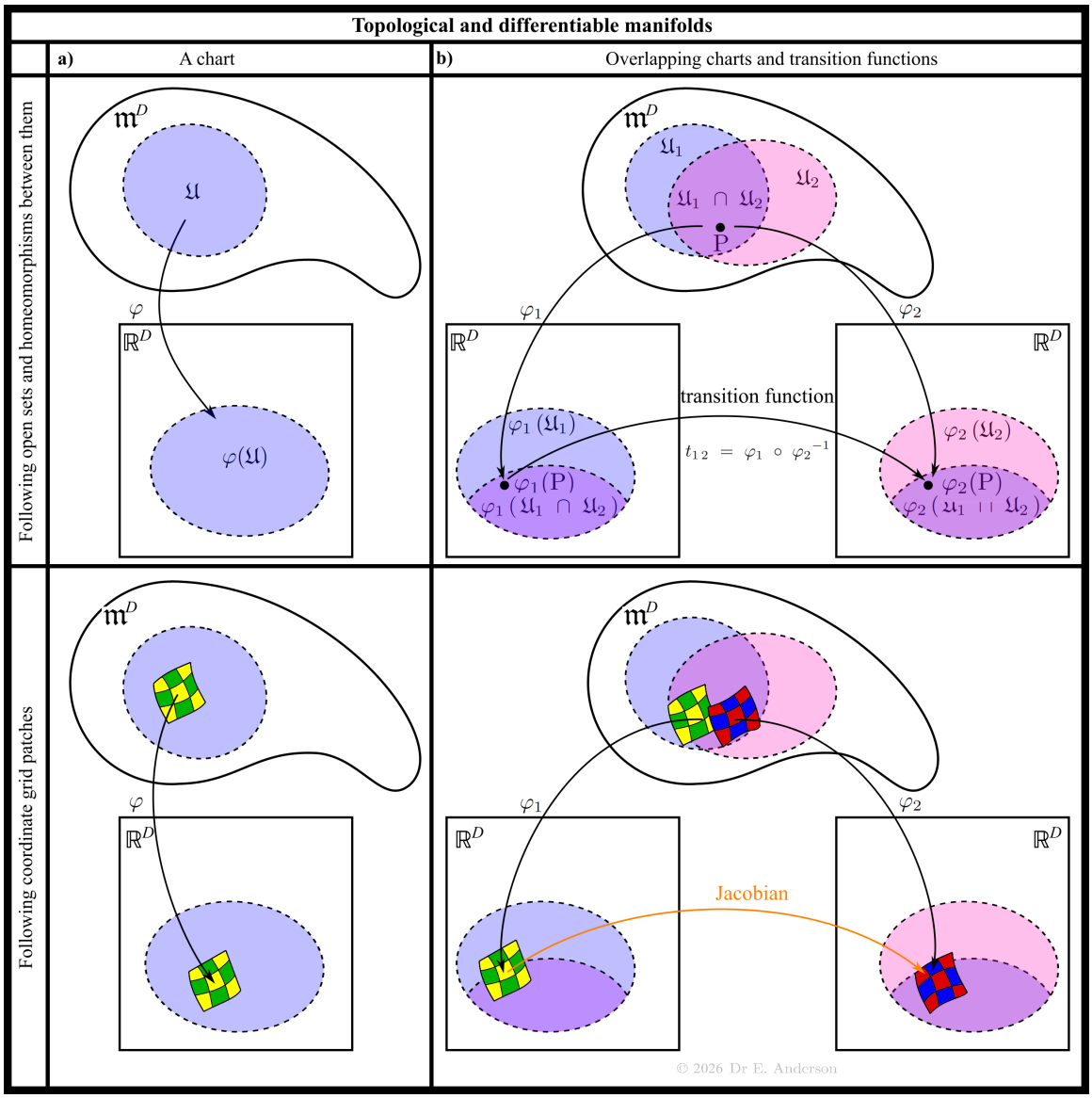


Figure 6:

So that the usual Calculus is available. By which the Jacobian's derivatives are meaningfully defined, at least for sufficiently well-behaved coordinate functions.

And so the flat-space general-curvilinear Tensor Calculus' Jacobian coordinate transformations extend to use on Differentiable manifolds. Providing us with a general-curvilinear Tensor Calculus on differentiable manifolds. Whose transformation group is now[1]

$$Diff\left(\mathfrak{m}^D\right) \ .$$

For use in Differentiable Geometry and its applications, including General Relativity.

1.2.5 The upstairs–downstairs distinction (u.d.d.)

Naming Remark 1 On the one hand, the above formula for the Jacobian matrix displays an *upstairs–downstairs distinction* of indices.

On the other hand, Cartesian tensors are usually presented without such a distinction, which we refer to as *plain indices*.

Aliases for u.d. indices include *raised* and *lowered* indices. U.d. is a more straightforward name. After where the indices are are located. As opposed to with reference to some raising or lowering action...

The alias *contra-* and *co-* indices is also widespread, following Sylvester [1]. 'Co' here refers to transforming like the frame, and 'contra' to transforming oppositely. This conveys a further piece of information relative to the above aliases. *Covariant* and *contravariant* are the general words of this kind. Another commonly encountered name is *covector*, short for *covariant vector*; *contravariant vectors* are usually just called vectors. In turn, *dual vector* and 1-*form* [11, 13, 103, 109], are further common aliases for covectors. While *column vector* and *row vector* are simple instances of vectors and covectors respectively.

At least the downstairs-upstairs and co-contra names continue to make sense in the context of index-free notation, of which the tensor networks notation [14, 21, 24, 26] that the current Encyclopaedia is written in is an example. With reference to the usual tensorial mantra of 'define and conceptualize index-free, for all that one usually evokes indices if calculations are required...'

Notational Remark 1 Three common styles of index-free notation are as follows. Use empty slots in place of indices. In the u.d.d. formulation, upstairs and downstairs slots need to be distinguished. Or use underlines in the plain case, to underlines and overlines in the u.d.d. formulation. Or use tensor networks, which have unlabelled legs in place of indices. Emanating out of the top of the tensor's box in place of overlines, or out of its bottom in place of underlines. This is the current Encyclopaedia's preferred choice of notation.

Remark 1 A piece of folklore, in particular among Physics majors, is that u.d.d. is a curved space(time) complication. This is incorrect.

Most obviously because it already features in curvilinear coordinates on flat space(time).

But trying to work with plain indices in the context of Special Relativity's inertial frames also incurs mistakes. Via say missing the sign change incurred by

$$\eta_{t\,t}\, V^t \;=\; -\, V_t \;.$$

Where $\boldsymbol{\eta}$ is the Minkowski metric and $\boldsymbol{V}$ is whatever 4-vector.

Let us leave it to the Reader to find a mistake incurred by neglecting to use u.d.d. in plane polar coordinates (**Exercise 1**).

All in all *each of curvilinear coordinates in flat space, and indefiniteness, also require making the u.d.d.*

Remark 3 In turn, this means that a second piece of folklore – that the theory of Lorentzian tensors is a very close copy of the theory of Cartesian tensors – comes with limitations. At least from the point of view that one usually learned Cartesian tensors in the plain formulation.

There is however another point of view. The above dismissal of the first piece of folklore leads to Cartesian tensors being highly-isolated in supporting a plain-index formulation. While Cartesian tensors support an equivalent u.d.d. formulation. So my chosen course of action is to give a u.d.d. formulation of Cartesian tensors. This is motivated by its much wider applicability. Upon which the

second piece of folklore is rendered rather more true, provided that it refers to u.d.d. formulations of Cartesian tensors...

Pedagogical Remark 1 U.d.d. formulations might not be presented in a course for first or second years. By choosing immediate use, simplicity and tradition. Over what happens when a proportion of the students more eventually encounter curvilinear, Special-Relativistic, Differential-Geometric and Generally-Relativistic material in some order.[5] The current Review furthermore specifically covers much of what could occur if a second course on Cartesian tensors were to be made available to third or fourth year students. As such, the u.d.d. formulation of Cartesian tensors makes for a very natural component of such a course.[6]

Pedagogical Remark 2 On the one hand, degree courses seldom offer second courses in Cartesian tensors. On the other hand, many graduate school programs make use of symmetries and representations. And another of the current Review's components concerns how tensors (not necessarily Cartesian) provide an interesting application of some of this material.

Furthermore, some graduate school programs make heavy use of tensor networks in various specialized areas. This is almost always done without drawing on how many core undergraduate subjects – by this stage very familiar material to the students – can be recast in tensor networks. Graduate school programs also rather more frequently introduce coordinate-free notation for conceptualization. In graduate-level courses, without drawing on how this can already be done for core undergraduate courses. Additionally such ventures done without tensor networks are quite rapidly vulnerable to ugly or broken coordinate-free notation.

Ergo more graduate-level programs should use tensor networks to mediate their wishes to be coordinate-free. While those students starting graduate school programs that do make heavy use of tensor networks would do well to see what happens when tensor networks are applied to the basic undergraduate courses. Which graduate schools do not provide. Providing one motivation for the current Encyclopaedia containing, and indeed starting just with, tensor networks in core basic courses.

Notational Remark 2 A separate benefit of u.d.d. is that it affords a more elegant tensor networks notation. Along the lines of Penrose's vertical variant [14]. Due to this, the current Encyclopaedia [27] has so far presented even Vector Algebra [29] Vector Algebra [30] and Linear Algebra [31] – subjects arguably simpler than Cartesian tensors – in u.d. notation. Without even displaying any plain-index tensor networks counterpart. In the current Review, however, we do for the first time present plain-index tensor networks, and furthermore compare them with the vertical variant.

[5] Plane, cylindrical and spherical polar coordinates are very useful throughout STEM and happen to be orthogonal-curvilinear. As such, students will often have already encountered orthogonally-curvilinear material prior to a first course in Cartesian tensors. Whether or not this material made active use of orthogonal-curvilinear features, or called them such, or abstracted to orthogonal-curvilinearity more generally.

[6] For this would be taught concurrently with Differential Geometry and General Relativity. And with advanced Special Relativity, or possibly after this.

1.3 Outline of contents

1.3.1 The main body

In Sec 3, we define rank for Cartesan tensors, in both plain and u.d.d. formulations. Let us refer to the case without as *plain tensors*. While using *total rank* to mean the sum of upstairs and downstairs ranks.

Cartesian tensor transformation laws are explained and tabulated in tensor networks form in Sec 2. Basic Tensor Algebra results are given and proven in tensor networks form in Sec 5. See [28, 32] for material at a comparable level to the current Review, the first on Cartesian tensors and the second on both these and tensor networks notation for them. Also see [19, 8, 7, 22, 12, 13, 15, 17, 2] for literature on further kinds of tensor, along the previous SSec's lines. Which the current Encyclopaedia shall be recasting in tensor networks notation over the next three years.
Both of the above Sections are mounted upon parts of the following arenaic framework.

In Sec 5, we consider A) standard arenas of tensors for fixed dimension and rank. Where an *arena* is the Mathematical space formed by the totality of some particular kind of Mathematical object, as motivated in SSSec 3.3.3. We furthermore entertain B) partly new Combinatorial tracking arenas (CTA), motivated in SSSec 4.4. Concentrating on all specific examples for total rank ≤ 4 . Rank gives Pascal's triangle and 2-part partition mathematics. Various meaningful quotients, and basic Graph-Theoretic compositions for rank-and-dimension, are also contemplated. We finally introduce new discrete-base continuous-attached-object general bundles that jointly model A) and B).

Index-exchange symmetries supported by rank- ≤ 4 tensors are listed in Sec 6. These symmetries are classified by partitions. We also cover the decomposition supported by these symmetries and the trace-tracefree decomposition. Both of these carry Repreresentation-Theoretic significance. We also include many more symmetry-enabled corollaries of the quotient theorem for tensors than is usual.

On the one hand, Cartesian tensors are taken to be plain in most previous literature. On the other hand, tensor networks are both more usually, and more simply and elegantly, presented using u.d.d. Indeed, the Penrose-type [14] vertical tensor networks that the body of the current Review uses, are a subcase of u.d.d. formulation. The above contrast motivates Sec 8 providing plain tensor network diagrams for the main body's topics. We include comparison for plain and u.d.d. tensor network notations.

Sec 7 next lays out tensor fields, their arenas, and the simple Cartesian tensor Calculus that these support. Finally classification of isotropic Cartesian tensors with rank ≤ 4 is provided in Sec **??** with supporting tensor network diagrams. For plain and then u.d.d. versions.

1.3.2 The Appendices

Some Order-Theoretic and Graph-Theoretic structure of Pascal's triangle and its quotient under reflection [71, 73] are exhibited in Appendix A. Both Sec 3's tensor rank arenas and Sec 6's tensor symmetries are substantially underpinned by partitions [69, 48, 54] and their arenas [78]. Motivating Appendix B to provide a unified account of these. With their own highly-developed Drawing and Visualization formalism: *Young diagrams* [46, 51, 53, 73]. And with a cut-off in its specific small examples at the partitions of the number 4 . So as to match the current Article's modelling limitation of covering only up to plain- or total-rank 4 tensors.

1.3.3 Conceptual underpinning for the current Review's range of Mathematics

A useful organizational observation at this point is as follows.

On the one hand, tensor networks provide a Combinatorial notation for our Cartesian tensor objects. And Young diagrams (partition Combinatorial notation) for their symmetries. With some support from the tensor product and direct-sum operations (Linear Algebra). As enabled by symmetry groups supporting representations via a well-known map from Group Theory to Linear Algebra: Representation Theory [101, 55].

But arenas seldom belong to the same 'area of Mathematics' as their objects. So it is rather likely that arena-level Combinatorial structure, conceptualization, and diagrammatic notation will be different for that for their objects.

Indeed, tensors' arenization for fixed dimension-and-rank is handled by Linear Algebra's vector spaces, duals and tensor products. And then Differential Geometry's fibre bundles for tensor fields. Order Theory sitting upon Graph Theory at the level of CTAs, revealing Pascal's triangle and further partition Mathematics alongside various basic graph-and-order compositions. And general bundles when tensors over nontrivial CTAs are in play. The symmetries' partitions form the standard Young lattice arena, to a rank arena picking out a simple conjugate pair of subarenas of this. Finally isotropy leans heavily on the CTA, placing interesting heat maps and subarena specifications thereupon.

And so, in an Encyclopaedia covering one 'easy end' of Modern Applied Topology, that the titular 'tensor and other multilinear array' objects are systematically handled using tensor networks does not cover the arenaic half of the subject. Which can however fortunately be handled by a fairly complete orchestra of basic Combinatorics [73]. Alongside several definitions concerning bundles (no details or proofs involving bundles required). Leaving the arenaic extension of the theory of Cartesian Tensors accessible to third-year undergraduates. And so here we are!

2 The Cartesian tensor transformation laws

2.1 $O(D)$ alias Cartesian tensors

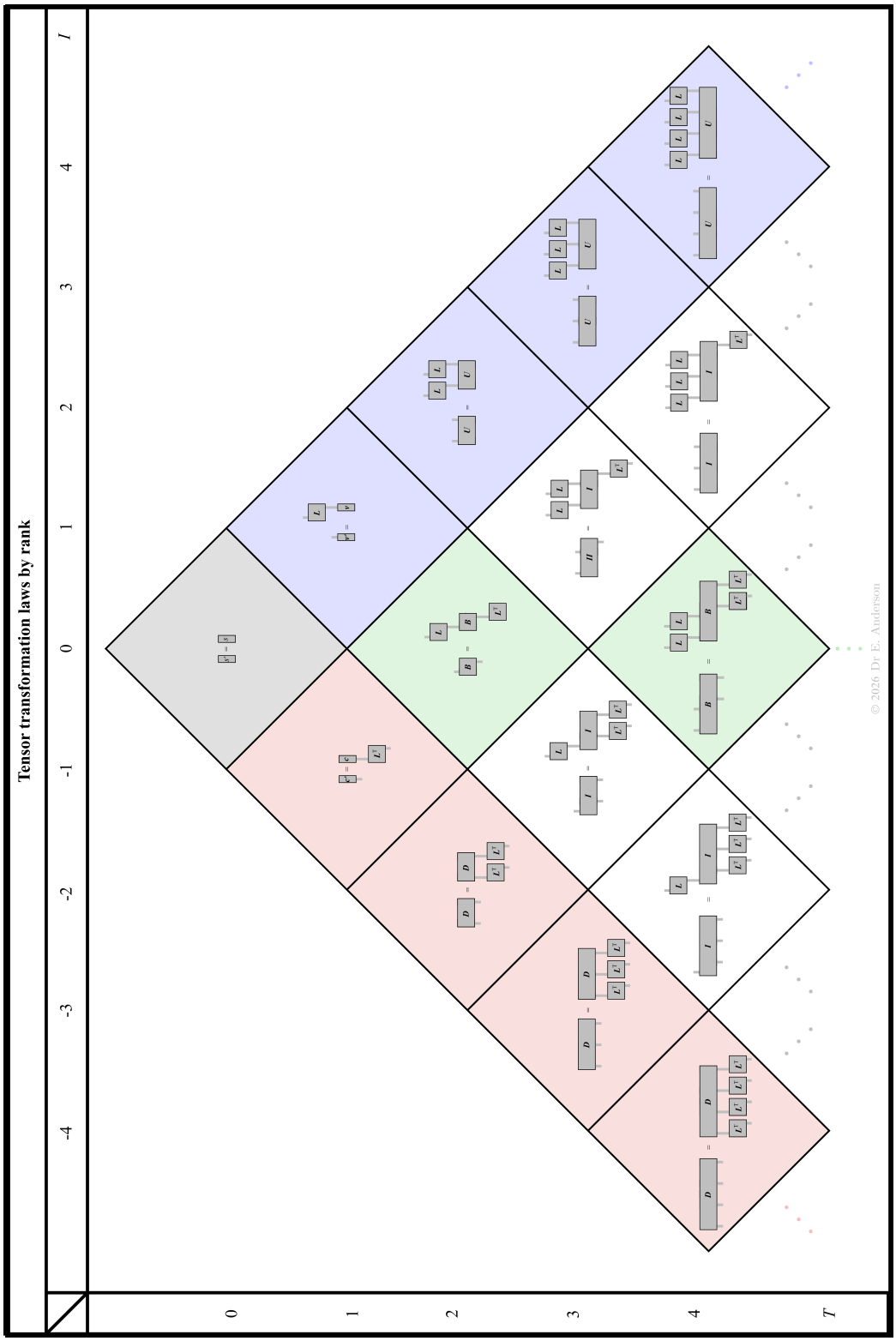


Figure 7:

Remark 1 Let us first use the transformation group $O(D)$, comprising both reflections and rotations. We also refer to the corresponding Cartesian tensors as $O(D)$-*tensors*.

Remark 2 Fig 7 presents the u.d.d. tensor transformation laws from $T = 0$ to 4 inclusive. Making u.d.d. In a Penrose-type vertical variant tensor networks notation. And displayed upon the arena of tensor transformation laws.

Remark 3 See also Fig 8 for a schematic depiction of the general-rank tensor transformation law, for subsequent use.

Remark 4 See instead Appendix 8.1 for the smaller repertoire of plain index tensor transformation laws. Using a type of plain tensor network. Upon the simpler arena of plain tensor ranks.

Remark 5 We explain both of these arenas in the next Section, via their being isomorphic to the corresponding arenas of tensor ranks.

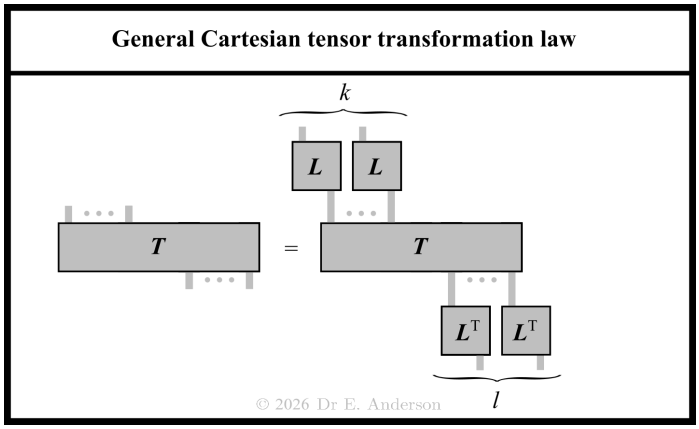


Figure 8:

2.2 $SO(D)$-tensors alias Cartesian pseudotensors

Structure 1 Let us next drop the reflections, leaving us with the special-orthogonal group $SO(D)$. The corresponding tensors are Cartesian $SO(D)$-*tensors*. Alias Cartesian *pseudotensors*, from the point of view of still considering reflections while permitting the tensor transformation law to generalize as follows. The *sign of the transformation*

$$\sigma(\boldsymbol{L})$$

is to feature as an extra factor in the transformation law. This sign is $+1$ for rotations, -1 for reflections, and multiplicative:

$$\sigma\left(\boldsymbol{L}_1 \boldsymbol{L}_2\right) \quad = \quad \sigma\left(\boldsymbol{L}_1\right) \sigma\left(\boldsymbol{L}_2\right) \ . \tag{8}$$

So it is $+1$ for the product of whatever number of rotations (for all that this is clearly still a rotation). $+1$ for the product of 2 reflections, which the Reader is invited to check that this is indeed also a rotation. And thus $\sigma = +1$ for the product of an even number of reflections, versus $\sigma = -1$ for the product of an odd number of reflections.

And finally and most pertinently, $\sigma = -1$ for the product of a reflection and a rotation. Which comprises the half of $O(D)$ that is not in $SO(D)$. Forming a second disconnected 'copy' of $SO(D)$. Albeit not now containing the identity element, and thus not constituting a subgroup. The Reader is also encouraged to check that this 'copy' is indeed instead a group coset [105].

Remark 1 Since this is the same modification in each case, let us not re-display this for all the smallest ranks. Rather illustrating it just for the pseudoscalar, pseudovector, and arbitrary rank (q, p)-pseudotensor in Fig 9. The $SO(D)$ and 'pseudo' points of view are compatible since for rotations the sign is always $+1$ and thus need not be included as an extra factor...

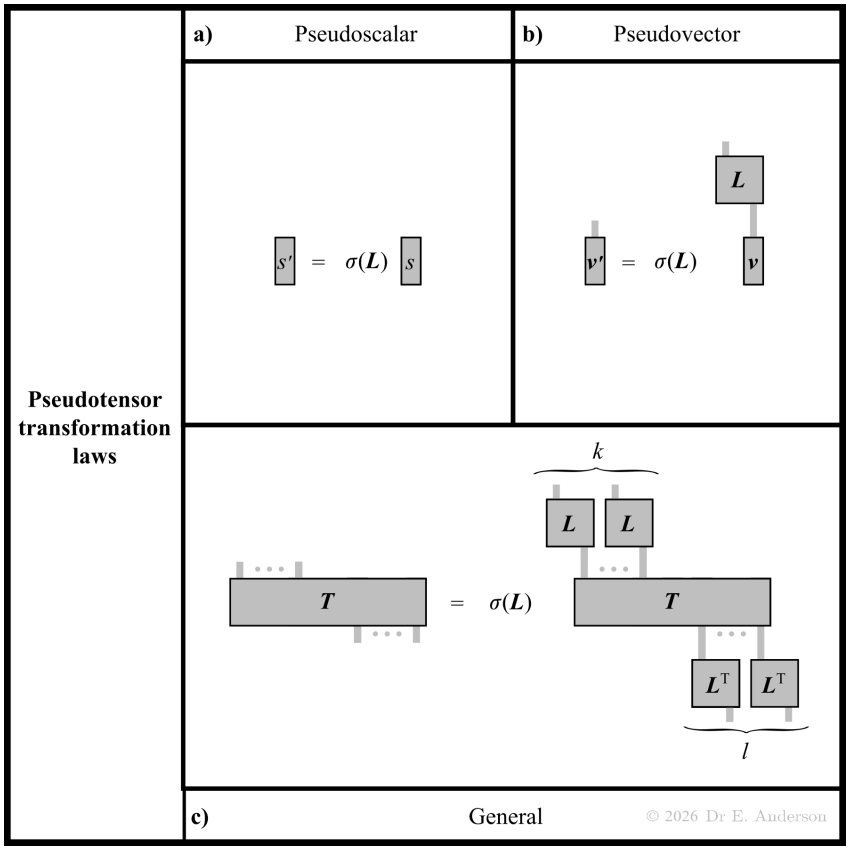


Figure 9:

3 Tensor arenas

3.1 Arenas of tensors for fixed dimension and rank

Structure 1 The plain Cartesian tensors' fixed dimension and rank arenas are as follows.

$$\mathfrak{Tensor}(\, D,\, P) \;:=\; \mathfrak{Tensor}\left(\, \mathbb{R}^D,\, P\,\right) \;:=\; \prod_{i\,=\,1}^{P} \mathbb{R}^D \;\;\cong\;\; \mathbb{R}^{D\,P}\,. \tag{9}$$

There are various ways in which the last simplification above is undesirable. One is that it is still $O(D)$ which acts. As opposed to the rapidly larger $O(P\,D)$ that would a priori be more natural on $\mathbb{R}^{P\,D}$.

Another is that a large and widely applicable generalization consists of replacing $\mathbb{R}^D$ with some other vector space

$$\mathfrak{V}\,.$$

Then the first two blockwise expressions continue to hold. While the remaining expressions may not pick out particularly meaningful objects.

Structure 2 For mixed-rank tensors, we strictly require

$$\begin{aligned}\mathfrak{Tensor}\left(\, \mathbb{R}^D,\, q,\, p\,\right) \;&:=\; \mathfrak{Tensor}(\, D,\, q,\, p\,) \;:=\; \left(\, \prod_{i\,=\,1}^{q} \mathbb{R}^D \right) \prod_{i\,=\,1}^{p} \mathbb{R}^{D\,*} \\ &=\;\; \mathbb{R}^{D\,q}\, \left(\mathbb{R}^{D\,*}\right)^p \;\;\cong\;\; \mathbb{R}^{D\,T}\,. \end{aligned} \tag{10}$$

Where the $*$ stands for dual space. Which enters since covectors are dual vectors.

There is less shelf life here to mixing the starred and unstarred copies of $\mathbb{R}^D$ in the last line. Where also

$$T \;:=\; q \;+\; p \tag{11}$$

is the *total rank* of our tensor. For all that for D finite, these are indeed isomorphic as vector spaces. Tenuous because each plays a different Geometrical role. For starters, the vectors belonging to the one and the covectors belonging to the other transform oppositely under coordinate transformations.

Structure 3 On differentiable manifolds,

$$\mathfrak{V}^D \;=\; \mathfrak{T}_{\mathrm{m}}\left(\mathfrak{m}^D\right)\,. \tag{12}$$

I.e. the tangent space at the point m in question. With dual space

$$\mathfrak{V}^{D\,*} \;=\; \mathfrak{T}^*_{\mathrm{m}}\left(\mathfrak{m}^D\right) \;\;: \tag{13}$$

the corresponding cotangent space.

Then

$$\mathfrak{Tensor}\left(\, \mathfrak{m}^D,\, q,\, p\,\right) \;:=\; \left(\, \prod_{i\,=\,1}^{q} \mathfrak{T}_{\mathrm{m}}\left(\mathfrak{m}^D\right) \right) \prod_{i\,=\,1}^{p} \mathfrak{T}^*_{\mathrm{m}}\left(\mathfrak{m}^D\right) \tag{14}$$

does not further simplify.

In the sense that

$$\mathfrak{T}_{\mathrm{m}}\left(\mathfrak{m}^D\right) \;\;\times\;\; \mathfrak{T}_{\mathrm{m}}\left(\mathfrak{m}^D\right)$$

is specifically the square of the tangent space rather than some further unified object of dimension $2\,D$.

3.2 Combinatorial tracking arenas (CTAs)

3.2.1 Introduction

Structure 1 In the current Article, the role of *Combinatorial tracking arena* (*CTA*) is played by the totality of one or both of dimensions and tensor ranks. Or of some meaningful quotient thereof. That these are Combinatorial is preliminarily-enabled by dimension and rank taking discrete values. And embodied by the resulting arenas exhibiting some patterns familiar from basic Combinatorics.

3.2.2 The arena of dimensions

Remark 1 The notion of dimension in the current Article is a conflation of the Linear- Algebraic and Flat-Geometric notions. The second of these notions extends to the notion of manifold dimension via the chart construct.

Structure 1 For continua, the *arena of dimensions* is

$$\mathfrak{Dim}_1(d) \ = \ \mathbb{N} \ . \tag{15}$$

If the discrete point is included – the connected case of dimension 0 – then the arena of dimensions is as follows.

$$\mathfrak{Dim}(d) \ := \ \mathfrak{Dim}_0(d) \ = \ \mathbb{N} \ . \tag{16}$$

Finally, if the empty set is included as well, then a schematic appending scheme is afforded by

$$\mathfrak{Dim}_{-1}(d) \ = \ \mathbb{N}_{-1} \ . \tag{17}$$

Where $\mathbb{N}_{-1}$ starts at -1 just like $\mathbb{N}_0$ starts at 0 . But the -1 value – from depicting the arena with the same spacing from 0 as 1 has – is more truly the following. A more loosely attached appendage of the empty set in its aspect as the *undimension*.

Of course, all of these arenas are isomorphic as sets to $\mathbb{N}$. By shifting the initial number up or down by 1 .

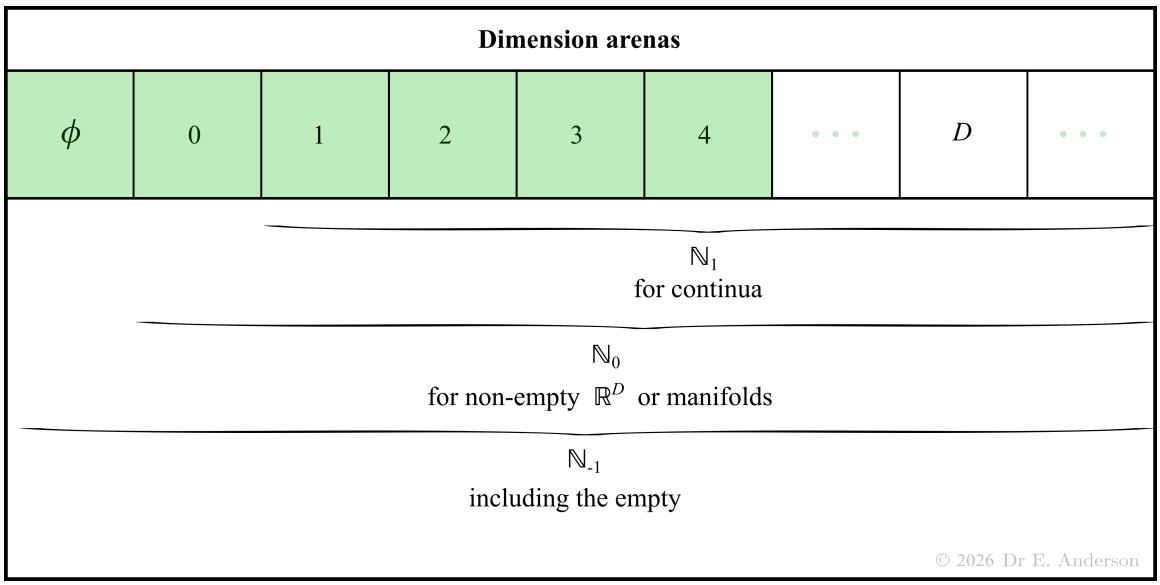


Figure 10:

3.2.3 Plain-rank arena

Structure 1 The *arena of plain tensor ranks* – is just

$$\mathfrak{Rank} \;=\; \mathbb{N}_0$$

(Fig 11). Starting with the rank-0 scalar and the rank-1 vector.

Arena of plain tensor ranks $\mathfrak{Rank}$

© 2026 Dr E. Anderson

0	1	2	3	4		P		$\mathbb{N}_0$
scalar s	vector	2-tensor	3-tensor	4-tensor	• • •	P-tensor	• • •	

Figure 11:

3.2.4 U.d.d. rank arena

Structure 3 The *arena of u.d.d. tensor ranks* is

$$\mathfrak{Rank}_u^d \;=\; {\mathbb{N}_0}^2$$

(Fig 12). With the vector now taken to be the object with 1 upstairs index. Versus a distinct notion of covector, the object with 1 downstairs index. The general case is a rank-(q, p) tensor. I.e. with p downstairs and q upstairs indices.[7]

Arena of upstairs-downstairs tensor ranks $\mathfrak{Rank}_u^d$

	0	1	2	3	4		q	$\mathbb{N}_0$
0	scalar s	vector	(2, 0)-tensor	(3, 0)-tensor	(4, 0)-tensor	• • •	(q, 0)-tensor	
1	covector	***B***	***I***	***I***	***I***		Imbalanced mixed-rank tensors: upstairs-dominant	
2	(0, 2)-tensor	***I***	***B***	***I***	***I***			
3	(0, 3)-tensor	***I***	***I***	***B***	***I***			
4	(0, 4)-tensor	***I***	***I***	***I***	***B***			
	⋮					⋱		
p	(0, p)-tensor	Imbalanced mixed-rank tensors: downstairs-dominant					Balanced mixed-rank tensors	
$\mathbb{N}_0$								

© 2026 Dr E. Anderson

Figure 12:

[7] This notation is consistent with the common practise of referring to totally-antisymmetric tensors with all-lowered indices as p-forms.

3.2.5 Adapted variables

Remark 1 Let us next introduce the following adapted variables [71, 73, 97]. The first is a promotion of a quantity we already introduced – the total rank –

$$T \;:=\; q \;+\; p \tag{18}$$

to now serve furthermore as an adapted variable. As partnered by the new (*signed*) *rank imbalance*

$$I \;:=\; q \;-\; p \;. \tag{19}$$

These adapted variables so happens to take the same form as the characteristic coordinates in each of the following basic and widely-encountered settings. The $2\,D$ flat-space scalar wave equation [99]. And the flat spacetime of Special Relativity [19]. In either case,

$$u_{\pm} \;=\; x \;\pm\; c\,t \;=\; x \;\pm\; t \;.$$

Where the last expression picks the units so that the wave-speed c is numerically 1 . For instance, in the Special Relativity case, picking the fundamental units sets the speed of light numerically to 1 . So we have unveiled that this characteristic structure is also realized at the level of tensor-rank CTAs...

Examples 2-4

$T \;=\; 2$ is minimum to possess a mixed tensor rank. This is balanced: rank $(\,1,\,1\,)$.

$T \;=\; 3$ is minimum to possess imbalanced mixed tensors. Here there are 2 types: rank $(\,2,\,1\,)$ and rank $(\,1,\,2\,)$.

$T \;=\; 4$ is minimum to exhibit both balanced and imbalanced mixed-rank tensors. The balanced case is of rank $(\,2,\,2\,)$. While the imbalanced cases are of rank $(\,3,\,1\,)$ and $(\,1,\,3\,)$.

Remark 2 Imbalanced ranks always come in pairs.

Remark 3 Rotating the arena clockwise through $\frac{\pi}{4}$ radians forms horizontal lines of constant T . Centred about the balanced-rank cases, scalars included at the summit. The purely upstairs and purely downstairs tensors then map into each other under reflection symmetry. Which also exchanges each imbalanced-rank pair.

Notational Remark 3 Fig 13 picks out the 2 pure lines and the balanced line. Using S. Sánchez' [71, 78] runnel notation for chain series in poset arenas. Alongside the remaining region of imbalanced mixed-tensor ranks: in 2 disconnected equal wedges. Our 3 privileged lines are concurrent at the scalar summit.

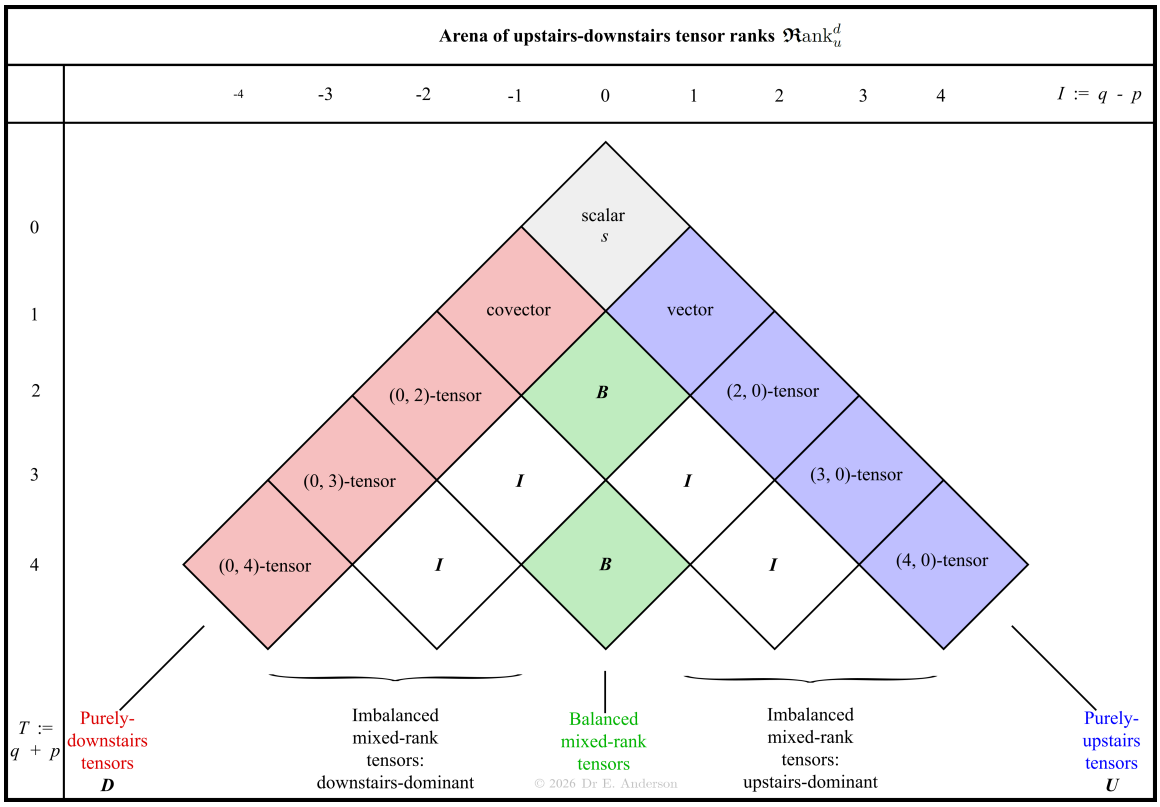


Figure 13:

The fully Order-Theoretic Visualization for the previous Figure is presented in Fig 14. This is a poset Hasse diagram [63] as applied to a Combinatorial arena, with runnels marked on it. Colour-coded in preparation for the quotienting 2 SSSecs along. With tangerine for the balanced rank $=$ *homoupdown* line. And lime-green and white for the imbalanced rank $=$ *heteroupdown* wedges. With white for the down-dominant representatives to lime-green for the up-dominant ones.

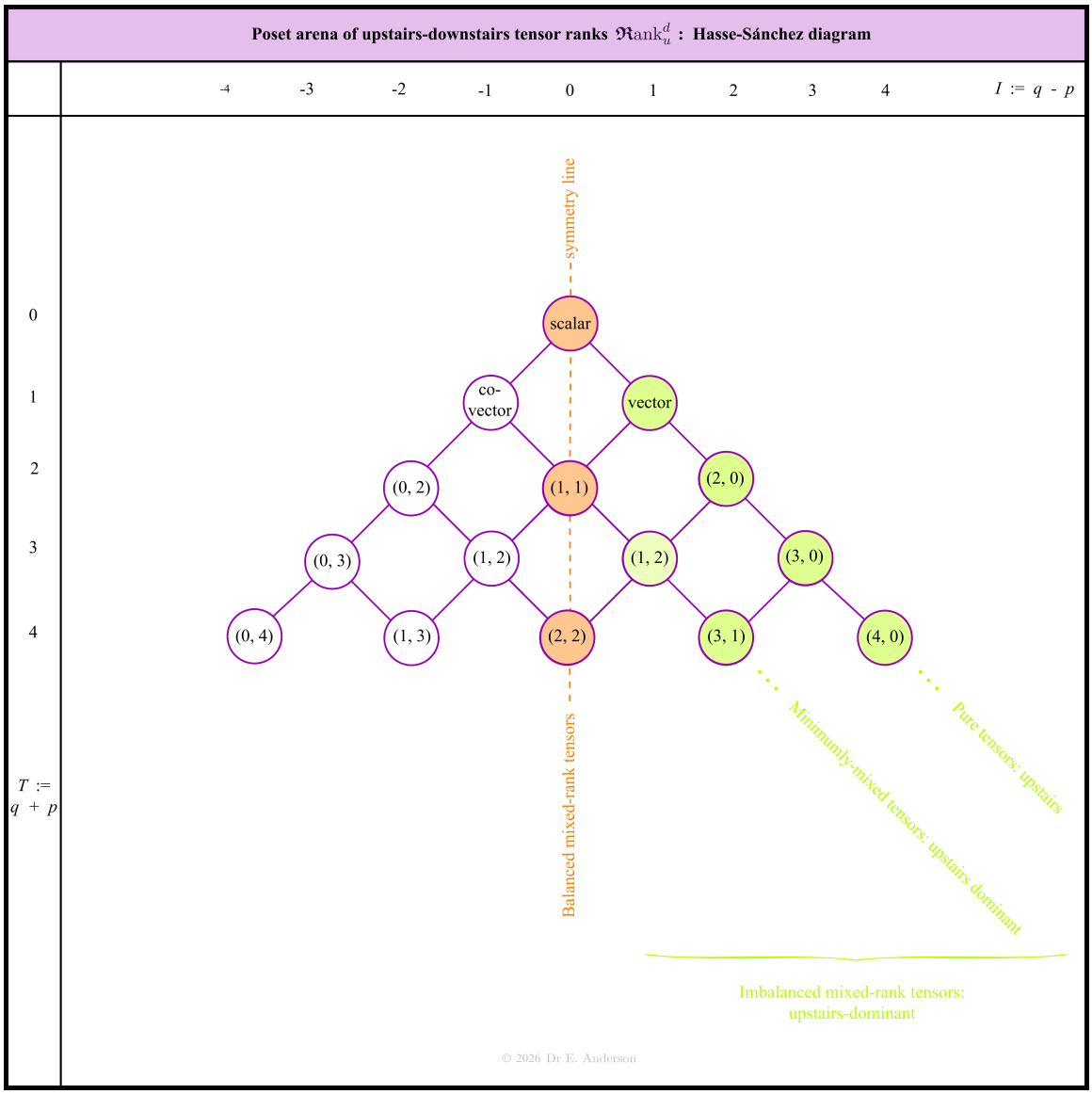


Figure 14:

3.2.6 0- and 1-D collapses

Remark 1 In 0- and 1-D , each higher-rank tensor is just a copy of the scalar. Thus identifying all of these copies,

$$\mathfrak{R}\mathrm{ank} \;=\; \mathrm{point} \;=\; \{0\} \;:$$

with the scalar rank as a natural label for it.

Remark 2 The undimension does not support any tensors at all. This could be accorded a single point, modelling the untensor arena interpretation of the empty set...

3.2.7 The u.d. distinguishable and yet meaningless quotient arena

Remark 1 The $C_{2\updownarrow}$ action becomes a reflection at the level of centred arenas. Leading to the quotient

$$\widetilde{\mathfrak{R}\mathrm{ank}_d^u} \;=\; \frac{\mathfrak{R}\mathrm{ank}_d^u}{C_{2\,\mathrm{ref}}} \tag{20}$$

corresponds to treating upstairs and downstairs as *distinguishable and yet meaningless labels* [73]. We exhibit this arena in Fig 15.

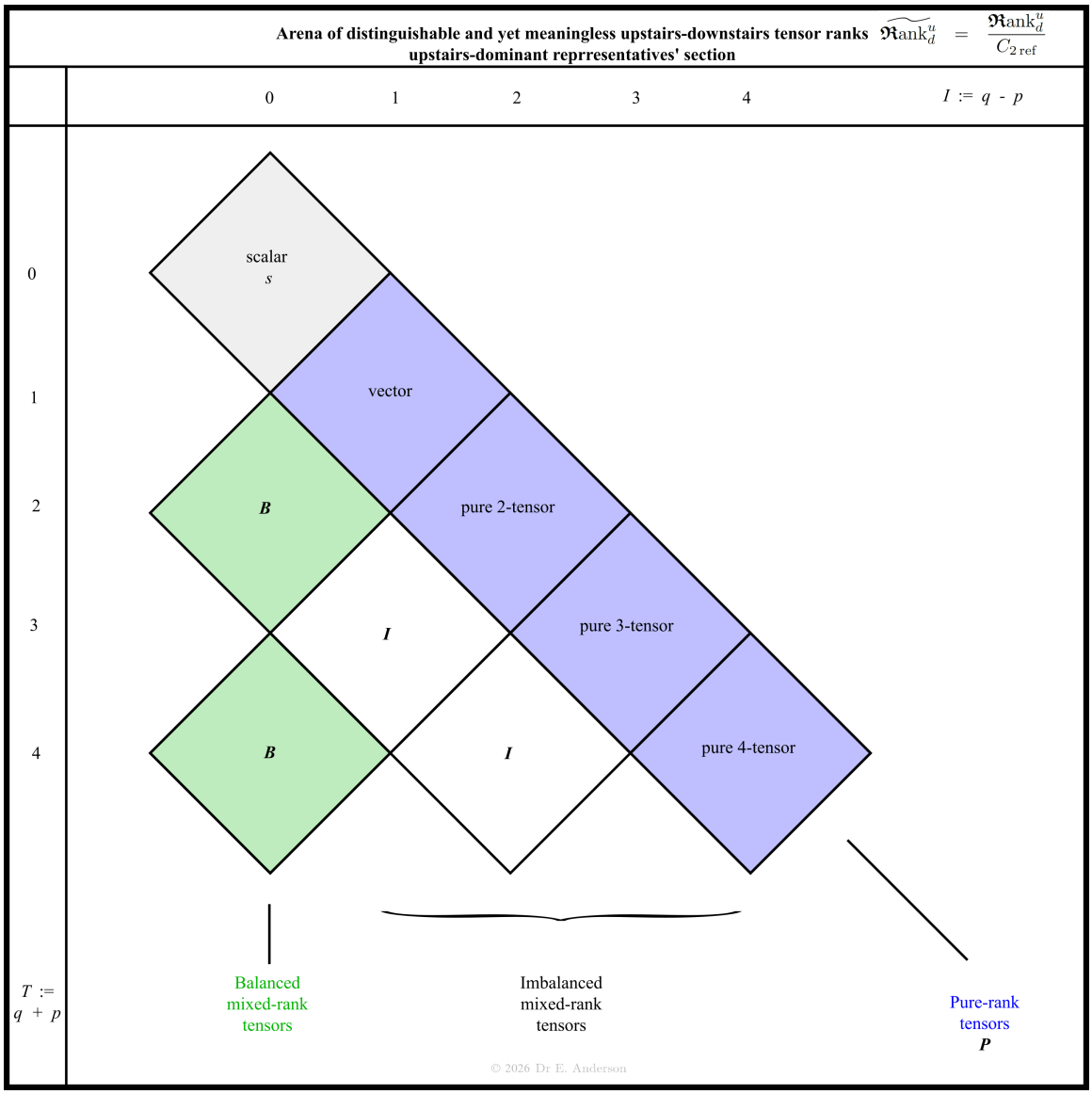


Figure 15:

2 privileged lines persist: pure and balanced. With now just the 1 connected wedge between these of imbalanced mixed-rank cases.

Motivation 1 Some Readers might quibble at this point that co- transforms with the frame and contra- oppositely. There is however also a notion of coframe. And now it is contra- which transforms with this to co- transforming oppositely. And in some contexts, which is the frame and which is the coframe is arbitrary...

Notational Remark 4 Exchange symmetries are usually denoted by $\leftrightarrow$. We however take licence to introduce $\updownarrow$ in this case, since the exchanged entities firstly are located upstairs and downstairs. Secondly, they are called upstairs and downstairs. And thirdly they are denoted by superscript u and subscript d on our rank arena symbol!

Remark 2 See finally Fig 16 for the corresponding fully Order-Theoretic Visualization.

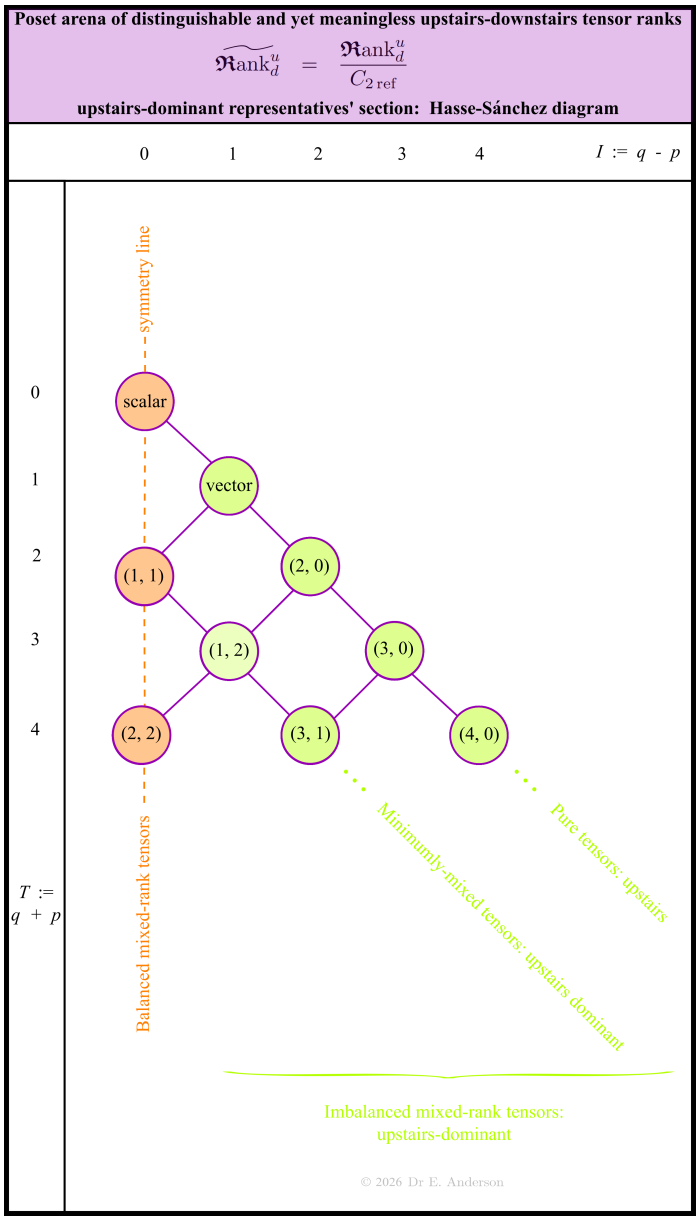


Figure 16:

3.2.8 Aside: fate of CTAs for manifolds

Remark 1 Here we have a problem in that 1 interpretation of bigger arena involves all manifolds in all dimensions. Easy classifications give us 0-, 1- and 2-D . But 3- and 4-D are wild and unknown. So this arena is too big for human civilization so far or in the foreseeable future...

Remark 2 A sop is to consider all the tensors on all the spheres. Or on all of the $\mathbb{T}^D$, $\mathbb{RP}^D$ or $\mathbb{CP}^K$ (with complex dimension K corresponding to real dimension $2\,D$). Each of these is analogous to the $\mathbb{R}^D$ case, in fact with exactly the same CTAs.

3.3 Order-Theoretic and Graph-Theoretic analysis of dimension and rank separately

3.3.1 The trivial cases

Remark 1 For dimension's $\mathbb{N}_0$ (give or take 1), we can assign the arrow of dimensional increase by 1.

While for plain rank's $\mathbb{N}_0$ specifically, we can assign the arrow of rank increase by 1. I.e. of appending 1 further index. Which Sec 5 shall further describe as tensoring in a vector. And Sec 6 as tensoring in a single Young box.

For dimensions, the cumulative arenas – $\leq P$ are chains, of length $P + 1$:

$$\mathfrak{Rank}[P] = \text{Chain}_{P+1} .$$

With underlying graphs P_{P+1}: the path of length $P + 1$. The full arenas are the semi-infinite chain and the corresponding semi-infinite path.

While for ranks, the cumulative arenas are chains of length $D + 1$. (Or D for just continua. Or $D + 2$ if including the undimension)!

Chains are Order-Theoretically trivial, in the sense of being total orders rather than partial orders (posets).

3.3.2 Full u.d.d. version



Figure 17:

Structure 1 We again assign the arrow of rank increase by 1. With now distinction between appending 1 upstairs index or 1 downstairs index.

This now produces a poset. See Fig 17 for the corresponding cumulative posets in its first column of subfigures. Whose edges are arrows encoding the add-index operation.

These posets are furthermore *semi-lattices* and *distributive* [73, 63, 67]. While if we prolong to infinite rank, a distributive lattice ensues. The meet operation is index-intersection, while the join operation is index-union. It is this join which fails, resulting in but a semi-lattice. Unless the poset is compensatingly semi-infinite. Which wipes out the lack of a unique lower bound in the cumulative approximations past just the point case.

Structure 2 Viewing $\mathfrak{Rank}_u^d$ furthermore as a poset, [60, 63], T furthermore serves as a *depth function*. The *level lines*[8]

$$T \; = \; \text{constant}$$

then supply cut-offs for our cumulative arenas.

Remark 1 For $T = 1$ to 4, a distinct double-irreducible (*DI*) poset is pinned under the actual poset. In each case, this is arrived at by defoliation: cutting off all leaves. This suffices for every case in the current Article (the so-called DI class P [74]). More generally one would need to serially-defoliate to eliminate all degree-1 vertices. And to pohomeomorph – carry out all licit removals of degree-2 vertices – so as to arrive at the DI. For the 'double' refers to [71] serial pohomeomorphing and serial defoliation in whichever order.

Remark 2 The corresponding underlying graphs are provided in the second column of subfigures. Starting with the point Pt, the 3-path P_3 and Buckle.

Remark 3 Getting to the DI now involves homeomorphs as well as defoliations, leading to a wider range of DI classes being exhibited. I.e. classes C and C_4 in addition to P and $T = 0$'s class D that says that the point is already a double-irreducible. Observe that this graph-level structural analysis works out differently from its poset-level counterpart. This is since the poset axioms forbid triangles. For our particular small arenas, every homeomorph turns out not to be a licit pohomoeomporph for this reason. It may also be useful to point out at this stage that graph homeomorphs are very standard [73, 62]. In contrast, poset homeomorphs – *pohomeomorphs*! – are a relatively new and unexplored analogue [71].

[8] Compare level curves, level surfaces and level sets [108].

3.3.3 Motivational aside on arenas

Motivation 1 The bigger picture is that given any type of Mathematical object, asking what the totality formed by objects of that type is a meaningful question. This is the *arenization question* of Modern Applied Topology [92, 94, 73, 97, 32].

'Early birds' – topics in which arenas were already being investigated before the 2010s advent of Modern Applied Topology – include the following. Physics and Dynamics' *configuration space* [81] and *phase space* [86]. Linear Algebra's *vector spaces*, *eigenspectra* and *eigenspaces* [80]. Functional Analysis' *function spaces* [91]. Matrix arenas that so happen to be (chunks of) Lie groups [82]. Or polygons forming complex-Projective arenas, be it in a Dynamical [84, 95], Statistical [90] or purely Flat-Geometric [97] context.

Motivation 2 Since Theoretical Computer Science has become the pre-eminent field of study of our times, let us double the previous paragraph with examples of arenas from this field. Arenas of trees [73], especially binary and/or rooted, play a key role in the traditional Computer Science of search-and sort [85]. The arena of doubly-stochastic matrices is the convex hull of the arena of permutation matrices [83, 93]. Rather analogously, the arena of Quantum mixed states is a convex set on the complex-Projective space of q-n-its [93]. A sizeable part of 'Big Data' involves ascribing Topological properties to large data sets. [94]'s model examples are arenas of barcodes. Some major aspects of Machine Learning can be summarized as (Multi-)Linear Algebra leading to the solution of large parameter-number variational problems. Consequently, (Multi-)Linear Algebra arenas play a role here, as do arenas of varied curves that are being extremized. See e.g. [24, 26] for tensor networks in the context of Machine Learning.

Motivation 3 The above allusion to Topology follows from how arenas are not just sets, but rather carry further levels of Mathematical structure. Rendering in particular what natural Topologies are carried by arenas to always be a pertinent question. More occasionally, arenas furthermore carry Differential-Geometric structure [90], or Order-Theoretic structure [73, 96], among other possibilities.

Motivation 4 On the one hand, arenas very often belong to a 'different branch of' Mathematics to the original objects... On the other hand, firstly many basic Combinatorial objects' arenas turn out to themselves be Combinatorial. Well-known ones include power sets – the totality of subsets of a finite set – and the Young lattice of partitions ([78] and Appendix B.2.2). Another simple case featuring in the current Article is reappraising Pascal's triangle as the arena of combinations. Secondly, many finite Linear Algebra arenas turn out to be Combinatorial [96, 32, 27]. In particular, this accounts for the current Article on Cartesian tensors containing various incidences of orders, graphs, combinations and partitions, all of which can be viewed as basic types of Combinatorics. More generally, it accounts for us being in the middle of a large-scale sweep to arenize all of basic Combinatorics [73, 79] and (Multi-)Linear Algebra [96, 32, 27] .

3.3.4 A first two Mathematical analogues of u.d.d. tensor rank arenas

Remark 1 The above families of posets and graphs are familiar from elsewhere in basic Mathematics. For instance, as arenas corresponding to Pascal's triangle and its quotient [73]. Or in the arena of eigenvalue signs for nondegenerate square matrices [96].

3.3.5 Corresponding quotient arenas

Remark 1 We now follow through with the above-mentioned quotient $C_{2\updownarrow}$ at the level of arenas. This is slightly over half the size of the unquotiented version. For rank-(p, p) tensors constitute fixed points: balanced alias homoupdown tensor ranks. The cumulative distributive semi-lattices in the third column of our Figure ensue. Their underlying graphs are provided in the fourth column. Now starting with the point Pt , the 2-path P , Claw alias 3-star S_3 , and the A-graph .

Remark 2 This quotient has a meaningful analogue both for Pascal's triangle: store only each row's distinct entries. And for matrix eigenvalue signs: regard $+$ and $-$ as distinguishable but meaningless. Corresponding for instance to the $+ + + -$ and $- - - +$ formulations of the Minkowski spacetime of Special Relativity being taken to only differ by convention...

3.3.6 Analogy 3: arena of partitions with ≤ 2 parts

Structure P More primarily, the above applications all concern *partitions with at most* 2 *parts* [73]. With partition conjugation playing the role of quotienting operation where applicable. See Appendix *.* for more about the corresponding arenas, including subsequent cumulative cases.

Modelling Assumption 1 For us for now, cutting off at $T = 4$ suffices, since we are not going further than the elasticity tensor or the Electromagnetic hexadecapole this year. Nor any further than the Riemann and Weyl curvature tensors [17, 103, 18] over the subsequent two years...

Remark 1 Basic counts for these follow by substituting T for N in Appendix B.1.4.

Remark 2 Our arena-level reflection symmetry can be interpreted as a vestige of the partition conjugation operation (see Appendix B.1.3).

Remark 3 Also see Appendix B.2.5 for $\mathfrak{Rank}$'s relation to $\mathfrak{Combi}$ and to various subarenas of partitions.

3.4 Dimension-and-rank arenas, including Order- and Graph- Theoretic analysis

3.4.1 Full plain version

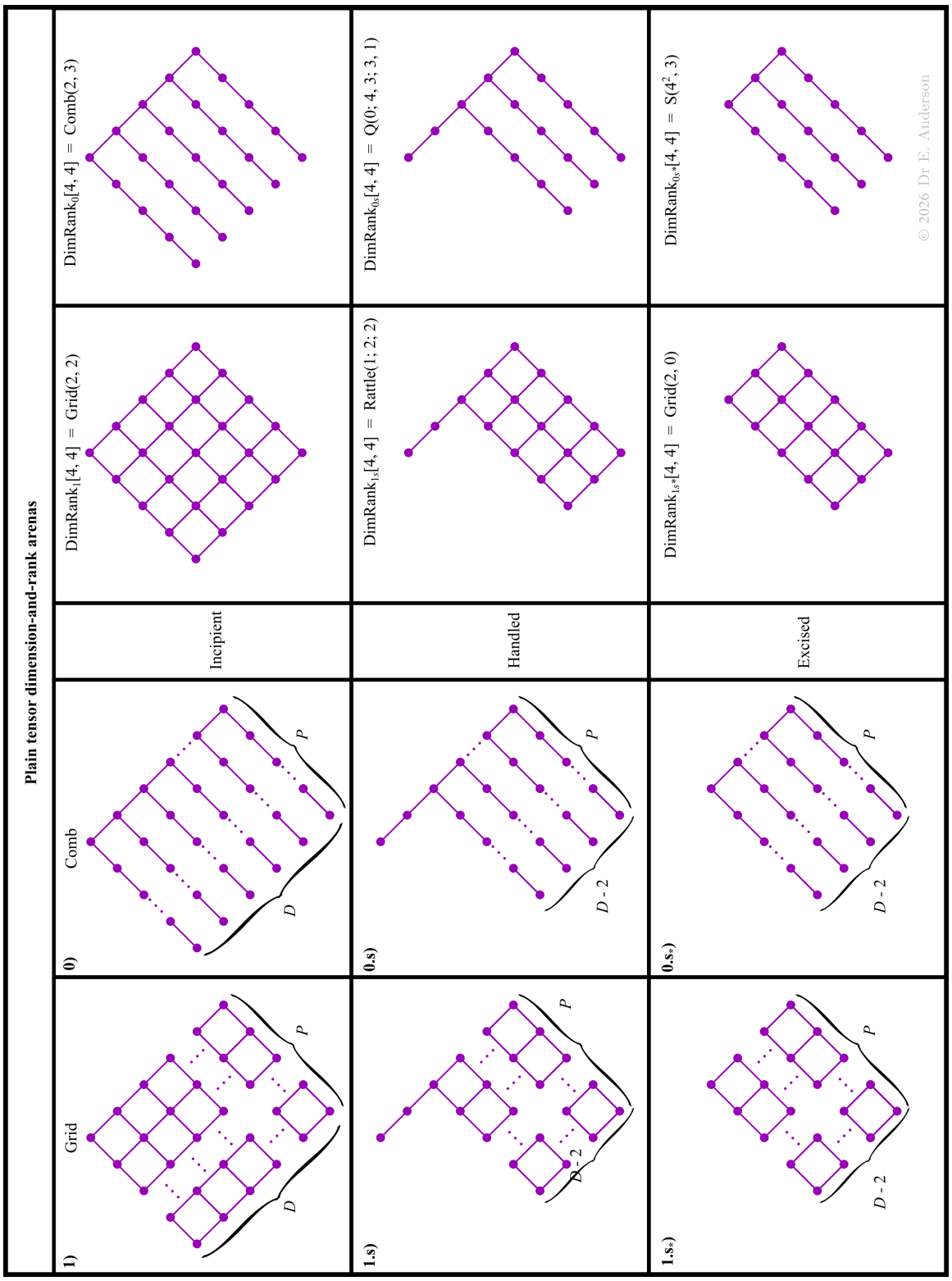


Figure 18:

Model 1 (Product) Suppose that the joint CTA is modelled by according to the arrows add-index

and extend dimension by 1 . Then we arrive at

$$\mathfrak{Dim}\mathfrak{Rank} \quad = \quad \mathbb{N}_0{}^2 \ .$$

Whose cumulative approximands are the diagonal-grid posets (Fig 18.1). Which are furthermore distributive lattices. And whose underlying graphs are the grid graphs,

$$\mathfrak{Dim}\mathfrak{Rank}[\, D,\, P\,] \quad = \quad \mathrm{Grid}_{D\,+\,1,\,P\,+\,1} \quad = \quad \mathrm{Grid}(\, D \ - \ 2,\, P \ - \ 2\,) \ . \tag{21}$$

Where the last parameter is *excess box size*, by which convention 2×2 boxes is the minimum grid,

$$\mathrm{Grid} \quad = \quad \mathrm{Square} \ . \tag{22}$$

Model 0 (Comb of paths) Here the same arrow types enter, but with the add-dimension arrow applied in a sparser generative manner: to scalars only. This returns the comb-of-paths posets (Subfig 0), which are furthermore at-most binary rooted trees. Whose cumulative approximands are

$$\mathfrak{Dim}\mathfrak{Rank}_0[\, D,\, P\,] \quad = \quad \mathrm{Comb}(\, D \ - \ 2,\, P\,) \ . \tag{23}$$

With excessive-variable calibration that there are to be ≥ 3 rows of teeth to have more than just a bent path. And where $P \ - \ 1$ is the *excess tooth length*.

Within the current Article's $(\, D,\, P\,) \ \leq \ (\, 4,\, 4\,)$ remit, these are confined to the following types. Paths, starlike, quadrupeds and scorpions. I.e. homeomorphs of P , $\mathrm{S} \ = \ \mathrm{Claw}$, P-of-Claws and P_3-of-Claws [73].

Model 0.s Here the 0-D and 1-D tensors are separately conflated. Giving a leading P handle, preceding the equal-length teeth of the comb (Subfig 0.c). It is somewhat less wasteful to introduce the unscalars at this point, by passing to a P_3 handle. As opposed to uniformly extending model 0 to include a priori distinct untensors for each rank... Here the cumulative approximands are

$$\mathfrak{Dim}\mathfrak{Rank}_0[\, D,\, P\,] \quad = \quad \mathrm{HandledComb}(\, H,\, D \ - \ 4,\, P\,) \ . \tag{24}$$

With this modelling, the current Article's remit now only gets as far as some homeomorphs of $\mathrm{Quadruped}$.

Model 1.s) Now the grid picks up a P (or P_3) handle. Thus forming the *rattle* family of Subfig 1.c). Whose cumulative approximands are

$$\mathfrak{Dim}\mathfrak{Rank}_{1.\mathrm{s}}[\, D,\, P\,] \quad = \quad \mathrm{Rattle}(\, H,\, D \ - \ 4,\, P \ - \ 2\,) \ . \tag{25}$$

Where the *excess handle length* parameter is 1 for P as handle. And 2 for P_3 as handle.

Model 0.s$_*$) Now we excise the handled comb's handle. This of course sends us back to just a comb, albeit with shifted tooth-number, as follows.

$$\mathfrak{Dim}\mathfrak{Rank}_{0.\mathrm{s}_*}[\, D,\, P\,] \quad = \quad \mathrm{Comb}(\, D \ - \ 4,\, P\,) \ . \tag{26}$$

See Subfig 0.s$_*$). With this modelling, the current Article's remit now only takes us as far as some starlike trees.

Model 1.s$_*$) Next we matchingly excise the rattle's handle. Sending us of course back to just a grid, albeit with modified parameter, as follows (Subfig 1.s$_*$).

$$\mathfrak{Dim}\mathfrak{Rank}_{1.\mathrm{c}_*}[\, D,\, P\,] \quad = \quad \mathrm{Grid}(\, D \ - \ 4,\, P \ - \ 2\,) \ . \tag{27}$$

Using this modelling, the current Article's remit again only takes us as far as some starlike trees.

Remark 1 The current Article's top grid, comb, handled comb, rattle, excised comb and excised grid are presented in the right-hand-side windowpane of subfigures.

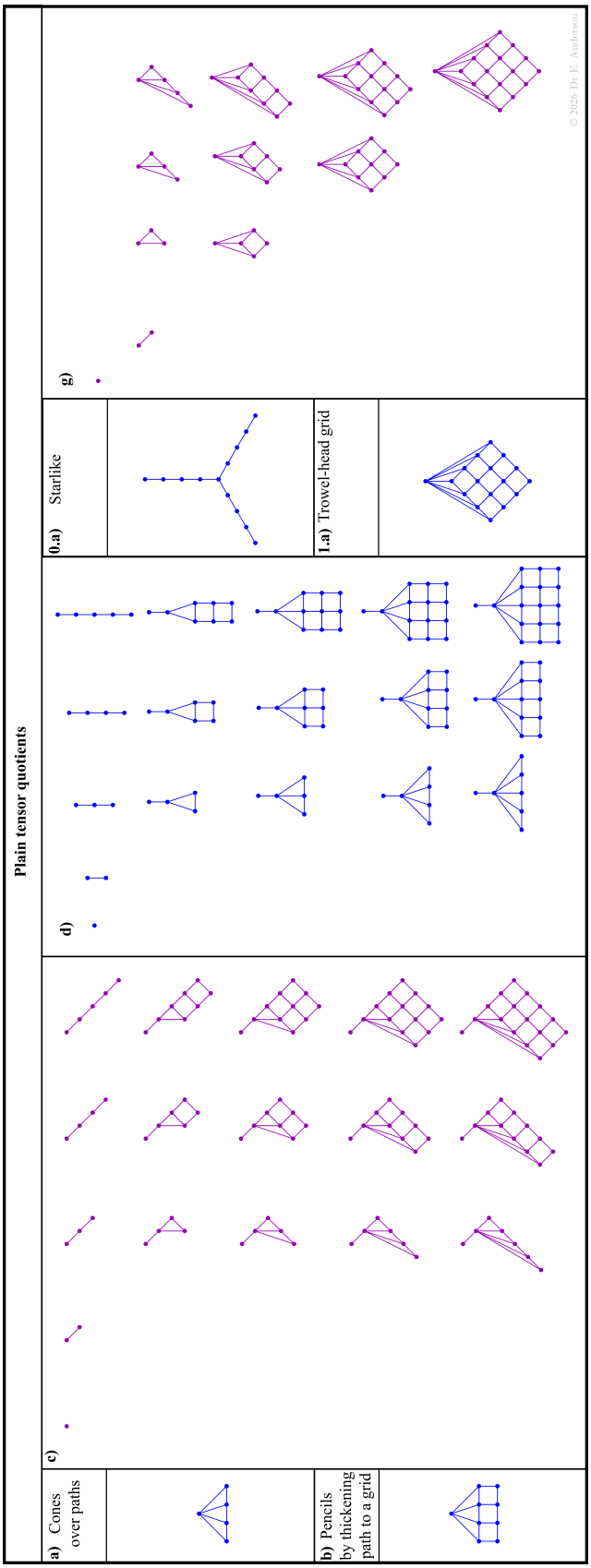

Plain tensor quotients
a) Cones over paths
b) Pencils by thickening path to a grid
c)
d)
0.a) Starlike
1.a) Trowel-head grid
g)
© 2026 Dr. E. Anderson

Figure 19:

3.4.2 Plain quotients

Model 1.q (Fans and Pencils) Suppose that the 1-D tensor ranks are identified in the grid model. And that the arrow from 1-D to 2-D P-tensors is preserved. Then 1-D 's identified point is linked to every 2-D rank. If no further dimensions are in play, then this returns the Fan subcase of cone in the Graph-Theoretic sense (19.a). While if subsequent dimensions are included, a direct-product pencil shaft grows away from the fan (Subfig b).

The cumulative approximands here are

$$\mathfrak{Dim}\mathfrak{Rank}^{d}_{u} \;\; 1.\mathrm{q}[\, D,\, P\,] \;\; = \;\; \mathrm{Pencil}[\, D \;-\; 2,\, P\,] \; . \tag{28}$$

Where the first parameter is the *excess pencil shaft length*.

Sticking to including 0-D but not the untensor, the current Article's cumulative approximands are tabulated in Subfig c) as orders. Yielding the underlying graphs in Subfig d).

Exercise 2 Work out their DI irreducibles and DI classes.

Remark 1 If the unobject is furthermore included, then our pencil is sharp: exposed lead P . While if instead we restrict attention to continua, our pencil is blunt: no exposed lead at all.

Structure 1 (additivity quotient) Let us also entertain quotienting down to collections of tensors within each of which addition is supported. This pinches all dimensions' scalars in the CTA to a single point.

Model 0.a (uniform starlike graphs) If this is applied to the comb model, a uniform starlike graph ensues (Subfig 0.a). With the scalars as the star-point nexus. The cumulative approximands here are

$$\mathfrak{Dim}\mathfrak{Rank}^{d}_{u} \;\; 1.\mathrm{a}[\, D,\, I\,] \;\; = \;\; S_I(\, I \;-\; 2\,) \; . \tag{29}$$

I.e. the I-pointed stars whose radiances are of excess length $I \;-\; 2$.

Model 1.a (Trowel-head grids [74]) But if applied to the grid model, Subfig f) ensues. These trowel-head grids are just parametrized in the manner of grids,

$$\mathrm{TrowelGrid}(\, D \;-\; 4,\, I \;-\; 3\,) \; . \tag{30}$$

Shifted by 2 in the first argument since $D \;=\; 0$ and 1 are no longer part of the grid. And by 1 in the second since $I \;=\; 0$'s scalars are not in the grid. The current Article's cumulative approximands are tabulated in Subfig g) as orders.

3.4.3 Full u.d.d. version

Model 1 (Pascal's prism) Now as per Fig 20.1),

$$\mathfrak{Dim}\mathfrak{Rank}^{d}_{u1} \;\; = \;\; \mathbb{N}_0 \;\times\; \mathrm{Pascal} \; . \tag{31}$$

With cumulative approximands

$$\mathfrak{Dim}\mathfrak{Rank}^{d}_{u1}[\, D,\, I\,] \;\; = \;\; [\, 0,\, D\,] \;\times\; \mathrm{Pascal}[I] \; . \tag{32}$$

Model 0 (Pascal's comb) While

$$\mathfrak{Dim}\mathfrak{Rank}^{d}_{u0} \;\; = \;\; \text{Pascal-Bunting} \; , \tag{33}$$

in the sense of Subfig 0). With cumulative approximands

$$\mathfrak{Dim}\mathfrak{Rank}^{d}_{u0}[\, D,\, I\,] \;\; = \;\; \mathrm{PascalBunting}[\, D \;-\; 3,\, I\,] \; . \tag{34}$$

Where the first parameter is the excess number of Pascal's triangles mounted onto the comb; 2 would just be a Bridge .

Model 0.s (Handled comb-of-Pascals) Here tensor-rank indistinguishability collapses the first 2 Pascal triangles down to points. Returning the handled comb of Pascals depicted in Subfig 0.s).

Model 1.s (Handled prism-of-Pascals) Here the same collapse returns Subfig 1.s).

Remark 1 Excising 0- and 1-D , one once again returns to the original structures – prism and comb – with values of D shifted down by 2 .

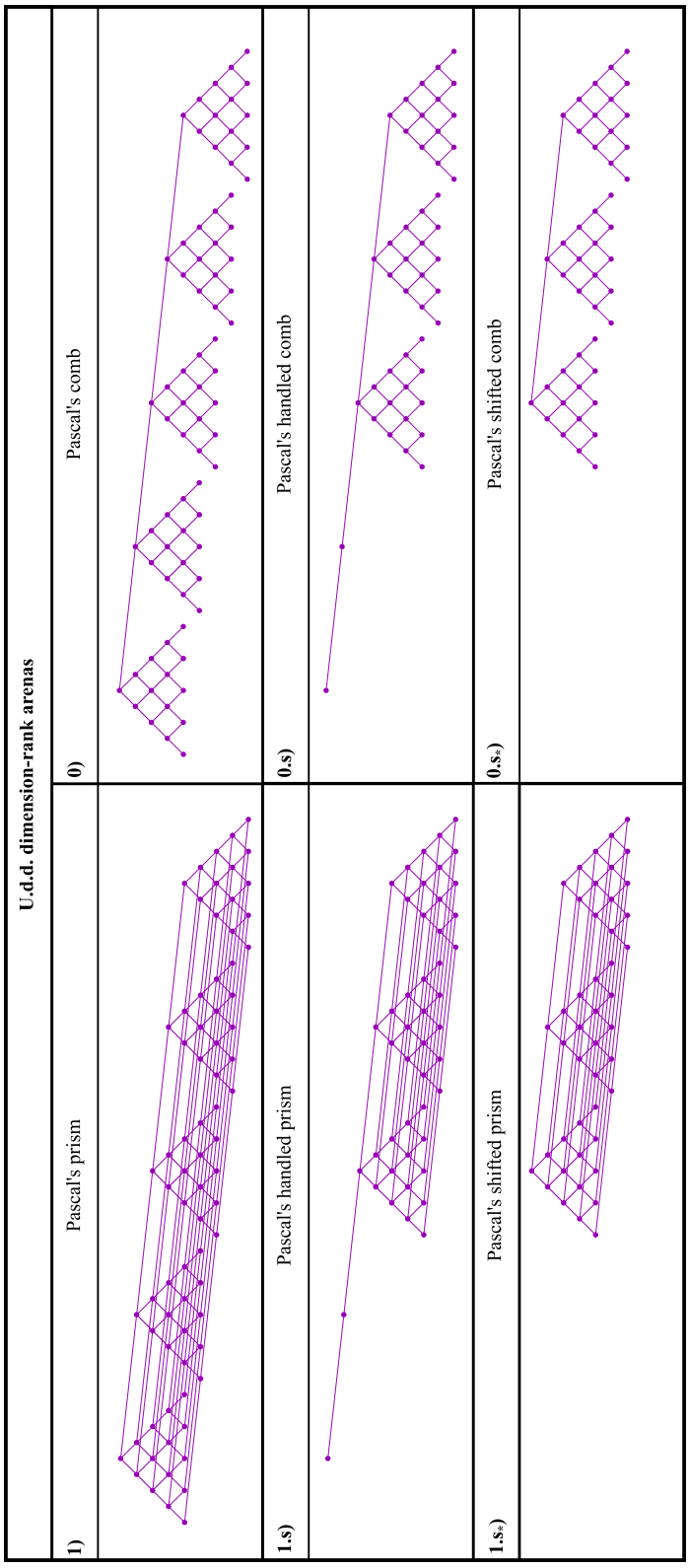


Figure 20:

3.4.4 Analogues of the preceding quotients

Model 1.q (Pascal's cones and pencils) The cumulative approximands here are firstly cones over Pascal's triangle. Down from which, a shaft of Pascal's prism is built, forming a pencil (Fig 21.a),

$$\mathfrak{Dim}\mathfrak{Rank}_u^d \quad 1.\mathrm{q}[\,D,\,I\,] \quad = \quad \mathrm{PascalPencil}[\,D\,-\,2,\,I\,]\,. \tag{35}$$

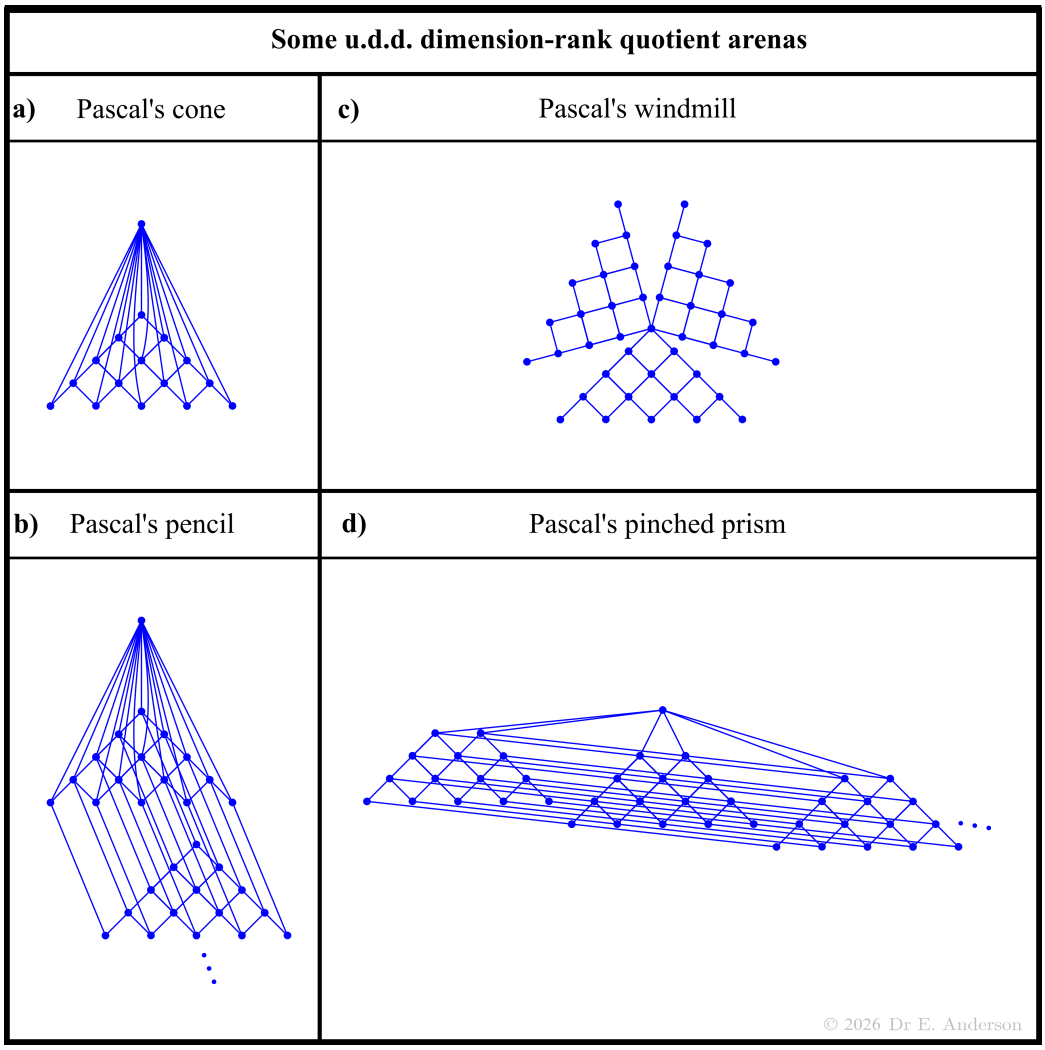


Figure 21:

Model 1.a (Pascal's windmills) The cumulative approximands here are windmills with $D - 1$ blades intersecting at the unqualified scalars' nexus point,

$$\mathfrak{Dim}\mathfrak{Rank}^{d}_{u} \;\; 1.\mathrm{a}[\, D,\, I\,] \;\; = \;\; \mathrm{PascalWindmill}[\, D \, - \, 1,\, I\,] \; . \tag{36}$$

See Subfig b).

Model 1.a (Pascal's pinched prism) See Subfig c).

3.4.5 Interplay with $\updownarrow$ quotient

Remark 1 This gives the $\widetilde{\text{Pascal}}$ counterparts of the previous two SSSecs, as per Fig 22.

Exercise 3^{-} Which of these pictures is the only one that can be assigned poset structure?

Pointer 1 While we are currently out of time to do so, further Order- and Graph-Theoretic structural analysis paralleling Fig 17 can also be applied to these last 5 SSSecs. This will eventually form a further Appendix in a subsequent version of the current Review. With some of the cases left as Exercises...

↕-quotient dimension-rank arenas
Pascal's prism
Pascal's comb
1)
0)
Pascal's handled prism
Pascal's handled comb
1,s)
0,s)
Pascal's shifted prism
Pascal's shifted comb
1,s,)
0,s,)
c)
Pascal's windmill
a) Pascal's cone
d)
b) Pascal's pencil
Pascal's pinched prism
© 2026 Dr. E. Anderson

Figure 22:

4 Bigger arenas of tensors

4.1 Definitions

Structure 1 One bigger arena of plain Cartesian tensors is

$$\mathfrak{Tensor}(D) \ .$$

Of tensors of all ranks on a fixed $\mathbb{R}^D$.

Another is

$$\mathfrak{Tensor}(R) \ :$$

the rank-R Cartesian tensors in no matter what dimension.

Finally we have the arena in Fig 23.a). I.e. the totality of ranks of Cartesian tensor throughout the totality of dimensions.

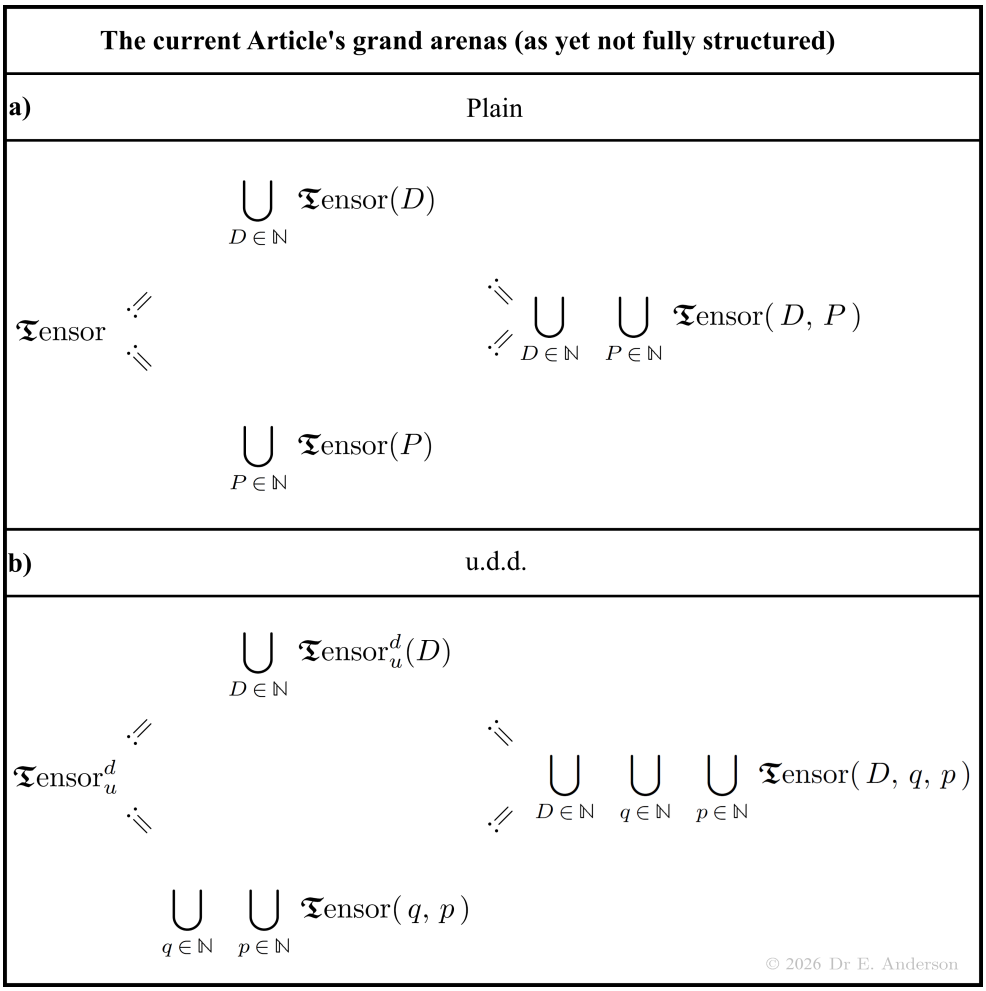


Figure 23:

Structure 2 One big arena of Cartesian u.d.d. tensors is

$$\mathfrak{Tensor}^{u}_{d}(D) \;:=\; \bigcup_{q \in \mathbb{N}} \; \bigcup_{p \in \mathbb{N}} \mathfrak{Tensor}(\, D,\, q,\, p\,) \;.$$

Of tensors of all ranks on a fixed $\mathbb{R}^D$.

Another is

$$\mathfrak{Tensor}^{d}_{u}(\, q,\, p\,) \;:=\; \bigcup_{D \in \mathbb{N}} \mathfrak{Tensor}(\, D,\, q,\, p\,) \;:$$

the rank-P Cartesian tensors in no matter what dimension.

Finally we have the arena in Subfig b). I.e. the totality of ranks of Cartesian tensor throughout the totality of dimensions.

4.2 Interlude on Bundle Theory

4.2.1 General bundles

Definition 1 Over each point $\mathfrak{b}$ in a base manifold $\mathfrak{B}$, we place an object $\mathfrak{O}_{\mathfrak{b}}$. Which the projection map π sends down to $\mathfrak{b}$ . . The overall structure $\mathfrak{E}$ thus formed is the *total space* of a *general bundle*. At least in the more conventional case, $\mathfrak{E}$ and the $\mathfrak{O}_{\mathfrak{b}}$ are manifolds as well.

Notational Remark 5

$$(\, \mathfrak{E},\, \pi,\, \mathfrak{B}\,) \tag{37}$$

is a common notation. A more complete one is

$$\left(\, \mathfrak{E},\, \pi,\, \mathfrak{B},\, \{\, \mathfrak{O}_{\mathfrak{b}}\,\}_{\mathfrak{b} \in \mathfrak{B}}\right) \;. \tag{38}$$

I.e. with specific reference to the collection of projected-down or 'attached' objects. See also Fig 24b).

Remark 1 The above definition is a bottom-up formulation. There is also a top-down formulation. In which a given $\mathfrak{E}$ is partitioned into[9] $\mathfrak{O}_{\mathfrak{b}}$. Each of which projects down to a point $\mathfrak{b}$. The totality of which points is $\mathfrak{B}$. In this approach, taking $\mathfrak{E}$ to be a manifold, and each $\mathfrak{O}_{\mathfrak{b}}$ as well, are concrete, simplifying and widely applicable modelling assumptions. While $\mathfrak{B}$ first enters as the partitions' pieces' index set.

Pointer 1 See e.g. within [89] for a brief introduction to general bundles, and [88] for further details.

[9] The current Review makes prominent use of integer partitions. Let us therefore clarify that 'partition' in a manifold context means a split into a disjoint union of pieces. I.e. our pieces are mutually-disjoint, and our split is collectively-exhaustive.

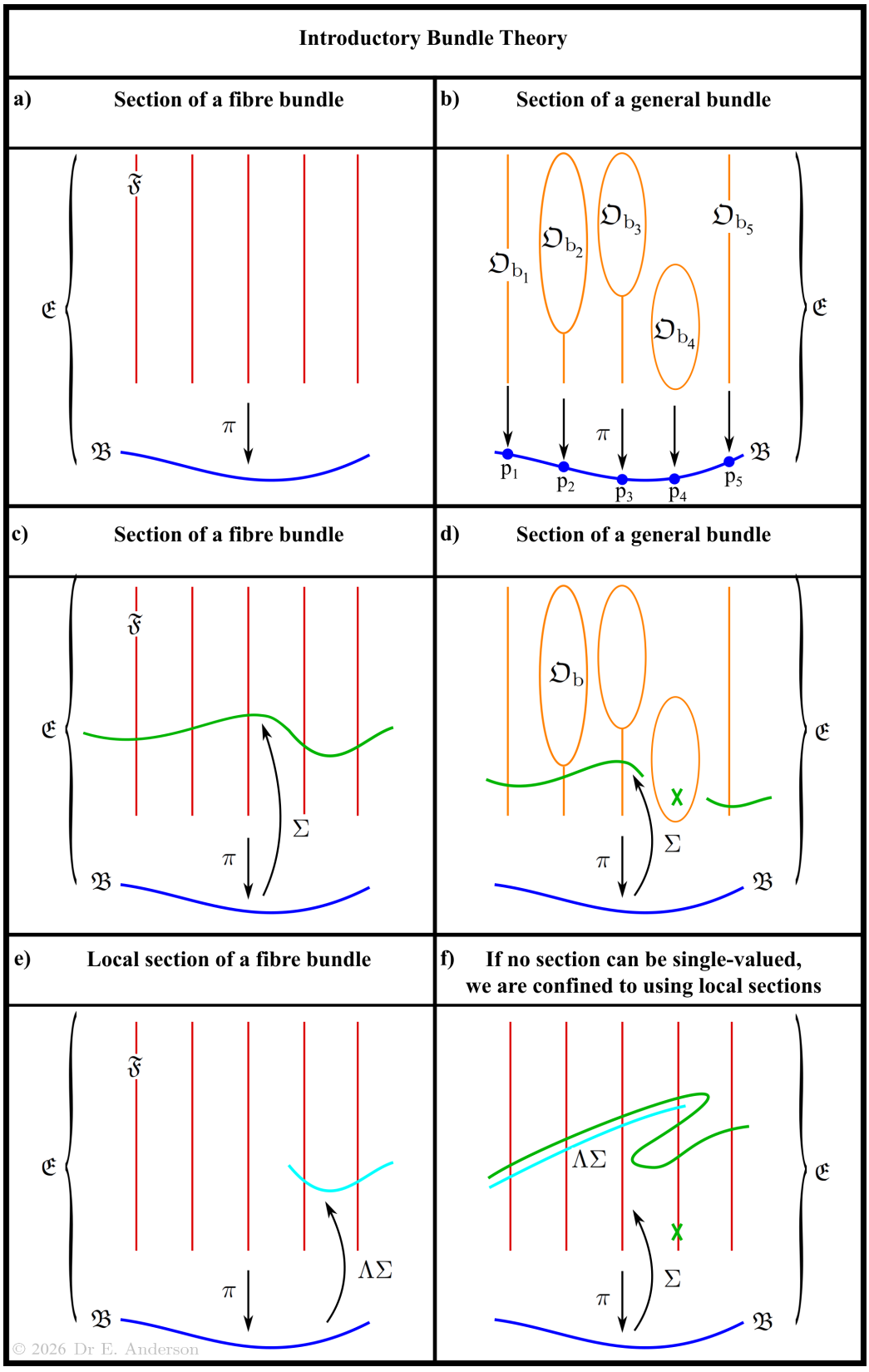

Introductory Bundle Theory
a) Section of a fibre bundle
b) Section of a general bundle
c) Section of a fibre bundle
d) Section of a general bundle
e) Local section of a fibre bundle
f) If no section can be single-valued, we are confined to using local sections
© 2026 Dr E. Anderson

Figure 24:

4.2.2 Fibre bundles

Structure 1 In a *fibre bundle*, furthermore all of the 'attached objects' are isomorphic *fibres*:

$$\mathfrak{O}_{\mathfrak{b}} \;\cong\; \mathfrak{F} \quad \forall\, \mathfrak{b} \,\in\, \mathfrak{B}\ . \tag{39}$$

See also Subfig a): a pictorial simplification of the general bundle case.

Fibre bundles carry further paraphenalia [87, 88]. The only one that need concern us in the current Review is that they also carry a *structure group* G . Acting on the fibres $\mathfrak{F}$.

Notational Remark 6

$$(\,\mathfrak{E},\,\pi,\,\mathfrak{B},\,\mathfrak{F}\,,G\,...\,) \tag{40}$$

is a common start on the notation for a fibre bundle.

A common simplification is for $\mathfrak{F}$ to itself coincide with the structure group G . Reducing matters to G acting on itself... In this case, the notation is abbreviated to

$$\mathfrak{p}(\,\mathfrak{B},\,G\,)\ . \tag{41}$$

As in effect a notation for $\mathfrak{E}$ viewed as such a fibre bundle. Which is termed the *principal* (*fibre*) *bundle*. Fibre bundles not sporting the above coincidence are often termed *associated bundles*.

4.2.3 Sections of a bundle

Definition 1 A *section* of a bundle is a map $\Sigma \,:\, \mathfrak{B} \,\longrightarrow\, \mathfrak{E}$ with the following property.

$$\pi \circ \Sigma \;=\; id_{\mathfrak{B}}\ . \tag{42}$$

Structure 1 Each attached object entering a section is cut precisely once. A *global section* Σ is one in which every object enters (Subfigs c-d). A *local section* $\Lambda\Sigma$ is one in which only some objects enter (Subfig e). For instance, some bundles are obstructed from having any global sections by all candidate sections being forced to cut $\geq\, 1$ object twice (Subfig f).

Exercise 4^{-} Formulate the fibre bundle subcase of global and local section.

4.2.4 Discretized base spaces

Structure 1 Some kinds of simplicial space [104] are conceptually similar to manifolds. Graphs are among these. And placing orders on these is not particularly disruptive [71]. Some can even be taken to be manifolds, albeit generally not differentiable ones due to the presence of corners. All in all, we are free to contemplate bundles over discrete spaces without and with orders.

4.3 Some plain examples of bigger tensor arenas

4.3.1 The full plain $\mathfrak{Tensor}(D)$

Structure 1 Here we place each fixed-rank tensor arena $\mathfrak{Tensor}(D, P)$. Above the corresponding dimension, that themselves form the chain $\mathfrak{Dim}(D) = \mathbb{N}_0$. These attached objects are entirely distinct. Except for scalars, for all the constant or pointwise $\mathfrak{Tensor}(D, 0)$ are identical copies. So this exception gives a fibre bundle while the other normative cases are all general bundles. With vectors minimal in realizing realize the persistent generic general-bundle structure. See Subfigs c-e) for the scalar, vector and general case.

4.3.2 The full plain $\mathfrak{Tensor}(R)$

Structure 2 Now instead we position each fixed-dimension tensor arena $\mathfrak{Tensor}(D, N)$ Above the corresponding rank, that themselves form the chain $\mathfrak{Rank}(P) = \mathbb{N}_0$. These attached objects are entirely distinct. Except for dimension 0 and 1 , for which they are all identical. Because all higher rank tensors collapse to scalars. So we have 2 identical fibre bundles, and then a heap of distinct general bundles

Finally, 2-D is minimum to realize the persistent generic general-bundle structure.

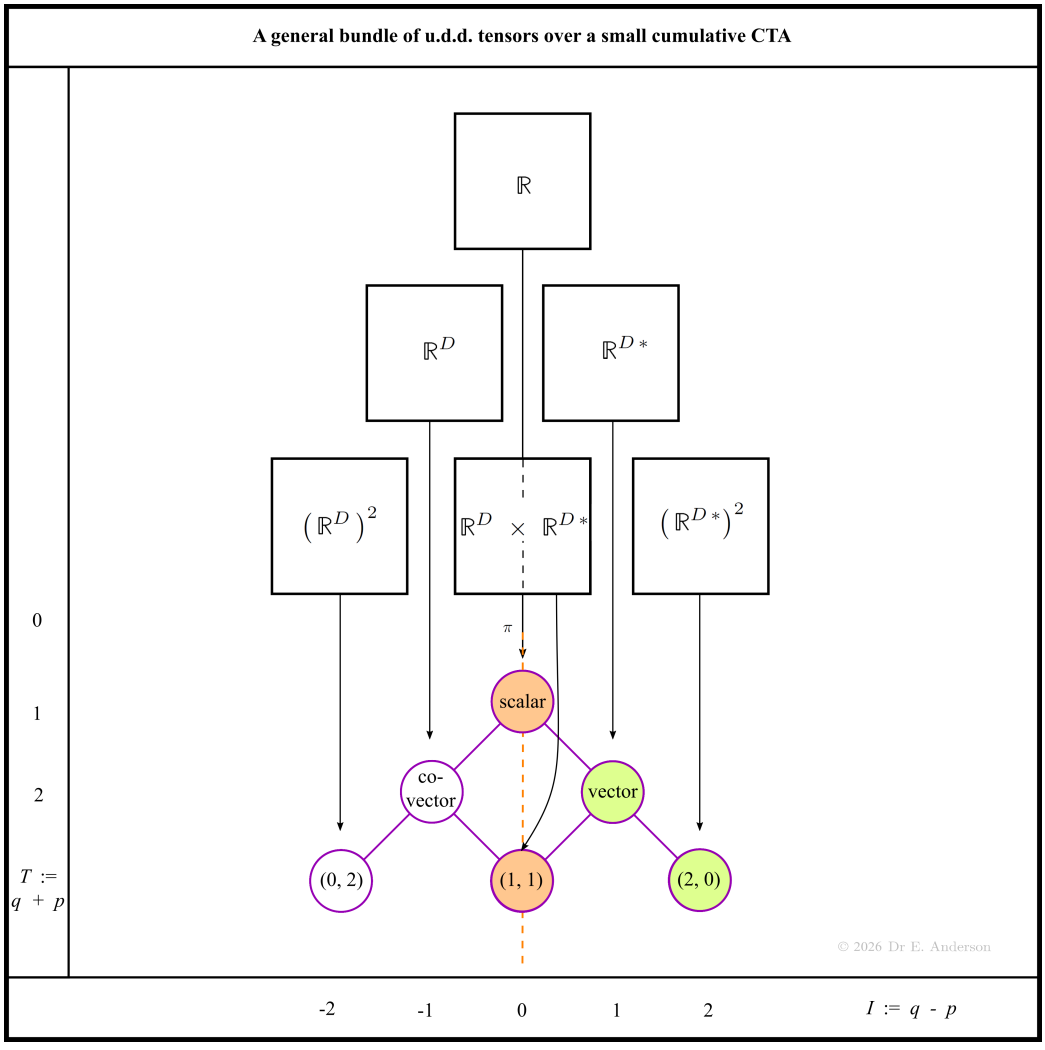


Figure 25:

4.3.3 Quotients so far

Remark 1 It furthermore makes sense to quotient each of the previous four fibre bundles down to a single scalar arena, each over a single point. For higher tensors are not distinguishably defined in 0- or 1-D . The base point then furthermore ceases to be a useful identifier. So we return to the scalar arena by itself.

In the case of the scalars in arbitrary-D , there often still is Physical distinction. Say between a scalar realized on a curve, a surface, or the whole space. Say total line charge versus total surface charge versus total charge enclosed in a volume element. If this is relevant, then one would not quotient.

Remark 2 One context for quotienting is to group together all terms that can be added, or, more generally, linearly combined.

Scalars can be added irrespective of the dimension each is defined on. This underpins how a particle Lagrangian can just be added to a background field Lagrangian. We correspondingly quotient $\mathfrak{T}\text{ensor}(\,D,\,0\,)$ to a single copy over a single point. And similarly for $\mathfrak{T}\text{ensor}$ and for u.d.d. counterparts.

4.4 Motivation for CTAs

4.4.1 $\mathfrak{T}\text{ensor}(R)$

Example R.1 In the next Section, we shall see that contraction, and products of tensors, interrelate tensors of distinct rank.

Structure R Some tensor splits decompose larger ranks into smaller ranks.

Example R.2 The *trace-tracefree split* that we shall cover in Sec 6.

Example R.3 Partial derivation sends a rank-R Cartesian tensor to a rank-$(\,R\,+\,1\,)$ one.

4.4.2 $\mathfrak{T}\text{ensor}(D)$

Structure D While other tensor splits decompose higher-D tensors into lower-D tensors.

Example D.1 One of the moves that can be used in setting up Hopf's little map [110] can be phrased as splitting a 4-D position vector into 2 2-D position vectors.

Example D.2 Hypersurface Geometry [20, 17, 27] underlies many examples. For instance in the local splitting of a manifold into a codimension-1 surface and a *normal-direction* vector field.

4.4.3 𝔗ensor

Example RD.1 *Duality* in dimension D maps rank-R to rank $D - R$.

Example RD.2 Physically interpreting Special Relativity provides numerous examples concurrently involving dimensional rank splits.

Splitting the *event* 4-vector $\overline{\boldsymbol{X}}$ into a 3-D position vector $\overrightarrow{\boldsymbol{x}}$ and a 1-D time scalar t .

The rank-2 4-D Minkowski spacetime metric $\underline{\underline{\boldsymbol{\eta}}}$ into the following. The rank-2 3-D Euclidean metric $\underrightarrow{\mathbb{I}}$ with components δ_{ij} . A constant 1-D time scalar, 1 . And the zero 3-vector of space-time cross-terms.

The *energy-momentum* 4-covector $\underline{\boldsymbol{P}}$ into the 3-D momentum covector $\underline{\boldsymbol{p}}$ and 1-D energy scalar E .

The rank-2 *energy-momentum-stress* 4-D tensor $\underline{\underline{\boldsymbol{T}}}$ into the following. The rank-2 3-D stress tensor $\underrightarrow{\underrightarrow{\boldsymbol{T}}}$ The 3-D momentum covector. And the 1-D energy scalar.

Or the rank-2 *electromagnetic field strength* 4-D tensor $\underline{\underline{\boldsymbol{F}}}$ into the following 3-D objects. The electric field covector $\underrightarrow{\boldsymbol{E}}$ and the magnetic field pseudocovector $\underrightarrow{\boldsymbol{B}}$.

Example RD.3 The further split of a manifold's D-dimensional metric $\underline{\underline{\boldsymbol{g}}}$ as induced by example D.1 into the following. A rank-2 hypersurface metric $\underrightarrow{\underrightarrow{\boldsymbol{h}}}$ of dimension $d := D - 1$. A *normal-displacement scalar* $g_{\perp\perp}$. And a *strutting d-vector* $\underrightarrow{\boldsymbol{g}}_{\perp}$ for how what were the hypersurface's coordinates are dragged around upon moving away from the hypersurface.

Example RD.4 The $3 + 1$ split of General Relativity in effect combines both of the previous examples. Interpreting the manifold as an in-general curved spacetime. The hypersurface as a spatial slice. And the normal-displacement to it as a time coordinate. Then the normal displacement scalar field is called the *lapse*, α . And the strutting 3-vector field is called the *shift*, $\underrightarrow{\boldsymbol{\beta}}$. In each case in Arnowitt, Deser and Misner's parlance and notation [9].

Let us leave the GR-coupled $3 + 1$ split versions of energy-momentum-stress and of Electromagnetism for a further occasion!

End Remark All in all, we need the CTAs and the big and grand arenas as the natural place in which to view these splits. And generalizations and yet further splits, which pervade the more advanced theory of tensors [27].

5 Cartesian tensor algebra

5.1 The full Cartesian case: $O(D)$

5.1.1 Linearity

Proposition 1 Let $\boldsymbol{T}$ and $\boldsymbol{S}$ be Cartesian tensors of the same rank. Then so are the following.
S)

$$\lambda \boldsymbol{T} \ . \tag{43}$$

A)

$$\boldsymbol{S} \ + \ \boldsymbol{T} \ . \tag{44}$$

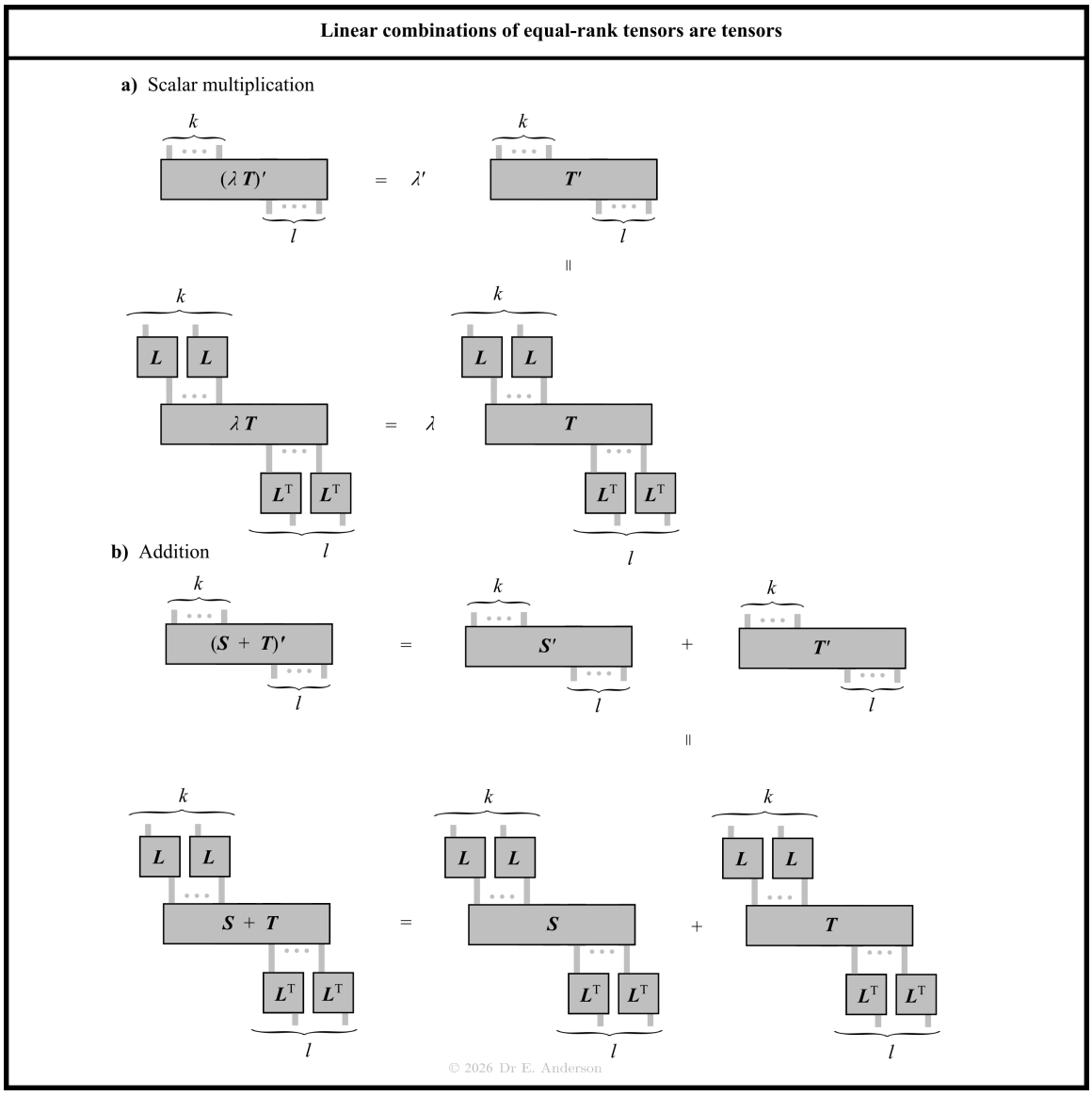


Figure 26:

Proof See Fig 26 for a tensor networks rendition.

S) In the first step, we unpackage the scalar multiple in the primed frame. In the second step, we apply the 2 corresponding tensor transformation laws. In the final step, we reform the scalar multiple in the unprimed frame.

A) In the first step, we unpackage the sum in the primed frame. In the second step, we apply the 2 corresponding tensor transformation laws. In the final step, we reform the sum in the unprimed frame. □

Corollary 1 L) Let $\boldsymbol{S}_{(i)}$ $(i) = (1)$ to (k) be Cartesian tensors of the same rank. Then so is the linear combination

$$\sum_{(i)\,=\,(1)}^{(k)} \lambda_{(i)}\, \boldsymbol{S}_{(i)} \ . \tag{45}$$

D) This includes the difference of any 2 equal-rank tensors,

$$\boldsymbol{S} \ - \ \boldsymbol{T} \ . \tag{46}$$

Remark 1 An alternative statement of Proposition 1 is that tensoriality is linear: scalar-multiplicative and additive. Corollary 1 then collects the implications that tensoriality is linearly-combinational and thus for instance subtractive.

Remark 2 Proposition 1 and the existence of the zero tensor of each dimension and rank establish the following. That $\mathfrak{T}\text{ensor}(\, D,\, P\,)$ and $\mathfrak{T}\text{ensor}^{d}_{u}(\, D,\, p,\, q\,)$ are vector spaces.

5.1.2 The outer product

Definition 1 The *outer product* of 2 tensors $\boldsymbol{T}$ and $\boldsymbol{U}$ is as follows. Their multiplicative concatenation without sharing any indices, $\boldsymbol{T}\boldsymbol{U}$. In say the u.d.d. case, this is a map

$$\mathfrak{T}\text{ensor}\,(\, D,\, q_1,\, p_1\,) \ \times \ \mathfrak{T}\text{ensor}\,(\, D,\, q_2,\, p_2\,) \ \longrightarrow \ \mathfrak{T}\text{ensor}\,(\, D,\, q_1 \ + \ q_2,\, p_1 \ + \ p_2\,) \ .$$

Proposition 2 Let $\boldsymbol{U}$ and $\boldsymbol{T}$ be tensors. Then so is their outer product $\boldsymbol{T}\boldsymbol{U}$.

Proof See Fig 27 for a tensor networks rendition. Where the first step unpackages the outer product in the primed frame. Each factor is prescribed to be a tensor, enabling step 2's tensor transformation laws. Finally we reform the outer product in the unprimed frame. □

Notational Remark 7 von Neumann's [4] *tensor product symbol* $\otimes$ has come to often be used to denote outer products in an more unambiguous manner. Now one writes not $\boldsymbol{T}\boldsymbol{U}$ but

$$\boldsymbol{T} \ \otimes \ \boldsymbol{U} \ .$$

See below for its further use. It has various further Algebraic and Categorical connotations, such as being dual to Linear Algebra's direct-sum operation $\oplus$.

Corollary 2 Let $\boldsymbol{T}_{(i)}$ $(i) = (1)$ to (k) be Cartesian tensors. Then so is

$$\prod_{(i)\,=\,(1)}^{(k)} \boldsymbol{T}_{(i)} \ = \ \bigotimes_{(i)\,=\,(1)}^{(k)} \boldsymbol{T}_{(i)} \ . \tag{47}$$

Remark 1 In other words, tensoriality is outer-productive. Binarily by Proposition 2 and finitely by Corollary 2. In von Neumann's conceptualization, moreover, this reduces just to tensoriality being tensor-productive!

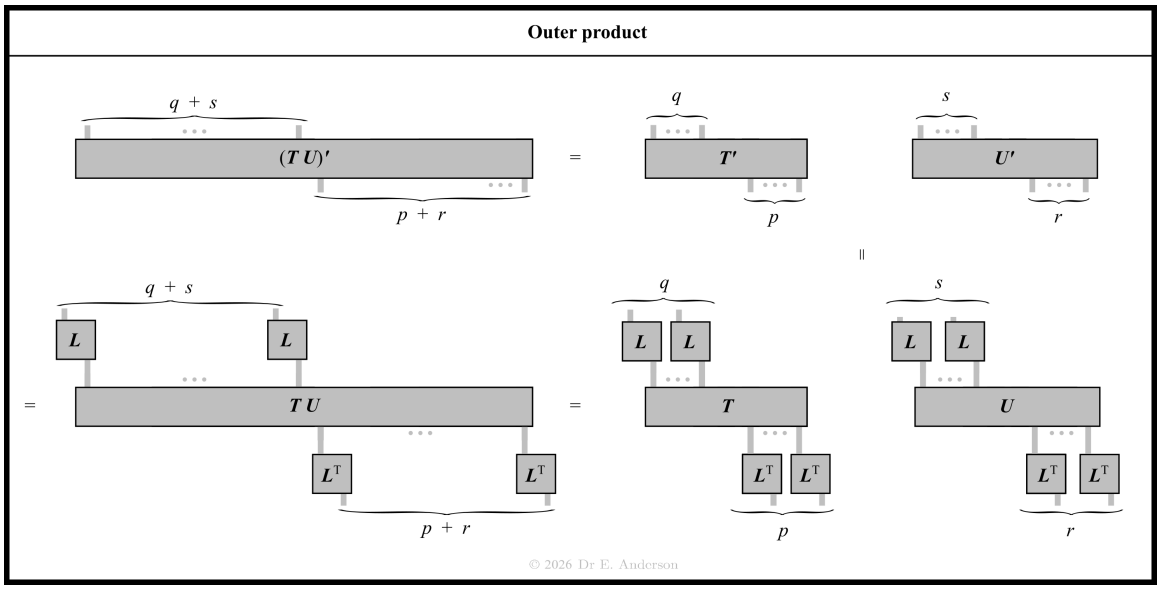


Figure 27:

5.1.3 Contraction and inner product. i. Definitions and notation

Structure 1 A *contraction* of a pair of indices involves summing over all values taken by each member of that pair.

E.g. in the u.d.d. case, this is overall a map

$$\mathfrak{Tensor}\,(\,D,\,q,\,p\,)\;\longrightarrow\;\mathfrak{Tensor}\,(\,D,\,q\;-\;1,\,p\;-\;1\,)\;.$$

Notational Remark 8 In longhand, this is just written as a sum. For instance

$$\sum_{j\,=\,1}^{d} T_{i\,j}\,U^{j\,k}\;.$$

A first shorthand is *Einstein's summation convention*. By which any repeated pair of indices in a tensor, or a product of tensors, is taken to carry such a sum. E.g.

$$T_{i\,j}\,U^{i\,k}\;.$$

A second shorthand is to join the pair of indices' legs in whichever tensor networks formulation. See for instance the top-right diagram in Fig 28.

A *dummy index change* is a common step in working with indices, consisting of the following. That changing any summed-over index pair's letter for any other at that point unused letter is a licit move. Clearly the tensor networks notation has no use for this move, since each joined leg in effect carries an arbitrary such index pair. In a manner that, when multiple joined legs are present, these are distinguishable from by which slots on which tensors each joins.

Structure 2 A (*singly-contracted*)*inner product* of tensors can be envisaged as an outer product followed by a single contraction. E.g. in the u.d.d. case, this is overall a map

$$\mathfrak{Tensor}\,(\,D,\,q_1,\,p_1\,)\;\times\;\mathfrak{Tensor}\,(\,D,\,q_2,\,p_2\,)\;\longrightarrow\;\mathfrak{Tensor}\,(\,D,\,q_1\;+\;q_2\;-\;1,\,p_1\;+\;p_2\;-\;1\,)\;.$$

Notational Remark 9 A standard notation is

$$\boldsymbol{T}\cdot\boldsymbol{U}\;.$$

For all that one may well need to take care to point out which index on $\boldsymbol{T}$ is contracted with which index on $\boldsymbol{U}$. A standard convention is that last-to-first is the default, requiring no further comment. Of course, using tensor networks notation, it is automatically clear which leg on $\boldsymbol{T}$ is being joined to which leg on $\boldsymbol{U}$.

Structure 3 An *l-fold-contracted inner product* can be envisaged as an outer product followed by an l-fold contraction. Which in turn can be viewed as l single contractions.

E.g. in the u.d.d. case, this is now overall a map

$$\mathfrak{Tensor}\,(\,D,\,q_1,\,p_1\,)\;\times\;\mathfrak{Tensor}\,(\,D,\,q_2,\,p_2\,)\;\longrightarrow\;\mathfrak{Tensor}\,(\,D,\,q_1\;+\;q_2\;-\;l,\,p_1\;+\;p_2\;-\;l\,)\;.$$

Notational Remark 10 A fairly standard notation for a 2-fold contracted inner product is

$$\boldsymbol{T}\;\bigcirc\;\boldsymbol{U}\;.$$

In some sources, the interior of this circle is shaded to form a bullet:

$$\boldsymbol{T}\;\bullet\;\boldsymbol{U}\;.$$

Higher l-fold versions feature considerably less in the literature. So notation for these is rather more down to each Author entertaining any such.

$$\boldsymbol{T}\;\textcircled{l}\;\boldsymbol{U}$$

is ours in this style.

The above matter of care multiplies with the number of indices getting contracted. The standard convention extends to last-l-into-first-l . The tensor network notation, however automatically renders clear which legs are being contracted with which others. Rendering this style increasingly superior with increasing l .

5.1.4 ii. Inner-product tensoriality

Proposition 3 Let $\boldsymbol{T}$ and $\boldsymbol{U}$ be tensors. Then so is any of their singly-contracted inner products $\boldsymbol{T}\cdot\boldsymbol{U}$.

Proof We consider this in Fig 29 in the case in which both input tensors have all-upstairs indices followed by all-downstairs indices. And the single contraction is between the last downstairs on the first and the first upstairs on the second.

The first step unpackages the inner product in the primed frame. The second applies the tensor transformation law to the uncontracted object. The third and fourth hit with orthogonality and the unit property respectively. The final step reforms the inner product in the unprimed frame. □

Corollary 3 Let $\boldsymbol{T}$ and $\boldsymbol{U}$ be tensors. Then so is any of their l-fold contracted inner products

$$\boldsymbol{T}\;\textcircled{l}\;\boldsymbol{U}\;.$$

Up to whichever value of l that their ranks support.

Remark 1 In other words, tensoriality is (l-fold) inner-productive. Binarily by Proposition 3 and Corollary 3. And also readily extendible to finitely, by connecting up legs across (k) objects.

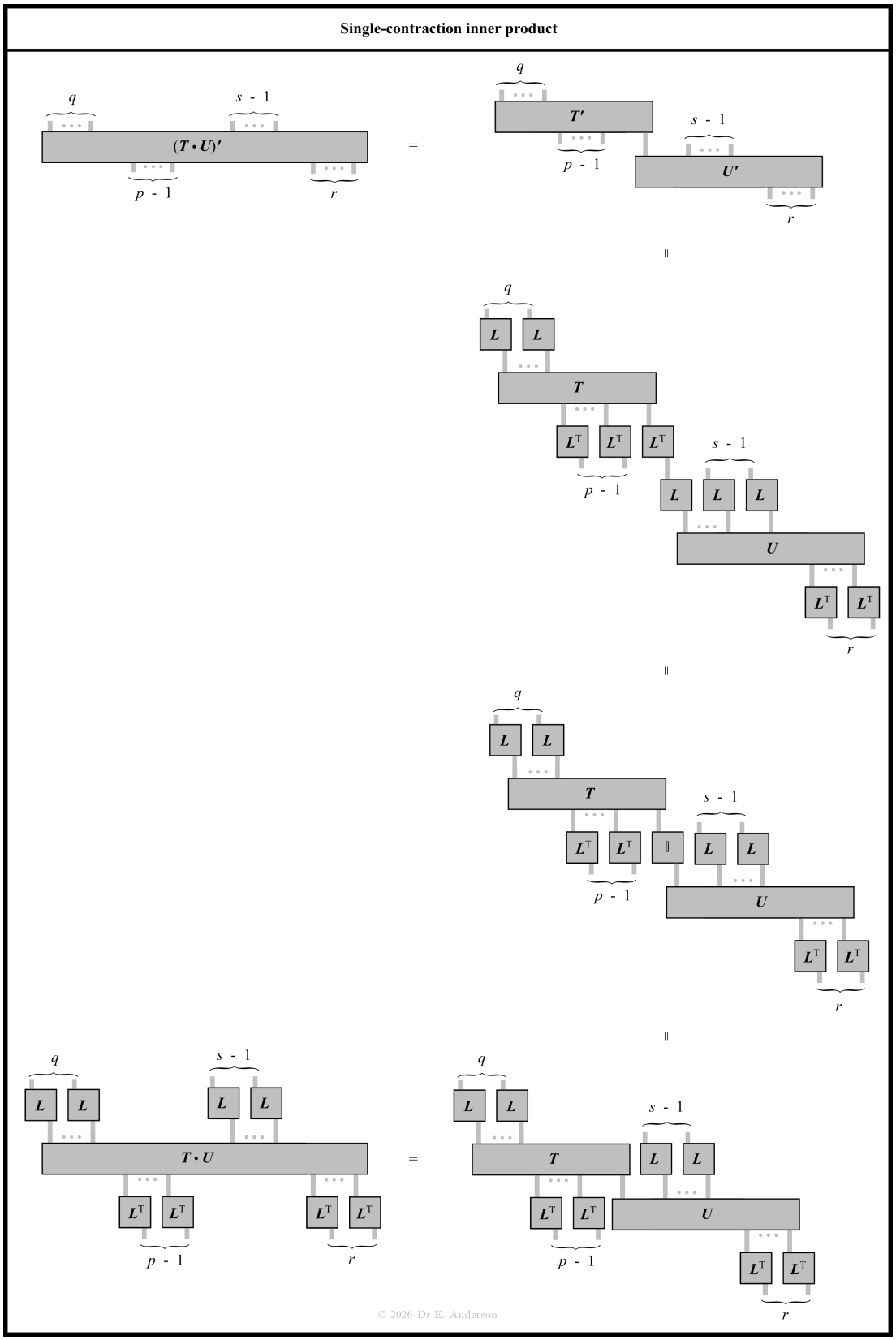


Figure 28:

5.1.5 iii. Self-contraction tensoriality

Proposition 4 The single self-contraction of a Cartesian tensor is itself a Cartesian tensor.

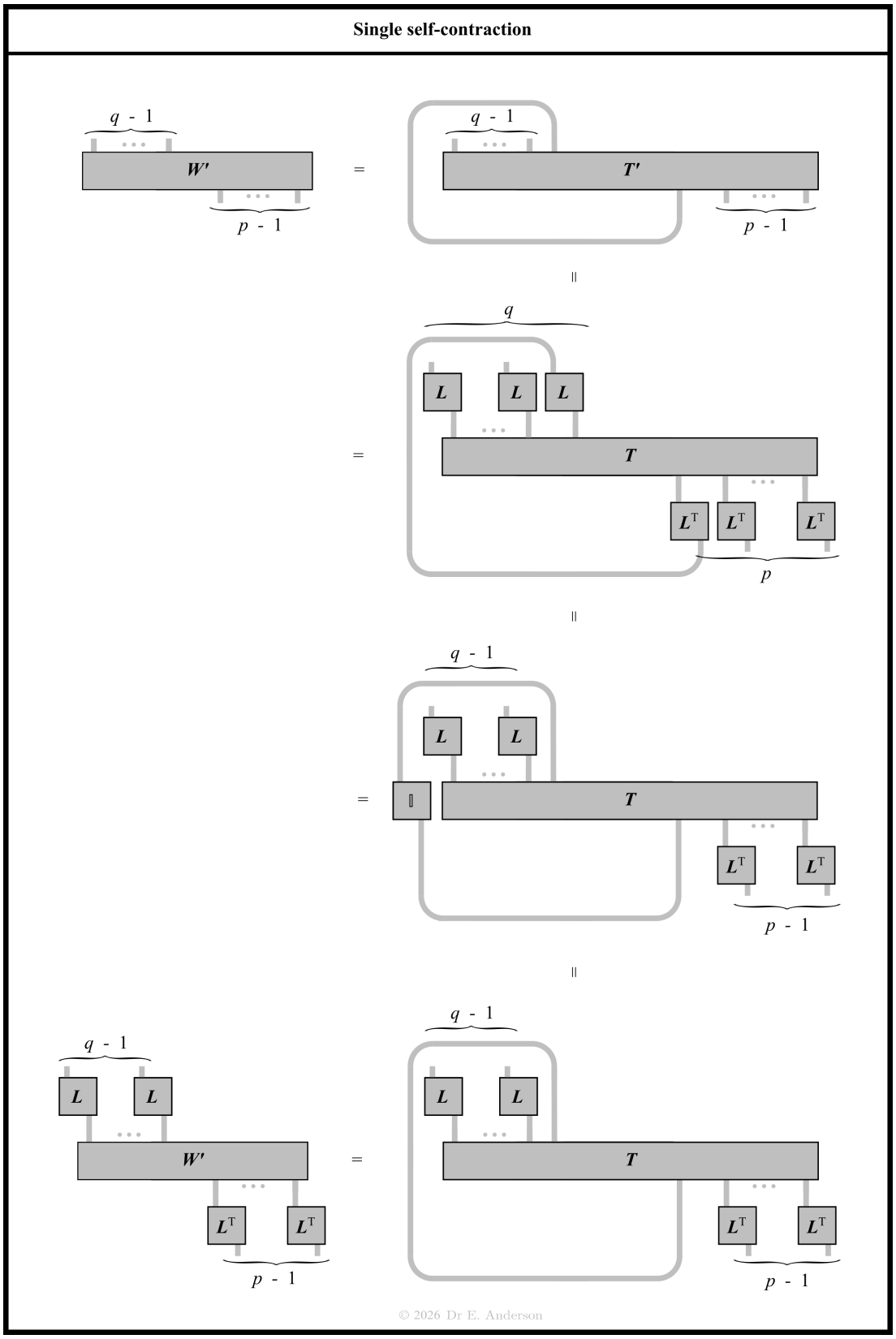


Figure 29:

Proof See Fig 28 for a tensor networks rendition. Where the first step unravels the self-contraction in the primed frame. The second applies the corresponding tensor transformation laws to each input tensor. The third and fourth hit with orthogonality and the unit property respectively. While the

last step reforms the self-contraction. □

Corollary 4 The l-fold self-contraction of a Cartesian tensor is itself a Cartesian tensor. Up to whichever value of l that its ranks support.

I.e. tensoriality is (l-fold) self-contractive.

Exercise 5 a)$^-$ Prove Corollaries 1 to 4.

b)$^-$ Explain how the scalar product of 2 vectors occurs as a subcase.

c) Prove Proposition 3 and Corollary 3 via use of Structures 1 and l.

Remark 1 In other words, tensoriality is self-contractive. One can also readily extend, whenever one requires, to the following. An l-fold contraction distributed in whichever manner over an n-ary outer product. Also one can interpret (l-fold) inner productivity as a composite of 2 tensorial operations: tensor product and contraction. By which it is itself tensorial...

5.1.6 The quotient theorem

Theorem Q (Quotient theorem for Cartesian tensors) Suppose that a Cartesian tensor $\boldsymbol{T}$ is the outer product of an arbitrary Cartesian tensor $\boldsymbol{A}$ and a candidate object $\boldsymbol{C}$. Then $\boldsymbol{C}$ is itself a Cartesian tensor.

Proof See Fig 30 for a tensor networks rendition. The first step realizes the outer product in the primed frame. Since $\boldsymbol{C}$ has not yet been established to be a tensor, let us denote its factor in the primed frame not by $\boldsymbol{C}'$ but by some $\boldsymbol{C}^*$ to be determined. Since $\boldsymbol{T}$ is a tensor, it obeys the tensor transformation law as per step 2. We undo the outer product in the unprimed frame in step 3.

In step 4, we equate the first and last of the above, factorizing $\boldsymbol{A}$ out. Since $\boldsymbol{A}$ is arbitrary, we can cancel out this common factor in step 5. Thus establishing that $\boldsymbol{C}^*$ is $\boldsymbol{C}'$, by which $\boldsymbol{C}$ itself meets the definition of Cartesian tensor. □

Remark 3 In other words, tensoriality is *quotientive*, provided that the candidate object's cofactor is an arbitrary tensor.

Remark 4 E.g. in the u.d.d. case, the arenaic framing of the quotient rule is as follows. $\boldsymbol{T} \in \mathfrak{T}\text{ensor}\,(\,D,\,q_t,\,p_t\,)$ And $\boldsymbol{A} \in \mathfrak{T}\text{ensor}\,(\,D,\,q_a,\,p_a\,)$. Then $\boldsymbol{C}$ is established to belong to $\mathfrak{T}\text{ensor}\,(\,D,\,q_t\,-\,q_a\,,\;\,p_t\,-\,p_a\,)$.

Exercise 6 The Reader is welcome to check that the (l)-fold inner product versions of Theorem Q also holds.

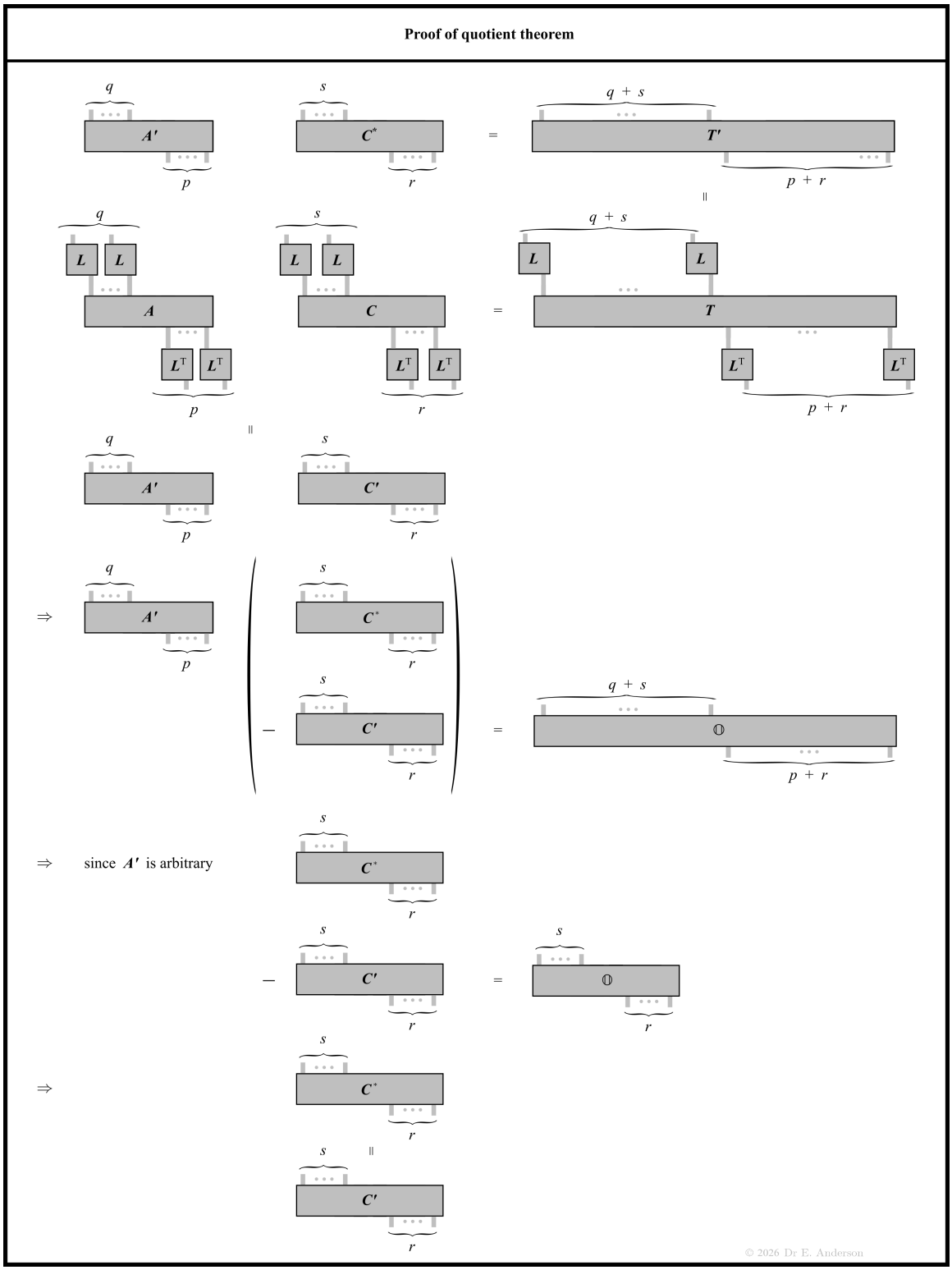

Proof of quotient theorem
q
s
q + s
A′
C*
=
T′
p
r
p + r
||
q
s
q + s
L
L
L
L
L
A
C
=
T
L^T
L^T
L^T
L^T
L^T
p
r
p + r
||
q
s
A′
C′
p
r
⇒
q
s
A′
C*
p
r
—
s
C′
r
q + s
=
𝕆
p + r
⇒
since A′ is arbitrary
s
C*
r
—
s
C′
r
=
s
𝕆
r
⇒
s
C*
r
s
||
C′
r
© 2026 Dr E. Anderson

Figure 30:

5.2 $SO(D)$ counterparts

5.2.1 Linearity

Proposition 1′ Let $\boldsymbol{P}$ and $\boldsymbol{O}$ be Cartesian pseudotensors of the same rank. Then so are the following.

S)

$$\lambda \boldsymbol{P} \ . \tag{48}$$

A)

$$\boldsymbol{P} \ + \ \boldsymbol{O} \ . \tag{49}$$

Corollary 1′

L) Let $\boldsymbol{P}_{(i)} \quad (i) \ = \ (1)$ to (k) be Cartesian pseudotensors of the same rank. Then so is

$$\sum_{(i) \ = \ (1)}^{(k)} \lambda_{(i)} \boldsymbol{P}_{(i)} \ . \tag{50}$$

D) This includes the difference of any 2 equal-rank pseudotensors,

$$\boldsymbol{S} \ - \ \boldsymbol{T} \ . \tag{51}$$

Remark 1 To mark $SO(D)$ dependence at the level of arenas, one can use such as

$$\mathfrak{T}\text{ensor}(\ D, \ q, \ p; \ SO(D)\ ) \ .$$

5.2.2 Outer product

Proposition 2′

E) Let $\boldsymbol{T}$ be a Cartesian tensor and $\boldsymbol{P}$ be a Cartesian pseudotensor. Then their outer product $\boldsymbol{T}\boldsymbol{P}$ is a Cartesian pseudotensor.

O) Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be Cartesian pseudotensors. Then their outer product $\boldsymbol{P}\boldsymbol{Q}$ is a Cartesian tensor

Corollary 2′

O) Let $\boldsymbol{P}_{(j)} \quad (j) \ = \ (1)$ to (l) be Cartesian pseudotensors. Then the (l)-fold outer product

$$\prod_{(j) \ = \ (1)}^{(l)} \boldsymbol{P}_{(j)} \quad = \quad \bigotimes_{(j) \ = \ (1)}^{(l)} \boldsymbol{P}_{(j)} \tag{52}$$

is itself a Cartesian pseudotensor for (l) odd. Versus a Cartesian tensor for (l) even.

E) Let $\boldsymbol{T}_{(i)} \quad (i) \ = \ (1)$ to (k) be Cartesian tensors. And $\boldsymbol{P}_{(i)} \quad (i) \ = \ (1)$ to (l) be Cartesian pseudotensors.

Then the $(\,(k) \ + \ (l)\,)$-fold outer product

$$\prod_{(i) \ = \ (1)}^{(k)} \boldsymbol{T}_{(i)} \ \times \ \prod_{(j) \ = \ (1)}^{(l)} \boldsymbol{P}_{(j)} \quad = \quad \bigotimes_{(i) \ = \ (1)}^{(k)} \boldsymbol{T}_{(i)} \ \otimes \ \bigotimes_{(j) \ = \ (1)}^{(l)} \boldsymbol{P}_{(j)} \tag{53}$$

– with whichever ordering to its factors – is the following.

A Cartesian pseudotensor for (l) odd. And a Cartesian tensor for (l) even.

5.2.3 Self-contraction

Proposition 4′ The single self-contraction of a Cartesian pseudotensor is itself a Cartesian pseudotensor.

Corollary 4′ The l-fold self-contraction of a Cartesian pseudotensor is itself a Cartesian pseudotensor. Up to whichever value of l that its ranks support.

5.2.4 Inner products

Proposition 3′

O) Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be pseudotensors. Then any of their singly-contracted inner products $\boldsymbol{P} \cdot \boldsymbol{Q}$ are tensors.

E) Let $\boldsymbol{T}$ be a Cartesian tensor and $\boldsymbol{P}$ a Cartesian pseudotensor. Then any of their once-contracted inner products $\boldsymbol{T} \cdot \boldsymbol{P}$ is a Cartesian pseudotensor.

Corollary 3′

O) Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be Cartesian pseudotensors. Then their -contracted inner products $\boldsymbol{P} \textcircled{l} \boldsymbol{Q}$ are Cartesian tensors.

E) Let $\boldsymbol{T}$ be a Cartesian tensor and $\boldsymbol{P}$ be a Cartesian pseudotensor. Then their l-fold contracted inner products $\boldsymbol{T} \textcircled{l} \boldsymbol{P}$ are Cartesian pseudotensors.

In each case up to whichever value of l that their ranks support.

5.2.5 Quotient theorem

Corollary Q.1 (Cartesian pseudotensors' corollary of the quotient theorem)

i) Suppose that a Cartesian pseudotensor $\boldsymbol{P}$ is the outer product of an arbitrary Cartesian tensor $\boldsymbol{A}$ and a candidate object $\boldsymbol{C}$. Then $\boldsymbol{C}$ is itself a Cartesian pseudotensor.

ii) Suppose that a Cartesian pseudotensor $\boldsymbol{P}$ is the outer product of an arbitrary Cartesian pseudotensor $\boldsymbol{E}$ and a candidate object $\boldsymbol{C}$. Then $\boldsymbol{C}$ is itself a Cartesian tensor.

iii) Suppose that a Cartesian tensor $\boldsymbol{T}$ is the outer product of an arbitrary Cartesian pseudotensor $\boldsymbol{E}$ and a candidate object $\boldsymbol{C}$. Then $\boldsymbol{C}$ is itself a Cartesian pseudotensor.

Exercise 7 [Long] Prove all of this Subsec's results piecemeal. Present your answers in the vertical tensor networks formulation.

5.3 Parity unification

5.3.1 Parity conservation

Remark 1 Propositions 1 and 4 are multiplicatively unary. As such, Propositions 1 and 1′ and can be unified into a single parity-conserving Proposition $1^\sharp$. And Propositions 4 and 4′ into another: Proposition $4^\sharp$. Where 'parity-conserving' means that the input parity is the same as the output parity. I.e. $+$ to $+$ for tensors, or $-$ to $-$ for pseudotensors. Absorbing Corollaries as well, we arrive at the following sharper unified statements.

Proposition $1^\sharp$ Cartesian parity-s tensoriality is scalar-multiplicative. And same-rank 'additive, subtractive and elsewise linearly-combineable.

Proposition $4^\sharp$ Cartesian parity-s tensoriality is l-fold self-contractive. Up to whichever value of l that its ranks support.

5.3.2 Parity multiplication. i. Binary

Remark 1 Entertaining the possibility of pseudotensoriality in Cartesian tensors' binary products confers 2×2 input-object parities. I.e.

$$++ \ , \quad +- \ , \quad -+ \ , \quad -- \ .$$

Since these are deployed productively, they result in the following respective output-object partities.

$$+ \ , \quad - \ , \quad - \ , \quad + \ .$$

Unqualified outer and inner products however treat their 2 input factors interchangeably. Thus the 2 above middle cases collapse to a single middle case. Which we denote by E) standing for h*e*terogeneous inputs. The first case – for tensors – has been left unlabelled. While the last case – in which both inputs are pseudotensors – is denoted above by O). Standing for h*o*mogeneous inputs.

Proposition 2$^\sharp$ Let $\boldsymbol{T}$ be a Cartesian parity-t tensor and $\boldsymbol{U}$ be a Cartesian parity-u tensor. Then their outer product $\boldsymbol{T}\boldsymbol{U}$ is a Cartesian parity-$t\,u$ tensor.

Proposition 3$^\sharp$ Let $\boldsymbol{T}$ be a Cartesian parity-t tensor and $\boldsymbol{U}$ be a Cartesian parity-u tensor. Then each of the l-fold-contracted inner products $\boldsymbol{T} \ \textcircled{l} \ \boldsymbol{U}$ supported is a Cartesian parity-$t\,u$ tensor.

5.3.3 ii. n-ary

Remark 1 Entertaining the possibility of pseudotensoriality in Cartesian tensors' n-ary products confers a large tower of input parities. However the single outputs only admit 2 parities. And these can readily be related to the input parity counts as follows. N_- input pseudotensors enforces output parity

$$(-)^{N_-} \ .$$

Irrespective of what N_+ of input tensors enter. This furthermore simplifies as follows.

$$(-)^{N_-} \quad = \quad \begin{cases} +1 & N_- \ \text{even} \\ -1 & N_- \ \text{odd} \ . \end{cases} \tag{54}$$

Corollary 2$^\sharp$ Let $\boldsymbol{T}_{(i)} \quad (i) = (1)$ to (f) be Cartesian parity-t_i tensors. Then the (f)-fold outer product

$$\prod_{(i) \, = \, (1)}^{(f)} \boldsymbol{T}_{(i)} \quad = \quad \bigotimes_{(i) \, = \, (1)}^{(k)} \boldsymbol{T}_{(i)} \tag{55}$$

– with whichever ordering to its factors – is the following. A Cartesian pseudotensor when N_- is odd. And a Cartesian tensor when N_+ is even.

Exercise 8 Prove this Subsection's results in a unified manner. Present your answers in the vertical tensor networks formulation.

End Remark 2 All in all, pseudotensoriality is thus linear and self-contractive. Furthermore, if one views pseudotensoriality as $SO(D)$-tensoriality, then these properties are immediate from their holding for whichever kind of G-tensor. While pseudotensoriality is outer- and inner-productive, and quotientive modulo the usual arbitrariness of cofactor, in each case modulo a parity factor behaving as stated. For all that we have postponed parity-unification of the quotient theorem and its pseudotensor corollary until the end of the next Section.

6 Tensor symmetries

6.1 The minimum setting: rank 2

6.1.1 Pairwise (anti)symmetry

Definition 1 Let $\boldsymbol{T}$ be a tensor. Suppose that exchanging a pair of $\boldsymbol{T}$'s indices, slots or legs preserves the value of $\boldsymbol{T}$. Then $\boldsymbol{T}$ is *symmetric* in our pair. If instead the sign is reversed, then $\boldsymbol{T}$ is *antisymmetric* in our pair.

Remark 1 A rank-2 object, is *symmetric* if

$$S_{a\,b} \;=\; S_{b\,a} \;.$$

While it is *antisymmetric* if

$$A_{a\,b} \;=\; -\,A_{b\,a} \;.$$

A simple index-free formulation is as follows.

$$\boldsymbol{S} \;=\; \boldsymbol{S}^{\mathrm{T}} \;,$$

$$\boldsymbol{A} \;=\; -\,\boldsymbol{A}^{\mathrm{T}} \;.$$

Remark 2 You probably covered in your first course on tensors that if a 2-tensor is (anti)symmetric in its indices in one frame, then it is likewise in all other frames. (Anti)symmetry are consequently well-defined tensorial properties. This rests in turn on the following. If a G-tensorial equation holds in one G-frame, then it holds in all other G-frames.

Modelling Assumption 1 For the rest of this Section, we assume that the field $\mathbb{F}$ we are working over is not of characteristic 2. Thus $2 \neq 0$ and so we can divide by 2. In the current Article's main context, $\mathbb{F} = \mathbb{R}$, so immediately $2 \neq 0$.

Lemma 1 A symmetric 2-tensor double-contracted into an antisymmetric 2-tensor returns zero.

Proof

$$\begin{aligned} S_{a\,b}\,A^{a\,b} \;&=\; S_{a\,b}\,\big(\;-\;A^{b\,a}\big) \;=\; -\,S_{b\,a}\,A_{a\,b} \;=\; -\,S_{a\,b}\,A_{a\,b} \\ &\Rightarrow\; 2\,S_{a\,b}\,A^{a\,b} \;=\; 0 \\ &\Rightarrow\; S_{a\,b}\,A^{a\,b} \;=\; 0 \;. \end{aligned}$$

Where the first step is by antisymmetry. The second makes 2 dummy-index changes. The third is by symmetry. The fourth groups like terms. And the fifth divides by 2, which we have guaranteed to be licit. □

Lemma 2 Every rank-2 tensor admits a decomposition into symmetric and antisymmetric parts.

Proof Our caveat ensures that division by 2 is licit. Then $0 = 1 - 1$ and rebracketing returns the following 2-term split.

$$\boldsymbol{T}_{a\,b} \;=\; \frac{1}{2}\,(T_{a\,b} \,+\, T_{b\,a}) \,+\, \frac{1}{2}\,(T_{a\,b} \,-\, T_{b\,a}) \;. \tag{56}$$

But the first term is symmetric and the second term is antisymmetric, so we have obtained such a sum. □

Structure 1 This split is underpinned by the following arena-level decomposition. Take our tensors act on the vector space $\mathfrak{V} = \mathbb{R}^D$. Then

$$\mathbb{R}^D \,\otimes\, \mathbb{R}^D \;=\; \mathfrak{Sym}\big(\mathbb{R}^D\big) \,\oplus\, \mathfrak{Antisym}\big(\mathbb{R}^D\big) \;=\; \Sigma^2\big(\mathbb{R}^D\big) \,\oplus\, \Lambda^2\big(\mathbb{R}^D\big) \;.$$

Of the last two, rather more sources use the notation $\Lambda^2(\mathfrak{V})$ [11, 103, 109]. This is because the theory of forms enjoys many applications.

6.1.2 Further notations

Notational Remark 11 The symmetric and antisymmetric (pairwise) *Bach brackets* notations are as follows.

$$T_{(\,a\,b\,)} \;=\; \frac{1}{2}\,(\,T_{a\,b}\;+\;T_{b\,a}\,)\;.$$

$$T_{[\,a\,b\,]} \;=\; \frac{1}{2}\,(\,T_{a\,b}\;-\;T_{b\,a}\,)\;.$$

Notational Remark 12 Let us refer to further indices, slots or legs on an object that do not participate in its symmetries, as *spectator indices*.

$$T_{a\,\ldots\,b\,i\,j}$$

has a parallel symmetric–antisymmetric split for whichever of its index pairs, with the other indices cast as spectators. In this way, we return to Definition 1's more general form. Suppose that the indices in the pair are not consecutive. Then it is customary to write the Bach brackets with internal bars bounding the spectators, for instance as follows.

$$T_{a\,\ldots\,(\,b\,|\,i\,|\,j\,)}$$

Bach brackets efficientize the notation.

Notational Remark 13 See Fig 31 for tensor network versions of each equation and notation covered so far in the current Section. With straight slash through some legs for symmetrization. And jagged slash for antisymmetrization. Spectator legs are not slashed. Suppose that some such are trapped between legs that are to feature along the same slash. Then this can be handled by the unslashed legs being drawn shorter than the slashed ones. So that the slashes miss them.

Exercise 9 It is furthermore possible for an index to be a spectator as regards one slash while participating in another! Produce the minimum counterexample to the above leg shortening not sufficing to depict this. Can you think of a way by which Bach brackets can get round this counterexample? And for tensor networks?

Notational Remark 14 We can furthermore cast each last expression in terms of permutations as follows.

$$T_{(a\,b)} \;=\; \frac{1}{|S_2|}\,\sum_{s\,\in\,S_2}\,T_{s(a)\,s(b)}\;. \tag{57}$$

And

$$T_{[a\,b]} \;=\; \frac{1}{|S_2|}\,\sum_{s\,\in\,S_2}\,\sigma(s)\,T_{s(a)\,s(b)}\;. \tag{58}$$

In each case with reference to the permutation group on 2 objects, S_2 . While the second case also brings in the sign $\sigma(s)$ of each element in our permutation group.

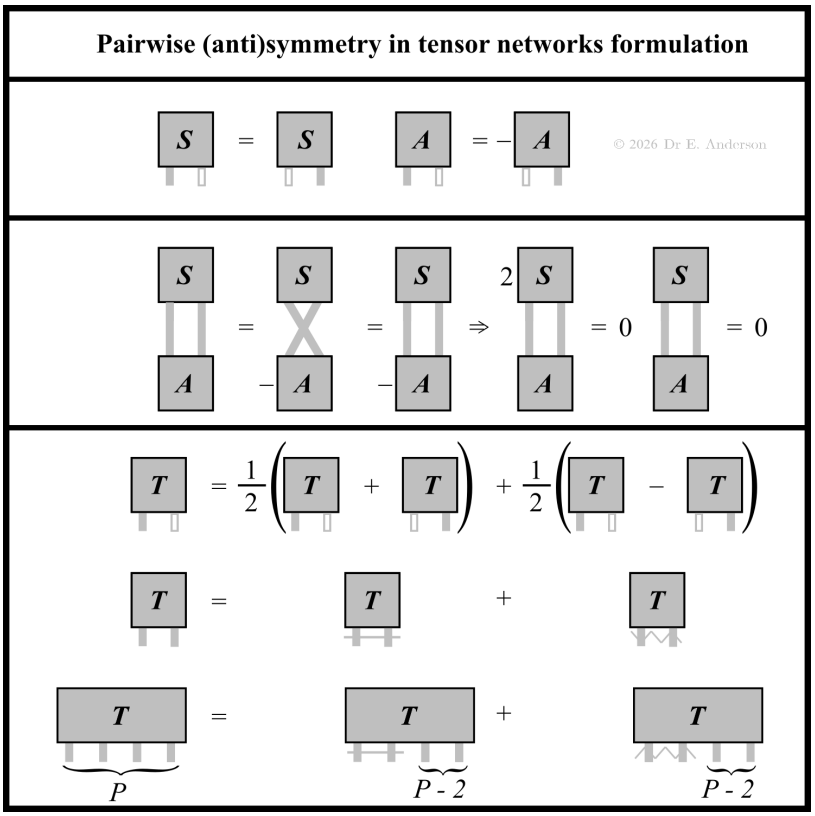


Figure 31:

6.1.3 The trace– tracefree and combined splits

Structure 1 The trace–tracefree split is as follows.

$$\boldsymbol{T} \;=\; \frac{T}{n}\,\mathbb{I} \;+\; \boldsymbol{T}^{\mathrm{T}}\;. \tag{59}$$

Thus inverting, an explicit form for the tracefree part is as follows.

$$\boldsymbol{T}^{\mathrm{T}} \;=\; \boldsymbol{T} \;-\; \frac{T}{n}\,\mathbb{I}\;. \tag{60}$$

Where $T \;=\; {}_{\mathrm{r}}\boldsymbol{T}$: the *trace* of $\boldsymbol{T}$. This is a map $\mathbb{R}^D \times \mathbb{R}^D \longrightarrow \mathbb{R}$.

Structure 2 The combined split – trace–tracefree split of the antisymmetric–symmetric split – is as follows.

$$\boldsymbol{T} \;=\; \boldsymbol{A} \;+\; \boldsymbol{S}^{\mathrm{T}} \;+\; \frac{T}{n}\,\mathbb{I}\;. \tag{61}$$

Notational Remark 15 See Fig 32 for tensor network versions of the current SSSec.

Exercise 10^{-} (square sums) Show that

$${}_{\mathrm{r}}\big(\boldsymbol{T}^2\big) \;=\; {}_{\mathrm{r}}\big(\boldsymbol{A}^2\big) \;+\; {}_{\mathrm{r}}\big(\boldsymbol{S}^2\big)\;.$$

$${}_{\mathrm{r}}\big(\boldsymbol{T}^2\big) \;=\; {}_{\mathrm{r}}\Big(\boldsymbol{T}^{\mathrm{T}\,2}\Big) \;+\; \frac{T^2}{n}\;.$$

$${}_{\mathrm{r}}\big(\boldsymbol{T}^2\big) \;=\; {}_{\mathrm{r}}\big(\boldsymbol{A}^2\big) \;+\; {}_{\mathrm{r}}\Big(\boldsymbol{S}^{\mathrm{T}\,2}\Big) \;+\; \frac{T^2}{n}\;.$$

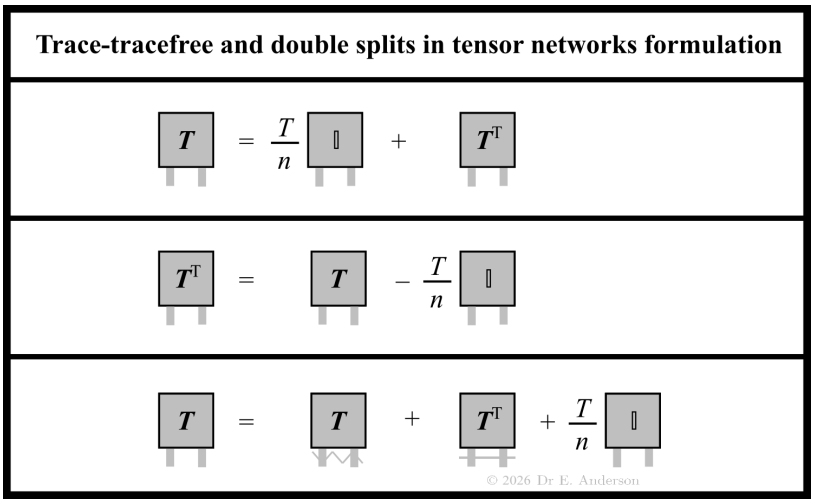


Figure 32:

6.2 A simpler extension

6.2.1 Minimum, total and partial (anti)symmetry

Remark 1 SSecs 5.1-2 considered the minimum nontrivial (anti)symmetrization: on 2 indices. One can however (anti)symmetrize larger sets of indices. If this is applied to all the indices present, it is total (anti)symmetrization In u.d.d. formulation , this is subject to the limitation that all the indices involved have to be on the same floor. Total (anti)symmetrization is the maximum possible amount and pairwise the minimum possible amount. Let us refer to all other instances as partial (anti)symmetrization.

Structure 1 The permutation group point of view provides a particularly clear picture of this generalization. In the total case, one passes from (57, 58) to the following.

$$T_{(\,a_1\,\ldots\,a_P\,)} \;=\; \frac{1}{|S_P|}\sum_{s\,\in\,S_P} T_{s(\,a_1)\,\ldots\,s(a_P)} \;=\; \frac{1}{P!}\sum_{s\,\in\,S_P} T_{s(\,a_1)\,\ldots\,s(a_P)}\;. \tag{62}$$

$$T_{[\,a_1\,\ldots\,a_P\,]} \;=\; \frac{1}{|S_P|}\sum_{s\,\in\,S_P} \sigma(s) T_{s(\,a_1)\,\ldots\,s(a_P)} \;=\; \frac{1}{P!}\sum_{s\,\in\,S_P} \sigma(s) T_{s(\,a_1)\,\ldots\,s(a_P)}\;. \tag{63}$$

While in the partial case with all M indices involved contiguous, say,

$$T_{(\,a_1\,\ldots\,a_M\,)\,a_{M\,+\,1}\,\ldots\,a_P\ldots} \;=\; \frac{1}{|S_M|}\sum_{s\,\in\,S_M} T_{s(\,a_1)\,\ldots\,s(a_P)\,a_{M\,+\,1}\,\ldots\,a_P} \;=$$

$$\frac{1}{M!}\sum_{s\,\in\,S_M} T_{s(\,a_1)\,\ldots\,s(a_P)\;a_{M\,+\,1}\,\ldots\,a_P}\;. \tag{64}$$

$$T_{[\,a_1\,\ldots\,a_M\,]\,a_{M\,+\,1}\,\ldots\,a_P\ldots} \;=\; \frac{1}{|S_M|}\sum_{s\,\in\,S_M} \sigma(s) T_{s(\,a_1)\,\ldots\,s(a_P)\,a_{M\,+\,1}\,\ldots\,a_P}$$

And

$$=\; \frac{1}{M!}\sum_{s\,\in\,S_M} \sigma(s) T_{s(\,a_1)\,\ldots\,s(a_P)\;a_{M\,+\,1}\,\ldots\,a_P}\;. \tag{65}$$

Example 3 The minimum example of total (anti)symmetry that is not just a pairwise (anti)symmetry is for $P\;=\;3$. Here

$$6\,T_{(\,a\,b\,c\,)} \;=\; \sum_{s\,\in\,S_3} T_{s(a)\,s(b)\,s(c)} \;=\; T_{a\,b\,c}\;+\;T_{b\,c\,a}\;+\;T_{c\,a\,b}\;+\;T_{a\,c\,b}\;+\;T_{b\,a\,c}\;+\;T_{c\,b\,a}\;. \tag{66}$$

$$6\,T_{[\,a\,b\,c\,]} \;=\; \sum_{s\,\in\,S_3} \sigma(s) T_{s(a)\,s(b)\,s(c)} \;=\; T_{a\,b\,c}\;+\;T_{b\,c\,a}\;+\;T_{c\,a\,b}\;-\;T_{a\,c\,b}\;-\;T_{b\,a\,c}\;-\;T_{c\,b\,a}\;. \tag{67}$$

Example 4 The minimum example of a partial (anti)symmetry that is neither pairwise nor total is for $P = 4$. Namely,

$$T_{(\,a\,b\,c\,)\,d} \;=\; \frac{1}{6}\sum_{s\,\in\,S_3} T_{s(a)\,s(b)\,s(c)\,d}\ . \tag{68}$$

$$T_{[\,a\,b\,c\,]\,d} \;=\; \frac{1}{6}\sum_{s\,\in\,S_3} \sigma(s) T_{s(a)\,s(b)\,s(c)\,d}\ . \tag{69}$$

And all other orderings of the index that is being left out, totalling 4 in each case.

Remark 2 See Fig 33 for a tensor networks rendition of the current Article.

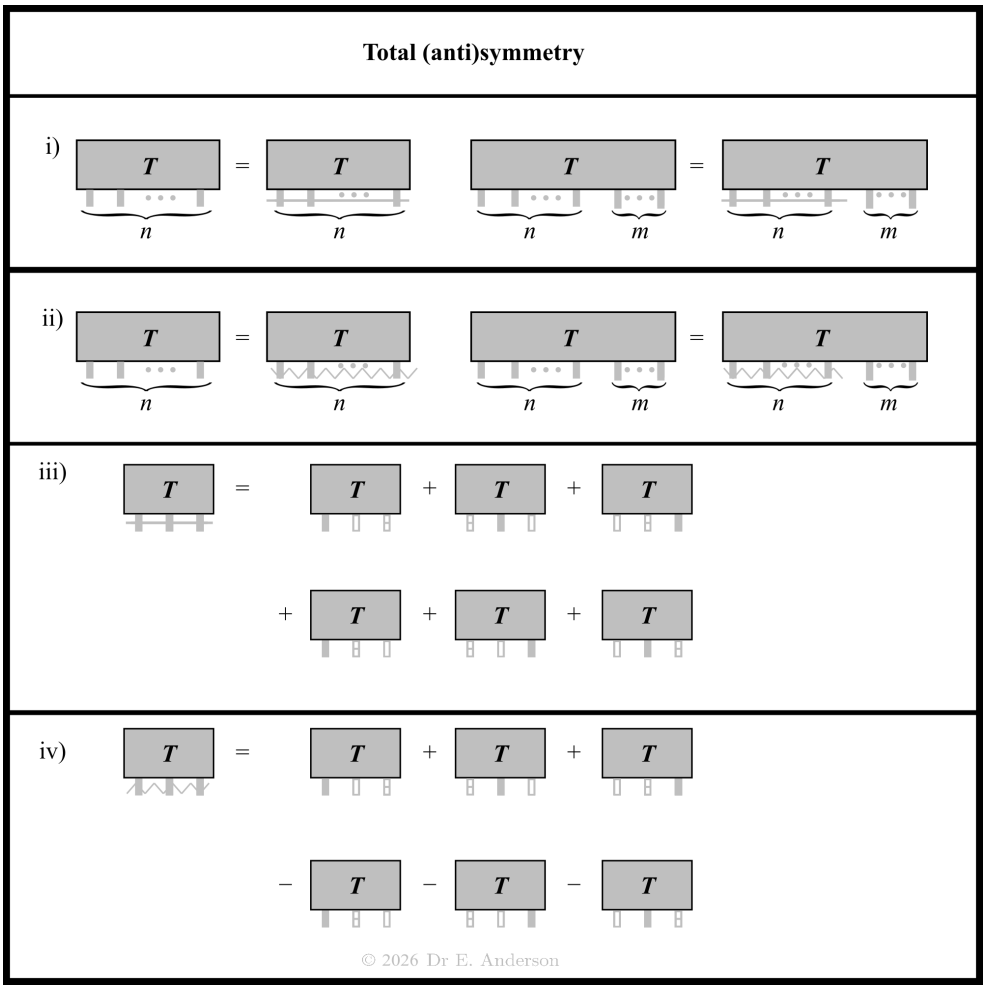


Figure 33:

6.2.2 Partition notation for symmetry type

Examples 0 and 1 The partition notation for the cases supporting no nontrivial symmetry between indices are as follows.

$$\emptyset$$

for scalars. And

$$\square$$

for vectors.

Example 2 The partition notation for pairwise (anti)symmetry is as follows.

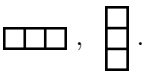


This suggests the following arena-level notation

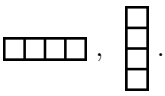
$$\mathfrak{V} \otimes \mathfrak{V} \;=\; \mathfrak{V}_{\square\square} \oplus \mathfrak{V}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \,. \tag{70}$$

Which then readily generalizes in parallel to the below developments.

Example 3 For 3-cycle symmetry,



Example 4 While for 4-cycle symmetry,



Structure 2 The vertical ones are the conjugate partitions of the horizontal ones of the same length. Using Appendix B's notation, we can jointly denote (anti)symmetry using

$$[\square\square] \;,\quad [\square\square\square] \;,\quad [\square\square\square\square] \;. \tag{71}$$

Remark 1 The current Section applies regardless of the notion of tensor involved. So long as all of the indices involved in an element of symmetry are of the same conceptual type, leaving them open to permutation. As regards the current Article's scope, then, the current section applies just as well to Cartesian pseudotensors as to Cartesian tensors.

Exercise 11 a) Show that in dimension D ,

$$\#(\square\square\square) \;=\; \frac{D(D+1)(D+2)}{3!} \;,\quad \#\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right) \;=\; \frac{D(D-1)(D-2)}{3!} \;,$$

$$\#(\square\square\square\square) \;=\; \frac{D(D+1)(D+2)(D+3)}{3!} \;,\quad \#\left(\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}\right) \;=\; \frac{D(D-1)(D-2)(D-3)}{3!} \;.$$

b) Prove the following.

Lemma 3 For rank $P := p + 1$

i)

$$\#\left(\underbrace{\square\ \cdots\ \square}_{P}\right) \;=\; \frac{D^{\overline{p}}}{P!} \;=\; \binom{D+p}{P} \;=\; \frac{1}{P!}\prod_{I=1}^{p} D + I \;. \tag{72}$$

ii)

$$\#\left(\begin{smallmatrix}\square\\\vdots\\\square\end{smallmatrix}P\right) \;=\; \frac{D^{\underline{p}}}{P!} \;=\; \binom{D}{P} \;=\; \frac{1}{P!}\prod_{I=1}^{p} D - I \;. \tag{73}$$

Where $D^{\overline{p}}$ and $D^{\underline{p}}$ are the *rising* and *falling factorials* respectively [59].

6.3 A harder extension: including mixed symmetries

6.3.1 Existence, minimum occurrence and persistence

Remark 1 That

$$\#(\square\square) \;+\; \#\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) \;=\; \#(\square^2)$$

underpins that

$$\square \;\otimes\; \square \;=\; \square\square \;\oplus\; \begin{smallmatrix}\square\\\square\end{smallmatrix} \;=\; [\square\square] \;. \tag{74}$$

These are of course reformulations of the basic split (56).

Remark 2 However, as regards

$$\bigotimes_{I\,=\,1}^{3} \square \,,$$

$$\#\left(\square\square\square\right) \;+\; \#\left(\begin{array}{|c|}\hline\;\;\\\hline\;\;\\\hline\;\;\\\hline\end{array}\right) \;<\; \#\left(\square^{3}\right) \,.$$

For

$$\frac{D\,(D\,+\,1)\,(D\,+\,2)}{3!} \;+\; \frac{D\,(D\,-\,1)\,(D\,-\,2)}{3!} \;=$$

$$\frac{D}{3}\left(\begin{array}{rcl} & D^2 \,+\, 3\,D \,+\, 2 \\ +\; & D^2 \,-\, 3\,D \,+\, 2 \end{array}\right) \;=\; \frac{D\,\left(D^2\,+\,2\right)}{3} \,.$$

And

$$D^3 \;-\; \frac{D}{3}\left(D^2 \,+\, 2\right) \;=\; \frac{D}{3}\left(3\,D^2 \,-\, D^2 \,-\, 2\right)$$

$$=\; \frac{2}{3}\,D\left(D^2 \,-\, 1\right) \;=\; \frac{2}{3}\,(D\,-\,1)\,D\,(D\,+\,1) \;<\; 0 \,.$$

Unless $D \,=\, 0$ or 1 . Neither of which however support nontrivial theories of tensors.

This points to the existence of $\geq\, 1$ mixed-symmetry species entering the following. The decomposition of our tensor triple product into a direct sum of irreducible pieces.

Remark 3 This existence of a nontrivial middle of mixed-symmetry species furthermore persists for all subsequent ranks. This is trivial from the partition representation. Alongside how fixed-N partition arenas for all subsequent N contain nontrivial middles [71].

All in all, only scalars, vectors and 2-tensors possess no mixed-symmetry parts entering the following.

$$(\text{ The } P\text{-tuple } \bigotimes \text{ of } \square\text{'s }) \text{ decomposition into } \bigotimes \text{ (irreducible pieces) .}$$

6.3.2 A start on determining the nature of the minimum example

Example $P \,=\, 3$ One further piece is supported by rank-3 tensors. In partition notation,

$$2\,1 \;=\; \begin{array}{|cc|}\hline\;\;&\;\;\\\hline\;\;\\\cline{1-1}\end{array} \;=\; v \,.$$

This is the minimum example of a mixed symmetry.

Exercise 12 a) The following split represents a correct if not yet symmetry-specifying formulation for a rank-3 tensor in 2-D .

$$T_{a\,b\,c} \;=\; T_{(a\,b\,c)} \;+\; T_{(a\,b\,c)\,=\,0} \,. \tag{75}$$

Investigate how many linearly-independent (LI) components each *word-type* – for us now $T_{a\,a\,a}$ and $T_{a\,a\,b}$ with no sums – contributes to each piece.

b) In $\geq$ 3-D, this split generalizes to

$$T_{a\,b\,c} \;=\; T_{(a\,b\,c)} \;+\; T_{[a\,b\,c]} \;+\; T_{\substack{(\;a\;b\;c\;)\,=\,0\\ [\;a\;b\;c\;]\,=\,0}} \,. \tag{76}$$

Which word-types are now supported, and how many LI components does each contribute to each piece?

c)$^{-}$ Write out each of the above pure pieces as vectors of matrices sporting a minimum number of LI components.

Exercise 13 Throughout this Exercise, work with a rank-3 tensor in 2-D. Trawl though all possible components to show that 1 antisymmetry is not enough to capture the mixed symmetry. While considering pairwise antisymmetry in each position in turn is too much. And that even after considering linear dependency (LD), this suggestion yields too many degrees of freedom (d.o.f.) and thus must be discarded.

6.3.3 Applying a subcase of a subcase of how to tensor Young diagrams

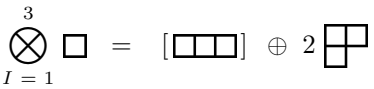
Remark 1 While we have so far been taking on faith that there is only 1 kind of mixed-symmetry piece for rank 3, a milder ambiguity impinges. Namely that 2 copies of  feature in the following. The triple tensor product of 's decomposition into a direct sum of irreducible pieces.

Remark 2 I.e.

(77)

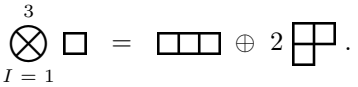
As follows from the simple rule for tensoring by atomos. Which Appendix B picks out as a subcase of Pieri's formula for tensoring by a 1-part partition. Which is in turn a subcase of the 'branching rule' conceptual type.

Remark 3 Applying Appendix B's furtherly shortened notation in terms of Young diagram conjugacy classes, (77) becomes the following.



(78)

Remark 4 But in 2-D, the following conjugation symmetry-breaking simplification occurs.



(79)

6.3.4 Determining the nature of the minimum example continued

Remark 1 Also Remark 2 of SSSec 6.3.1's count needs to be halved, yielding the following.

$$\#\left(\begin{array}{|c|c|}\hline \multicolumn{2}{|c|}{}\\\hline &\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\right) \;=\; \frac{D\left(D^2-1\right)}{3} \;=\; \frac{(D-1)D(D+1)}{3}\,. \tag{80}$$

Exercise 14 a) Show that the overlapping-index combination
i) antisymmetry in last pair

$$T_{abc} \;=\; T_{a[bc]}\,,$$

ii) symmetry in first pair

$$T_{abc} \;=\; T_{(ab)c}$$

is not a viable candidate for .

b) But that the combination
i) antisymmetry in last pair

$$T_{abc} \;=\; T_{a[bc]}\,, \tag{81}$$

ii) zero antisymmetric 3-cycle

$$T_{[abc]} \;=\; \mathbb{0}_{abc} \tag{82}$$

supports the correct number of components.

c) How about the symmetry–antisymmetry reversal of ii)? I.e.
i) symmetry in the last pair

$$T_{abc} \;=\; T_{a(bc)}\,. \tag{83}$$

ii) zero symmetric 3-cycle

$$T_{(abc)} \;=\; \mathbb{0}_{abc}\,. \tag{84}$$

Remark 2 See Fig 34 for a tensor networks rendition of b)'s pair of symmetries.

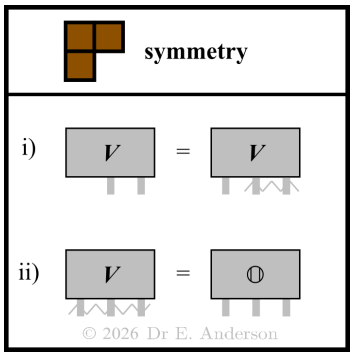
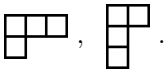


Figure 34:

6.3.5 Rank 4

Remark 1 Ensure that you have covered Exercise B.1 by this stage.

Example 4 Rank $P = 4$ is minimum to sport multiple types of mixed symmetry. Comprising 1 heteroconjugate pair,



(85)

And a rather more interesting example,



(86)

Among the famous Differential-Geometric tensors, the all-downstairs Riemann and Weyl curvature tensors enjoy this particular mixed symmetry.

Notational Remark 16 Following on from discussion in [29], the Kulkarni–Nomizu product encodes the same symmetry. These feature for instance in the 2 trace pieces by which the Weyl tensor differs from the Riemann tensor. The following related truer names and notation is suggested. *Square symmetry* and *square product*. after the corresponding Young diagram for the symmetry. To be denoted by this Young diagram itself. In a smaller size for the product for comensurability with other product- and more generally binary-relation-symbols.

The 2 correction terms in setting up the Weyl curvature tensor are then the following homogeneous and heterogeneous uses of the new symbol.

$$\boldsymbol{g} \boxtimes \boldsymbol{g} \ , \quad \boldsymbol{g} \boxtimes \boldsymbol{R} \ .$$

Each term furthermore comes with a dimension-dependent proportionality factor.] Observe that in rotating the square by $\frac{\pi}{4}$ radians, we not only get it to contain a boxed multiplication symbol. But also attain Appendix B's more advanced notation that is manifestly conjugation-invariant.

Remark 2  corresponds to the following symmetries. ‘

i) Pairwise antisymmetry.

$$T_{abcd} \ = \ T_{ab[cd]} \ .$$

ii) Index-pair exchange symmetry, with 1 of the exchanged pairs carrying the previous antisymmetry.

$$T_{abcd} \ = \ T_{cdab} \ .$$

iii) Zero 3-cycle condition.

$$T_{a[bcd]} \ = \ 0_{abcd} \ .$$

Which, in the context of the Riemann curvature tensor, is called the *first Bianchi identity* [17, 13]. See Fig 35 for a tensor networks rendition.

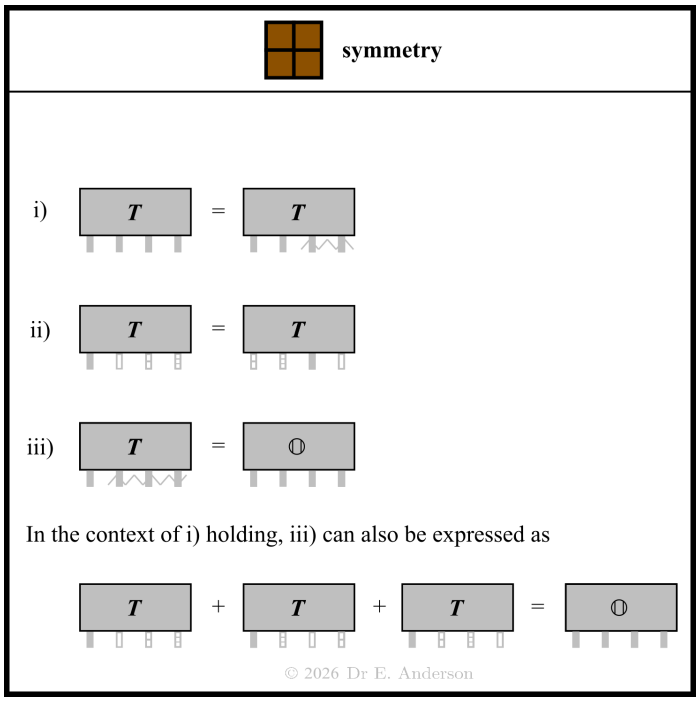


Figure 35:

6.3.6 Exercises

Exercise 15 Show that

$$\#\left(\square\!\square\right) \;=\; \frac{D^2\left(D^2\,-\,1\right)}{12} \;=\; \frac{(D\,-\,1)\,D^2\,(D\,+\,1)}{12}\,. \tag{87}$$

Exercise 16^{-} Show that

$$\#\left(\square\!\square\right) \overset{2\text{-}D}{=} 1\,, \tag{88}$$

$$\#\left(\square\!\square\!\square\right) \overset{2\text{-}D}{=} 3\,. \tag{89}$$

Exercise 17^{-} a) Find

$$\#\left(\square\!\square\right)$$

in 3- and 4-D . b) Show that

$$\#\left(\square\!\square\right) \overset{3\text{-}D}{=} \#\left(\square\!\square\right). \tag{90}$$

But

$$\#\left(\square\!\square\right) \overset{4\text{-}D}{>} \#\left(\square\!\square\right). \tag{91}$$

Pointer 1 (90) corresponds for instance to the 3-D Riemann curvature tensor containing no information in excess of its first contraction the Ricci curvature tensor. But not in 4-D by (91), so there is a strong sense in which $D = 4$ is minimum for distinctive study of the Riemann curvature tensor. Furthermore, the Weyl curvature tensor encodes this further information. So this remarkable object is realized in 4-D , but is just zero in 3-D . While 88 corresponds to the Ricci curvature tensor containing no more information than its own contraction the Ricci scalar.

Exercise 18 Show that in arbitrary-D ,

$$\#\left[\square\!\square\!\square\right] \;=\; \frac{D^2\left(D^2\,-\,1\right)}{6} \;=\; \frac{(D\,-\,1)\,D^2\,(D\,+\,1)}{6}\,. \tag{92}$$

Exercise 19 Find where among the 2-D $\square\!\square$'s components the single supported d.o.f. resides.

6.3.7 More advanced Exercises

Exercise 20 [71] (Extent of mixed domination)

Let $\mathfrak{p} := \#(\text{pure d.o.f.})$ and $\mathfrak{M} := \#(\text{mixed d.o.f.})$.

Show that for rank $P = 3$,

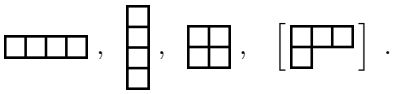
$$\begin{array}{rcl} \mathfrak{p} & = & \mathfrak{M} \quad \text{for} \quad D = 0 \quad \text{or} \quad 2 \\ \mathfrak{p} & > & \mathfrak{M} \quad \text{for} \quad D = 1 \\ \mathfrak{p} & < & \mathfrak{M} \quad \text{for} \quad D \geq 3 \qquad . \end{array} \tag{93}$$

While for $P = 4$,

$$\begin{array}{rcl} \mathfrak{p} & = & \mathfrak{M} \quad \text{for} \quad D = 0 \quad \text{or} \quad 1 \\ \mathfrak{p} & < & \mathfrak{M} \quad \text{for} \quad D \geq 2 \qquad . \end{array} \tag{94}$$

Remark 1 Thus failing to spot that mixed symmetries exist rapidly and permanently degenerates into missing *most of* general tensors' d.o.f.

Exercise 21 For rank-4 tensors, work out how many components each word-type contributes to each of the following.



In each of 2-, 3- and 4-D .

Pointer 2 On the one hand, the abovementioned Riemann to Weyl interconversion is a rank-4 example of the analogue of the trace-tracefree split. On the other hand, let us leave systematic consideration of rank-3 and -4 analogues of the trace-tracefree split to a later version of the current Review.

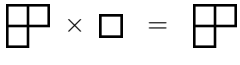
Exercise 22^{-} Extend the elementary course's result that pairwise (anti)symmetry is tensorial to arbitrary symmetry. Also establish that the value of the trace, and being tracefree, are tensorial.

6.4 Symmetries of u.d.d. tensors

Remark 1 Suppose that we are given a tensor with q upstairs and p downstairs indices. Then a priori we can only consider permuting upstairs indices separately from permuting downstairs indices. Thus each u.d.d. tensor carries an upstairs-index permutation-symmetry class and a separate downstairs one. This can be encoded as the direct alias Cartesian product of each's partition's Young diagram.

Remark 2 If a metric is furthermore present, then one is further entitled to place all the indices on the same floor. Be it by the metric's lowering property or by its inverse's raising property. Then one can permute whichever indices, and one is back to encoding these using a single Young diagram. At least in some settings, however, purely downstairs and 'mixed-floor' versions of the tensor as different objects. Indeed as characterized by their possessing distinct symmetries from each other.

Example 1 A common example on manifolds is that Metric Geometry's all-downstairs Riemann curvature tensor carries more symmetries than the following. Its Affinely-natural counterpart with 1 raised index. By  to  . In this case, since there is only 1 upstairs index, there is no room for upstairs to contribute a nontrivial factor to the above Cartesian product . Via



6.5 Interplay with quotient rule

6.5.1 Contractions with symmetric pieces

Corollary Q.2 i) Let $\boldsymbol{S}$ be a rank-2 tensor that is symmetric but elsewise arbitrary. Suppose that this is double-contracted with a candidate object $\boldsymbol{C}$. And that the outcome is a tensor $\boldsymbol{T}$. Then $\boldsymbol{C}$ symmetrized over its contracted pair of indices, which we denote by $\boldsymbol{SC}$, is itself a tensor.

ii) Let $\boldsymbol{A}$ be a rank-2 tensor that is antisymmetric but elsewise arbitrary. Suppose that this is double-contracted with a candidate object $\boldsymbol{C}$. And that the outcome is a tensor $\boldsymbol{T}$. Then $\boldsymbol{C}$ antisymmetrized over its contracted pair of indices is itself a tensor.

Proof i) See Fig 36 for a tensor networks rendition. We cannot immediately cancel off $\boldsymbol{S}$, since it is not arbitrary. But $\boldsymbol{SC}$ double-contracted into an antisymmetric but elsewise arbitrary tensor $\boldsymbol{A}$ is zero by symmetry–antisymmetry. So the given $\boldsymbol{SC}$ double-contracted into $\boldsymbol{S}$ is equal to $\boldsymbol{SC}$ double-contracted into $\boldsymbol{S} + \boldsymbol{A} = \boldsymbol{Arb}$. Whose second factor is indeed now arbitrary, and thus can be cancelled off. So the standard quotient theorem applies to $\boldsymbol{SC}$, giving the claimed result. □

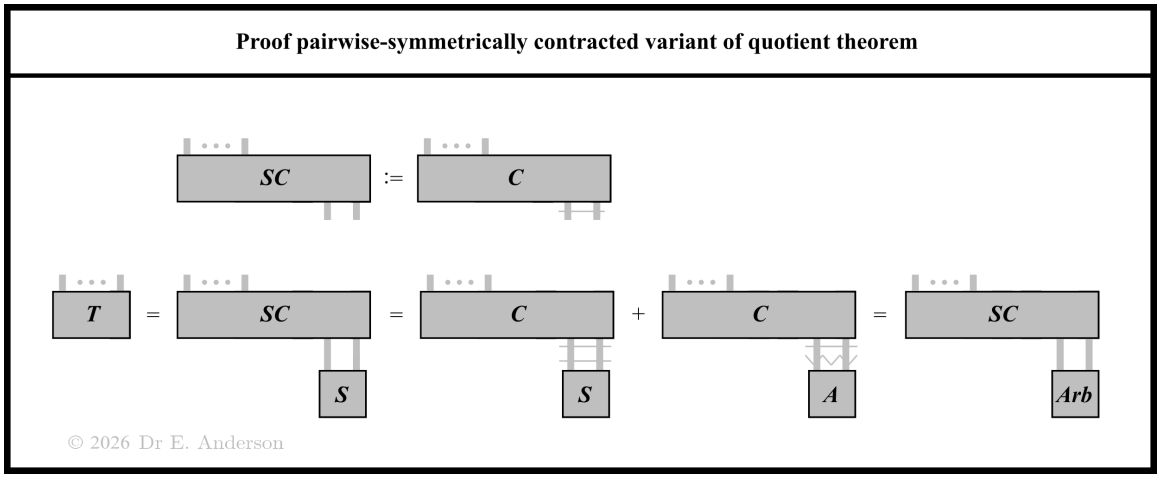


Figure 36:

Corollary Q.3 (with spectator indices) i) Let $\boldsymbol{S}$ be a rank-k tensor that is symmetric in 2 of its indices but elsewise arbitrary. Suppose that this is doubly-contracted with a candidate object $\boldsymbol{C}$. And that the outcome is a tensor $\boldsymbol{T}$. Then $\boldsymbol{C}$ symmetrized over its doubly-contracted indices is a tensor.

ii) The symmetry → antisymmetry counterpart of this also holds.

Exercise 23 Prove Corollaries Q.1.ii) and 2. Present your answers in tensor networks form.

Corollary Q.4 i) Let $\boldsymbol{S}$ be a rank-l tensor that is l-fold symmetric but elsewise arbitrary. Suppose that this is l-fold contracted with a candidate object $\boldsymbol{C}$. And that the outcome is a tensor $\boldsymbol{T}$. Then $\boldsymbol{C}$ symmetrized over its contracted l-fold of indices is a tensor.

ii) The symmetry → antisymmetry counterpart of this also holds.

iii) Both of these extend to include spectator indices as well.

Remark 1 From now on, we build in spectator indices without further comment.

Corollary Q.5 (general-symmetry extension of quotient theorem for tensors) Suppose that a tensor $\boldsymbol{M}$ carries the Young diagram $Y(\mathrm{p}(N))$ of symmetries. And has an l-subset of

these indices contracted into a candidate object $\boldsymbol{C}$. Let this l-subset furthermore realize the Young subdiagram $Y(\,\mathrm{p}(l)\,)$. Finally suppose that our product $\boldsymbol{M}\ \textcircled{l}\ \boldsymbol{C}$ is a tensor. Then the $Y(\,\mathrm{p}(l)\,)$-symmetrization of the contracted indices on $\boldsymbol{C}$ is itself a tensor.

Exercise 24^{+} Prove Corollaries Q.3-4. Present your answers in tensor networks form.

6.5.2 Interplay with trace–tracefree split

Corollary Q.6 (trace-tracefree case) a) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the tracefree index pair of an elsewise arbitrary tensor $\boldsymbol{\mathcal{T}}$. So as to form a tensor $\boldsymbol{T}$. Then

$$\boldsymbol{C}^{\mathrm{T}} \ := \ (\text{ tracefree part over an index pair of } \boldsymbol{C}\ )$$

is itself a tensor.

b) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the trace term's index pair of an elsewise arbitrary tensor $\mathbb{I}\,\boldsymbol{\mathcal{F}}$. Where $\boldsymbol{\mathcal{F}}$ carries purely spectator indices. So as to form a tensor $\boldsymbol{T}$. Then

$$\boldsymbol{C}^{\mathrm{t}} \ := \ (\text{ trace part over an index pair of } \boldsymbol{C}\ )$$

is itself a tensor.

Corollary Q.7 (double-split case) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the symmetric-tracefree index pair of an elsewise arbitrary tensor $\boldsymbol{\mathcal{S}}^{\mathrm{T}}$. So as to form a tensor $\boldsymbol{T}$. Then

$$\boldsymbol{C}^{\mathrm{ST}} \ := \ (\text{ tracefree part over an index pair of } \boldsymbol{C}\ )$$

is itself a tensor.

Remark 1 This result superposes 2 different types of split: symmetric-antisymmetric and trace-tracefree. It has no new partner results. For the antisymmetric partner works with just a symmetric concomitant piece. And so is covered by corollary Q.3. And the trace partner works with just an unsplit trace-free concomitant piece. And so is covered by corollary Q.6.

6.5.3 Interplay with pseudotensors

Corollary Q.7 (general-symmetry extension of quotient theorem for pseudotensors)

i) Suppose that a tensor $\boldsymbol{M}$ carries the Young diagram $Y(\,\mathrm{p}(N)\,)$ of symmetries. And has an l-subset of these indices l-fold contracted into a candidate object $\boldsymbol{C}$. Let this l-subset furthermore realize the Young subdiagram $Y(\,\mathrm{p}(l)\,)$. Finally suppose that our product $\boldsymbol{M}\ \textcircled{l}\ \boldsymbol{C}$ is a pseudotensor. Then the $Y(\,\mathrm{p}(l)\,)$-symmetrization of the contracted indices on $\boldsymbol{C}$ is itself a pseudotensor.

ii) Suppose that a pseudotensor $\boldsymbol{M}$ carries the Young diagram $Y(\,\mathrm{p}(N)\,)$ of symmetries. And has an l-subset of these indices l-fold contracted into a candidate object $\boldsymbol{C}$. Let this l-subset furthermore realize the Young subdiagram $Y(\,\mathrm{p}(l)\,)$. Finally suppose that our product $\boldsymbol{M}\ \textcircled{l}\ \boldsymbol{C}$ is a tensor. Then the $Y(\,\mathrm{p}(l)\,)$-symmetrization of the contracted indices on $\boldsymbol{C}$ is itself a pseudotensor.

iii) Suppose that a pseudotensor $\boldsymbol{M}$ carries the Young diagram $Y(\,\mathrm{p}(N)\,)$ of symmetries. And has an l-subset of these indices l-fold contracted into a candidate object $\boldsymbol{C}$. Let this l-subset furthermore realize the Young subdiagram $Y(\,\mathrm{p}(l)\,)$. Finally suppose that our product $\boldsymbol{M}\ \textcircled{l}\ \boldsymbol{C}$ is a pseudotensor. Then the $Y(\,\mathrm{p}(l)\,)$-symmetrization of the contracted indices on $\boldsymbol{C}$ is itself a tensor.

Corollary Q.8 a).i) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the tracefree index pair of an elsewise arbitrary tensor $\mathcal{T}$. So as to form a pseudotensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{T}}$ is itself a pseudotensor.

ii) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the tracefree index pair of an elsewise arbitrary pseudotensor $\mathcal{T}$. So as to form a tensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{T}}$ is itself a tensor.

iii) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the tracefree index pair of an elsewise arbitrary pseudotensor $\boldsymbol{\mathcal{T}}$. So as to form a pseudotensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{T}}$ is itself a tensor.

b).i) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the trace term's index pair on an elsewise arbitrary tensor $\mathbb{I}\,\boldsymbol{\mathcal{F}}$. Where $\boldsymbol{\mathcal{F}}$ carries purely spectator indices. So as to form a pseudotensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{t}}$ is itself a pseudotensor.

ii) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the trace term's index pair on an elsewise arbitrary pseudotensor $\mathbb{I}\,\boldsymbol{\mathcal{F}}$. So as to form a tensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{t}}$ is itself a pseudotensor.

iii) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the trace term's index pair on an elsewise arbitrary pseudotensor $\mathbb{I}\,\boldsymbol{\mathcal{F}}$. So as to form a pseudotensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{t}}$ is itself a tensor.

c) i) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the symmetric-tracefree index pair of an elsewise arbitrary tensor $\boldsymbol{\mathcal{S}}^{\mathrm{T}}$. So as to form a pseudotensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{ST}}$ is itself a pseudotensor.

ii) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the symmetric-tracefree index pair of an elsewise arbitrary pseudotensor $\boldsymbol{\mathcal{S}}^{\mathrm{T}}$. So as to form a tensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{ST}}$ is itself a pseudotensor.

iii) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the symmetric-tracefree index pair of an elsewise arbitrary pseudotensor $\boldsymbol{\mathcal{S}}^{\mathrm{T}}$. So as to form a pseudotensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{ST}}$ is itself a tensor.

Exercise 25 [Long] Prove Corollaries Q.7–8 piecemeal. Present your answers in tensor networks form.

6.5.4 Parity reformulation of four quartets of quotient theorems and corollaries

Remark 1 Entertaining the possibility of pseudotensoriality in the quotient theorem for Cartesian tensors now involves all 4 distinct parity cases. For, as compared with the previous product case, the quotient makes heterogeneous use of its inputs: outer product and arbitrary factor. In the order used in SSSec 5.3.2, the first is tensorial, while the others are what we previously labelled as cases i) to iii).

Notational Remark 17 Indeed truer notation for these cases follows from their inputs: $+-$, $-+$ and $--$ respectively.

Theorem Q$^\sharp$ (Parity unification of Cartesian (pseudo)tensor quotient theorem and corollaries) Suppose that a Cartesian parity-t tensor $\boldsymbol{T}$ is the outer product of the following. An arbitrary Cartesian parity-a tensor $\boldsymbol{A}$ And a candidate object $\boldsymbol{C}$. Then $\boldsymbol{C}$ is itself a Cartesian parity-$a\,t$ tensor.

Corollary Q.7$^\sharp$ (Parity unification of Cartesian (pseudo)tensor quotient theorem's general-symmetry extension)

Suppose that a Cartesian parity-m tensor $\boldsymbol{M}$ carries the Young diagram $Y(\,\mathrm{p}(N)\,)$ of symmetries. And has an l-subset of these indices l-fold contracted into a candidate object $\boldsymbol{C}$. Let this l-subset furthermore realize the Young subdiagram $Y(\,\mathrm{p}(l)\,)$. Finally suppose that our product $\boldsymbol{T} = \boldsymbol{M} \,\textcircled{l}\, \boldsymbol{C}$ is a Cartesian parity-t tensor. Then the $Y(\,\mathrm{p}(l)\,)$-symmetrization of the contracted indices on $\boldsymbol{C}$ is itself a Cartesian parity $m\,t$-tensor.

Corollary Q.8$^\sharp$ a) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the tracefree index pair of an elsewise arbitrary parity-a tensor $\boldsymbol{\mathcal{A}}$. So as to form a parity-t tensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{T}}$ is itself a parity-$a\,t$ tensor.

b) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the trace term's index pair of an elsewise arbitrary parity-b tensor $\mathbb{I}\,\boldsymbol{\mathcal{B}}$. Where $\boldsymbol{\mathcal{B}}$ carries purely spectator indices. So as to form a parity-t tensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{t}}$ is itself a parity-$b\,t$ tensor.

c) i) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the symmetric-tracefree index pair of an elsewise arbitrary parity-s tensor $\boldsymbol{\mathcal{S}}^{\mathrm{T}}$. So as to form a parity-t tensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{ST}}$ is itself a parity-$s\,t$ tensor.

Exercise 26 Prove this SSec's results in a unified manner. Present your answers in tensor networks form.

Exercise 27 Work out the arenas involved in each result.

7 Tensor fields and Cartesian Tensor Calculus

7.1 Tensor fields and tensor bundles

7.1.1 Introduction

Remark 1 View what we have covered so far as the constant or pointwise case. In contrast with the following.

Structure F A *tensor field* over a Mathematical space $\mathfrak{X}$ is a tensor defined pointwise over that space, i.e permitting values to differ from point to point. Compare the more basic and consequently more widely encountered *scalar field* and *vector field*.

Motivation 1 We need tensor fields to describe much of Differential Geometry and Physics. Entertaining tensor fields supports not only Tensor Algebra but also Tensor Calculus [7, 22, 17].

Remark 2 For 0-D, in the connected case of a single point, there is no room for tensor fields to differ from pointwise tensors. Since there is also no room for distinct tensor ranks, this collapses to scalar fields being no different from scalars in connected 0-D.

7.1.2 Scalar and vector field arenas

Remark 1 Let us consider these first, for their simplifying feature of not supporting any plain to u.d.d. ambiguity.

Example 0 The arena corresponding to a $\mathbb{R}$-valued scalar field on a space $\mathfrak{X}$ is some function space over $\mathfrak{X}$. For instance, picking the *smooth* (= infinitely differentiable) functions,

$$\mathfrak{Scalar Field}(\mathfrak{X}) := \mathfrak{C}^\infty(\mathfrak{X}, \mathbb{R}) . \tag{95}$$

Then setting $\mathfrak{X} = \mathbb{R}^D$ this specializes to the following arena.

$$\mathfrak{Scalar Field}\left(\mathbb{R}^D\right) := \mathfrak{Scalar Field}(D, 0) = \mathfrak{C}^\infty\left(\mathbb{R}^D, \mathbb{R}\right) . \tag{96}$$

For scalar fields, it makes little difference which type of transformation-group notion is in play. One could use

$$\mathfrak{pseudoScalar Field}$$

to render explicit the discrete reflection ambiguity.

Example 1 With vectors, however, one needs to be more specific. For now some particular group's transformation matrix is to feature in the corresponding tensor transformation law. So specifically the Cartesian vector arena of fixed dimension is

$$\mathfrak{Vector Field}\left(\mathbb{R}^D\right) := \mathfrak{Vector Field}(D) = \mathfrak{C}^\infty\left(\mathbb{R}^D; \mathbb{R}^D\right) . \tag{97}$$

This dependency notation indicates that $\mathbb{R}^D$ carries a $\mathbb{R}^D$ vector space's worth of vectors.

Remark 1 If one did not exhibit this full argumentation, then one would be writing the same function space notation for 2 distinct function spaces. Which could well lead to confusion and errors...

Remark 2 Most sources do not make this second argument explicit. Nor do they mention how this second argument keeps scalars and vectors distinct. Nor explain the empty product. The vectorial case is widely selected for its relation to derivatives, via the tangent vectors to a curve. This is the first part of how Tensor Calculus on manifolds [12, 17, 18] goes about acquiring a notion of derivative.

Remark 3 The above choices of function space, and many others, are vector spaces. They are furthermore infinite-dimensional.

7.1.3 Continuing in tensor field arena notation

Notational Remark 18 Let us now rewrite the scalar field arena (96) as

$$\mathfrak{Tensor Field}\left(\mathbb{R}^D,\,0\right) \;:=\; \mathfrak{Tensor Field}(\,D,\,0\,) \;=\; \mathfrak{C}^\infty\left(\mathbb{R}^D,\,\mathbb{R}\right)\,. \tag{98}$$

And the plain vector field arena (97) as

$$\mathfrak{Tensor Field}\left(\mathbb{R}^D,\,1\right) \;:=\; \mathfrak{Tensor Field}(\,D,\,1\,) \;=\; \mathfrak{C}^\infty\left(\mathbb{R}^D,\,\mathbb{R}^D\right)\,. \tag{99}$$

While the u.d.d. version is

$$\mathfrak{Tensor Field}\left(\mathbb{R}^D,\,1,\,0\right) \;:=\; \mathfrak{Tensor Field}(\,D,\,1,\,0) \;=\; \mathfrak{C}^\infty\left(\mathbb{R}^D,\,\mathbb{R}^D\right)\,. \tag{100}$$

Example $(0,\,1)$ Whereas the covector field arena is

$$\mathfrak{Tensor Field}\left(\mathbb{R}^D,\,0,\,1\right) \;:=\; \mathfrak{Tensor Field}(\,D,\,1\,) \;=\; \mathfrak{C}^\infty\left(\mathbb{R}^D,\,\mathbb{R}^{D^*}\right)\,. \tag{101}$$

Remark 1 $\mathfrak{C}^{\infty\,*}$ is well-known not to be isomorphic to $\mathfrak{C}^\infty$. But what we are actually dualizing is an input finite vector space, which is isomorphic to its dual. So whether plain or u.d. tensors are in use has no effect on the output function space for vector fields. Including whether one picks the upstairs vector or the downstairs covector to be the re-encodement of the plain vector.

7.1.4 Arenas of tensor fields of arbitrary fixed rank

Remark 1 In the plain formulation,

$$\mathfrak{Tensor Field}(\,D,\,P\,) \;=\; \prod_{i\,=\,1}^{P}\,\mathfrak{C}^\infty\left(\mathbb{R}^D,\,\mathbb{R}^D\right)\,. \tag{102}$$

Where we are building up the rank-P tensor field by tensoring up (see Sec 5.1.2) P vector fields. And where the empty product corresponding to the scalar is set to return (96).

Remark 2 While in the u.d.d. formulation,

$$\mathfrak{Tensor Field}(\,D,\,q,\,p\,) \;:=\; \prod_{i\,=\,1}^{q}\,\mathfrak{C}^\infty\left(\mathbb{R}^D,\,\mathbb{R}^D\right) \;\times\; \prod_{i\,=\,1}^{p}\,\mathfrak{C}^\infty\left(\mathbb{R}^D,\,\mathbb{R}^{D\,*}\right) \;\cong\; \prod_{i\,=\,1}^{T}\,\mathfrak{C}^\infty\left(\mathbb{R}^D,\,\mathbb{R}^D\right)\,.$$

7.1.5 (Co)tangent bundle approach to the above arenas

Remark 1 First courses on Cartesian tensors do not usually mention the arenas that tensors of each rank form. Second or third courses including tensors are often on Differential Geometry or General Relativity. These often do mention such arenas, but for a general manifold, as we present below. The current SSSec's idea is to restrict these to $\mathbb{R}^D$ with just orthogonal transformations so as to recover arenas of Cartesian tensors within a much better-known setting.

Structure 1 [17, 18] At a point $\mathrm{p} \in \mathfrak{m}$, the vectors form the *tangent space*

$$\mathfrak{T}_{\mathrm{p}}\,\mathfrak{m}\,. \tag{103}$$

Next collecting up all of the manifold's points, we arrive at the *tangent bundle*

$$\mathfrak{T}\,\mathfrak{m} \;=\; \bigcup_{p\,\in\,\mathfrak{m}}\,\mathfrak{T}_{\mathrm{p}}\mathfrak{m}\,. \tag{104}$$

While the covectors form the *cotangent space*

$$\mathfrak{T}^*_{\mathrm{p}}\,\mathfrak{m}\,. \tag{105}$$

Which is the dual space of the tangent space viewed as a vector space. And is isomorphic to this vector space itself for finite-D $\mathfrak{m}$. Also now collecting up all of the manifold's points, we arrive at the *cotangent bundle*

$$\mathfrak{T}^{*}\,\mathfrak{m} \;=\; \bigcup_{p\,\in\,\mathfrak{m}} \mathfrak{T}^{*}_{\mathrm{p}}\mathfrak{m}\;. \tag{106}$$

Next, tensor up q copies of the tangent space and p copies of the cotangent space. So as to obtain the following arena of tensors at a point.

$$\bigotimes_{i\,=\,1}^{q} \mathfrak{T}_{\mathrm{p}}\mathfrak{m} \;\otimes\; \bigotimes_{i\,=\,1}^{p} \mathfrak{T}^{*}_{\mathrm{p}}\mathfrak{m}\;. \tag{107}$$

The corresponding arena of rank-$(\,q,\,p\,)$ tensor fields is then

$$\bigotimes_{i\,=\,1}^{q} \mathfrak{T}\mathfrak{m} \;\otimes\; \bigotimes_{i\,=\,1}^{p} \mathfrak{T}^{*}\mathfrak{m}\;. \tag{108}$$

Each of the above bundles is furthermore a fibre bundle. Finally conflate q and p contributions to recover the general plain case.

7.1.6 With sections recovering individual tensor fields

Remark 1 In this more Mathematically sopishticated modelling, a real scalar field means a *section*

$$\phi \;:\; \mathfrak{m} \;\longrightarrow\; \mathfrak{m}(\mathbb{R})\;. \tag{109}$$

Each smooth real such is a function pertaining to

$$\mathfrak{C}^{\infty}(\,\mathfrak{m},\mathfrak{m}(\mathbb{R})\,)\;. \tag{110}$$

I.e. assigning a unique real value to each point of our manifold. (110) is then a (name-providing) notation for the arena of scalar fields. As restricted to the Differential-Geometrically useful smooth subcase.

A real vector field is a section

$$\mathcal{V} \;:\; \mathfrak{m} \;\longrightarrow\; \mathfrak{T}\mathfrak{m}\;. \tag{111}$$

Now pertaining to

$$\mathfrak{C}^{\infty}(\,\mathfrak{m},\mathfrak{T}\mathfrak{m}(\mathbb{R})\,)\;. \tag{112}$$

And so on for each rank of tensor field.

7.1.7 Route 2 to Cartesian tensor fields and their arenas

Remark 1 To get back to the current Section's incipient route, set $\mathfrak{m}^{D} \;=\; \mathbb{R}^{D}$ throughout. Use that now

$$\mathfrak{T}_{\mathrm{p}}\,\left(\,\mathbb{R}^{D}\,\right) \;\;\cong\;\; \mathbb{R}^{D}\;. \tag{113}$$

Also that the more general structure group collapses to $O(D)$. And a bit of poetic license with the notation. Remembering that our second much more widely known approach pins down precisely what everything means.

7.2 Interplay with CTAs

Remark 1 In the arena of scalar fields in all dimensions, all the attached objects are consequently distinct. Thus among tensor fields, it is scalar fields that are minimum to exhibit general bundle behaviour!

Structure 1 We can furthermore place objects over 2 base spaces at once. Firstly, the model of space $\mathbb{R}^{D}$ and secondly the discrete CTA model. The next sub-version of this Article shall draw a picture of a small and yet suitably nontrivial example of this... 37 As well as providing the missing 4-part figure mosaic for the Subsection outlining Bundle Theory...

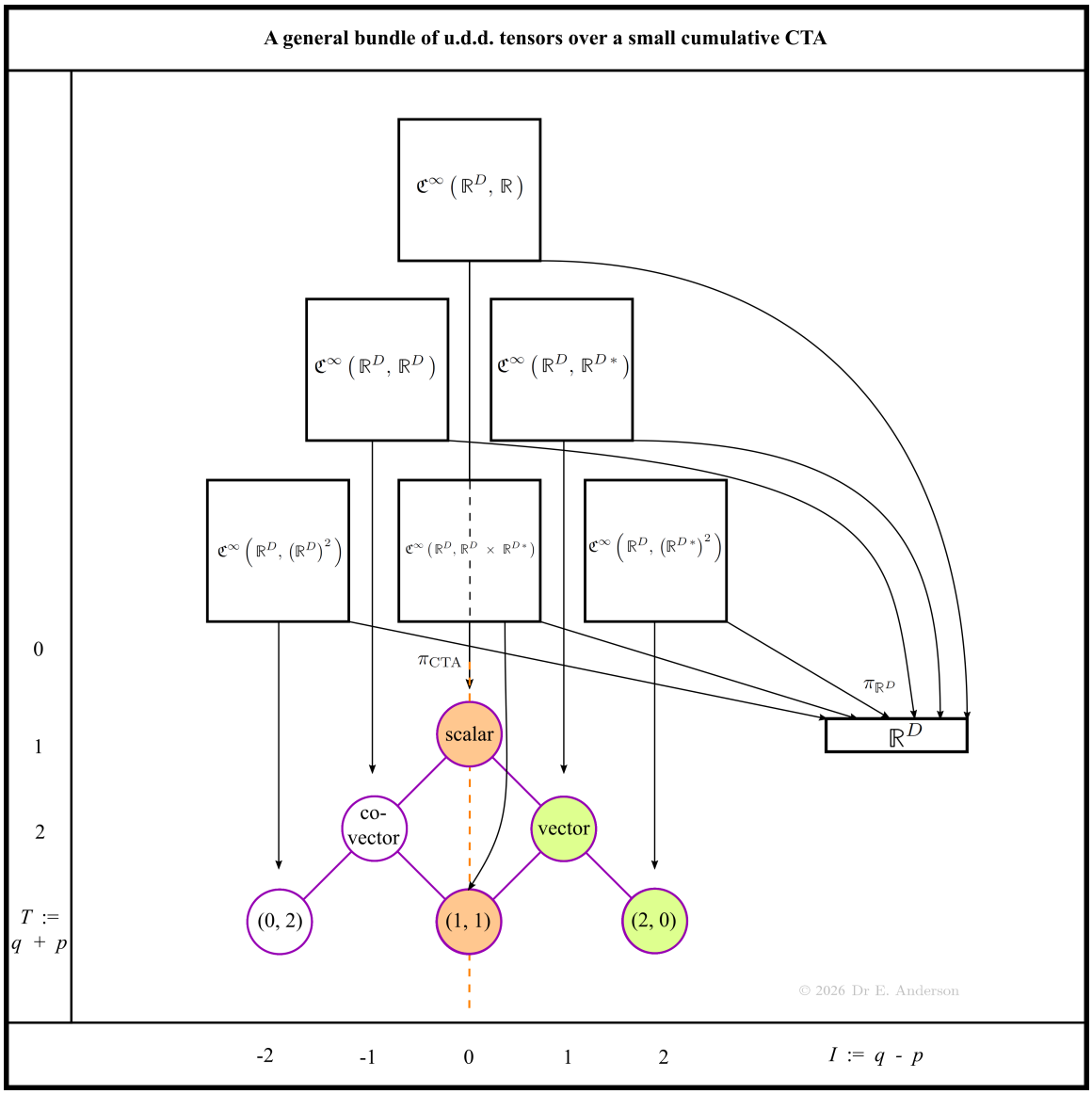

A general bundle of u.d.d. tensors over a small cumulative CTA
$\mathfrak{C}^\infty\left(\mathbb{R}^D, \mathbb{R}\right)$
$\mathfrak{C}^\infty\left(\mathbb{R}^D, \mathbb{R}^D\right)$
$\mathfrak{C}^\infty\left(\mathbb{R}^D, \mathbb{R}^{D*}\right)$
$\mathfrak{C}^\infty\left(\mathbb{R}^D, \left(\mathbb{R}^D\right)^2\right)$
$\mathfrak{C}^\infty\left(\mathbb{R}^D, \mathbb{R}^D \times \mathbb{R}^{D*}\right)$
$\mathfrak{C}^\infty\left(\mathbb{R}^D, \left(\mathbb{R}^{D*}\right)^2\right)$
π_{CTA}
$\pi_{\mathbb{R}^D}$
$\mathbb{R}^D$
scalar
co-vector
vector
(0, 2)
(1, 1)
(2, 0)
0
1
2
$T := q + p$
-2
-1
0
1
2
$I := q - p$
© 2026 Dr E. Anderson

Figure 37:

7.3 Cartesian Tensor Calculus

7.3.1 Grad is a covector

Remark 1 By the chain rule, grad is a covector. And so it transforms as per Fig 38.a).

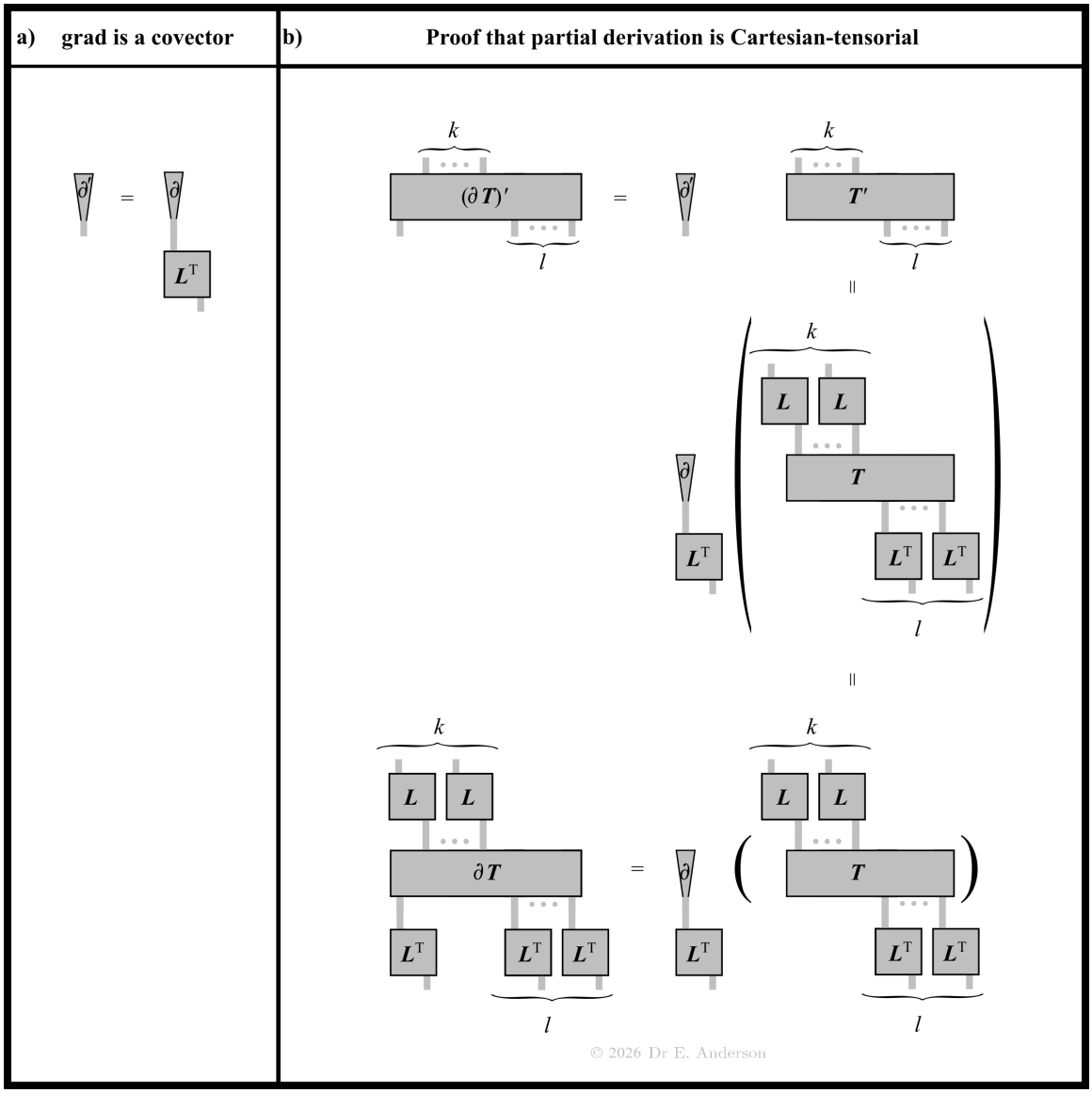


Figure 38:

7.3.2 Partial derivation is tensorial

Proposition 9 Partial derivation is Cartesian-tensorial.

Proof This is depicted in Subfig b) in tensor networks notation. In the first step, we unpack partial derivation in the primed frame. In the second, we apply each object's Cartesian tensor transformation laws. In the third, we can pull the transformation matrices outside $\boldsymbol{\nabla}$ since they are constant. Thus realizing an easy subcase of the product rule. In the final step, we reform partial derivation in the unprimed frame. □

7.3.3 Cartesian Tensor Calculus counterpart of the quotient theorem

Proposition 10$^\sharp$ Suppose that the partial derivative of a candidate object is a Cartesian s-parity tensor. Then the candidate object is itself a Cartesian s-parity tensor for some choice of constant of integration.

Exercise 28 Prove this. Also establish its counterpart for tensor divergence rather than tensor gradient. Finally split each of these two results up into Cartesian tensor and pseudotensor statements.

7.3.4 Tensor Calculus subtleties

Pointer 1 $GL(D)$ tensors share the above good fortunes. Curvilinear and curved ones do not. The partial derivative is here only tensorial when acting upon the scalar, thus avoiding any transformation matrices entering...

The *Affine connection* however cancels out this nontensoriality, yielding a tensorial *covariant derivative*.

Forming this derivative amounts to parallel transport of vectors.

It induces a covariant derivative on covectors.

Finally, the (co)vector covariant derivatives team up to induce covariant derivatives of arbitrary-rank tensors.

Pointer 2 In contrast, the *Lie derivative* [18, 5] involves no Affine structure. At the cost of introducing a vector field for the Lie derivative to Lie-drag along.

8 Plain tensor networks

8.1 Tensor transformation laws

Remark 1 See Fig 39 for the corresponding tensor transformation laws.

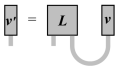
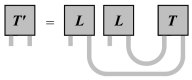
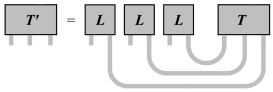
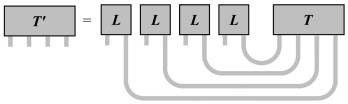
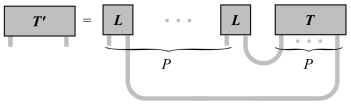
rank and name	Plain tensor transformation laws	name
0 scalar		
1 vector		
2 2-tensor		
3 3-tensor		
4 4-tensor		
⋮	⋮	
P P-tensor		P-pseudotensor

Figure 39:

8.2 Tensor Algebra

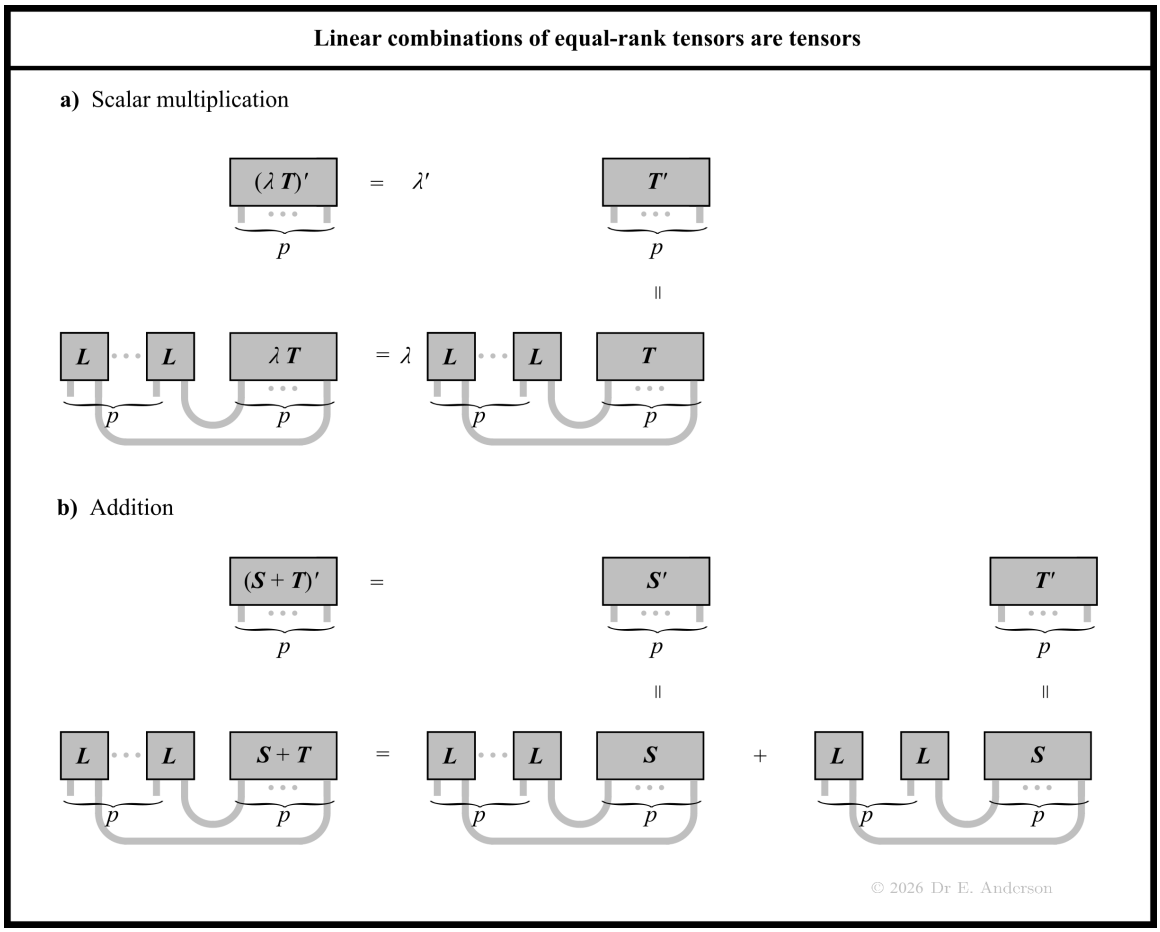


Figure 40:

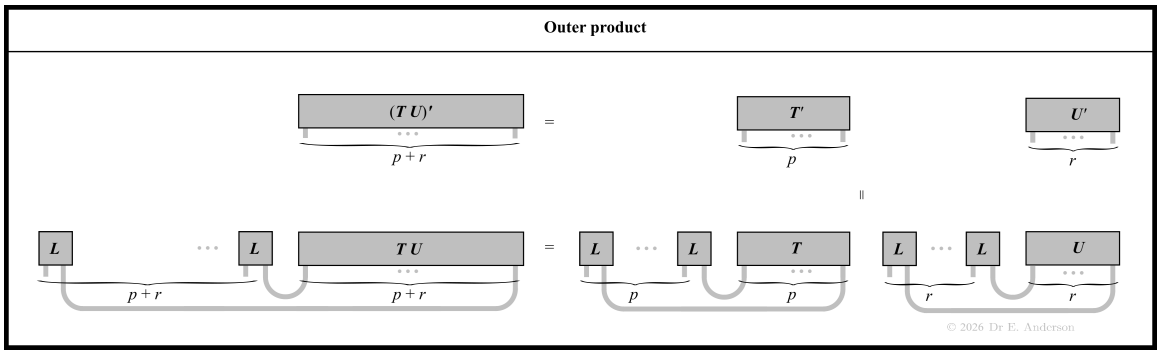


Figure 41:

Remark 1 See Fig 40 for the corresponding proof that tensoriality is linear. Fig 41 for its outer-productivity. Fig 42 for its inner-productivity in the single-contracted case. Fig 43 for its single internal contraction. And Fig 44 for its quotient theorem.

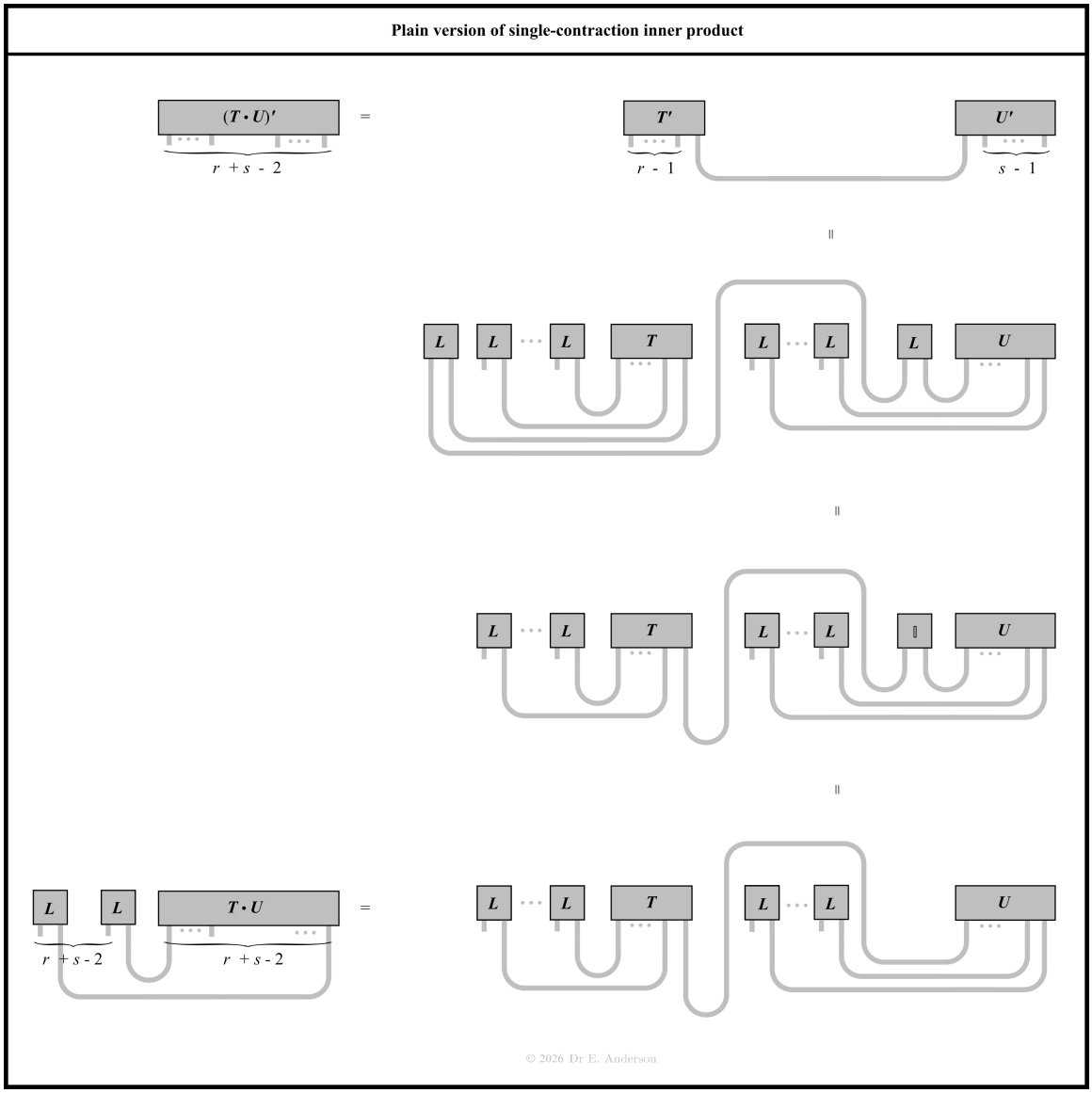


Figure 42:

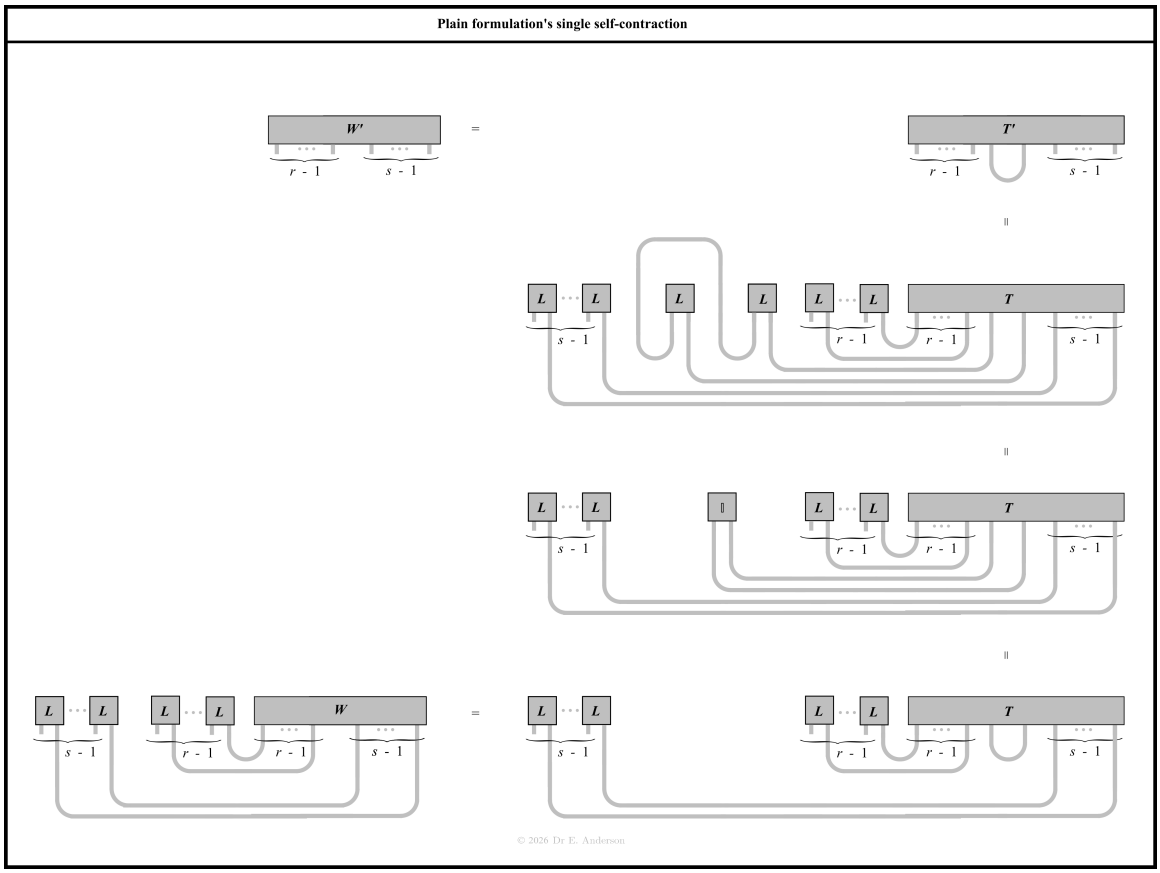

Plain formulation's single self-contraction
W′
=
r - 1
s - 1
T′
L
T
W
=
© 2026 Dr E. Anderson

Figure 43:

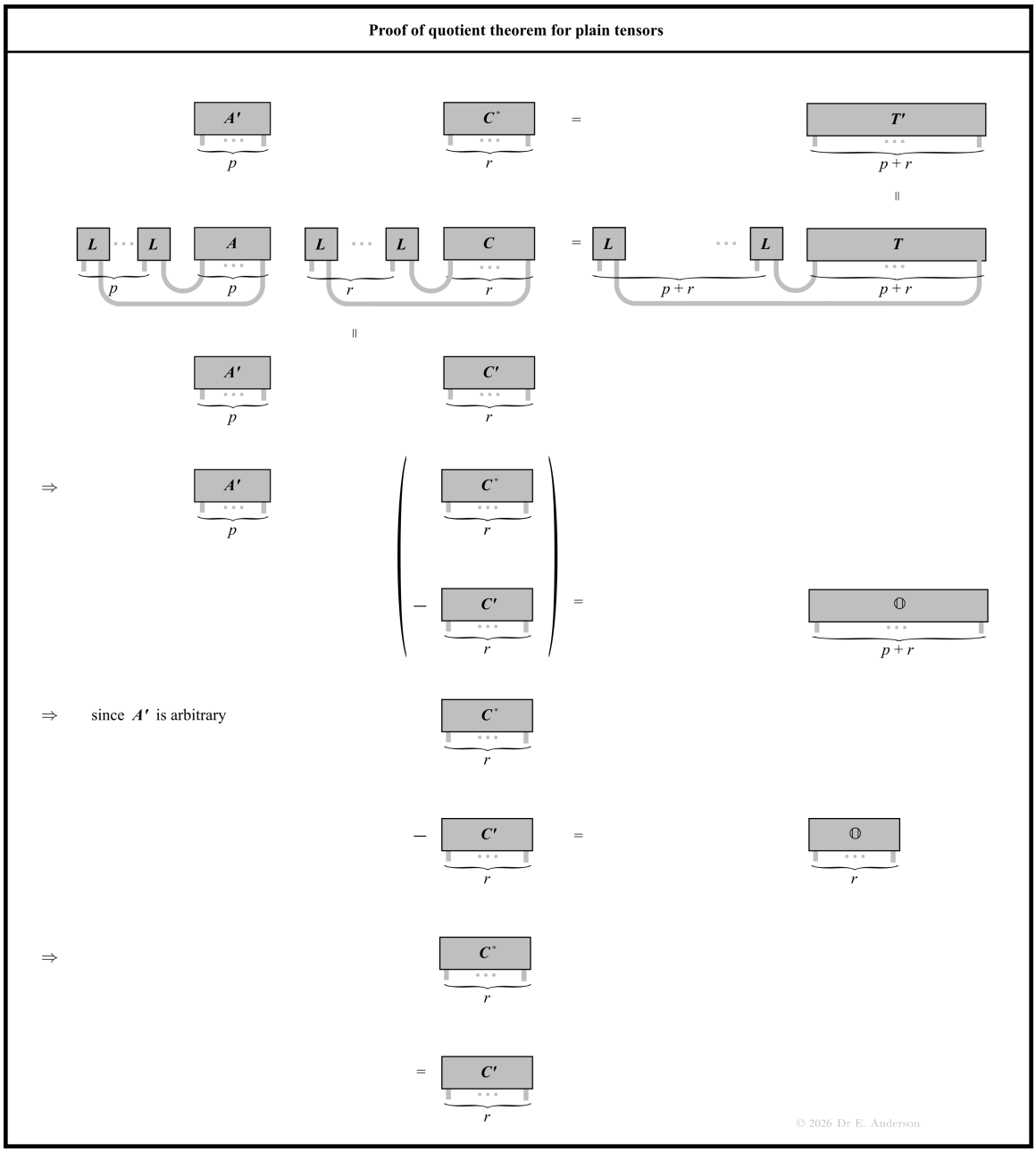

Proof of quotient theorem for plain tensors
A′
p
C*
r
=
T′
p + r
L
A
C
T
p
r
p + r
C′
⇒
since A′ is arbitrary
0
© 2026 Dr E. Anderson

Figure 44:

8.3 Tensor symmetries

Remark 1 See Fig 45 for the sole distinct plain counterpart of Sec 6's first two figures.

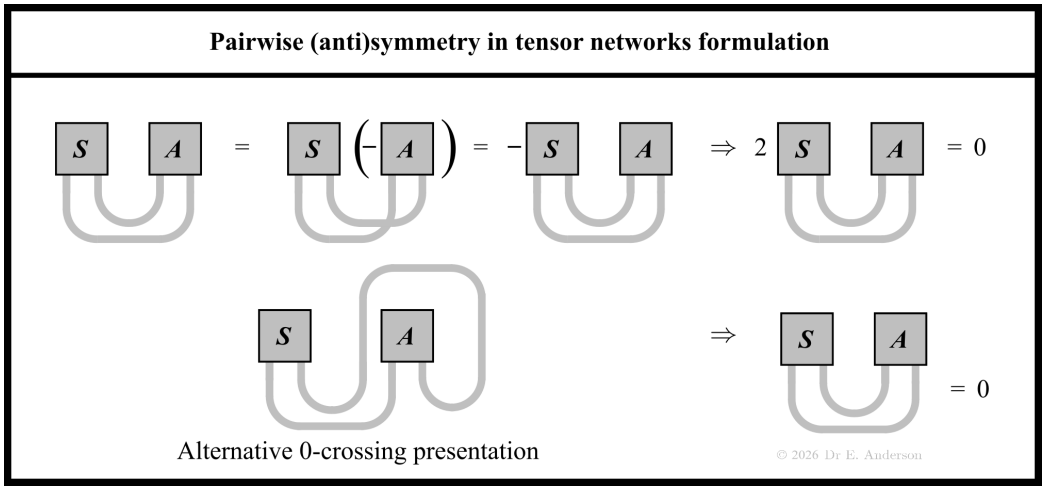


Figure 45:

8.4 Tensor Calculus

Remark 1 See Fig 46 for the plain counterpart of Sec 7's proof's figure.

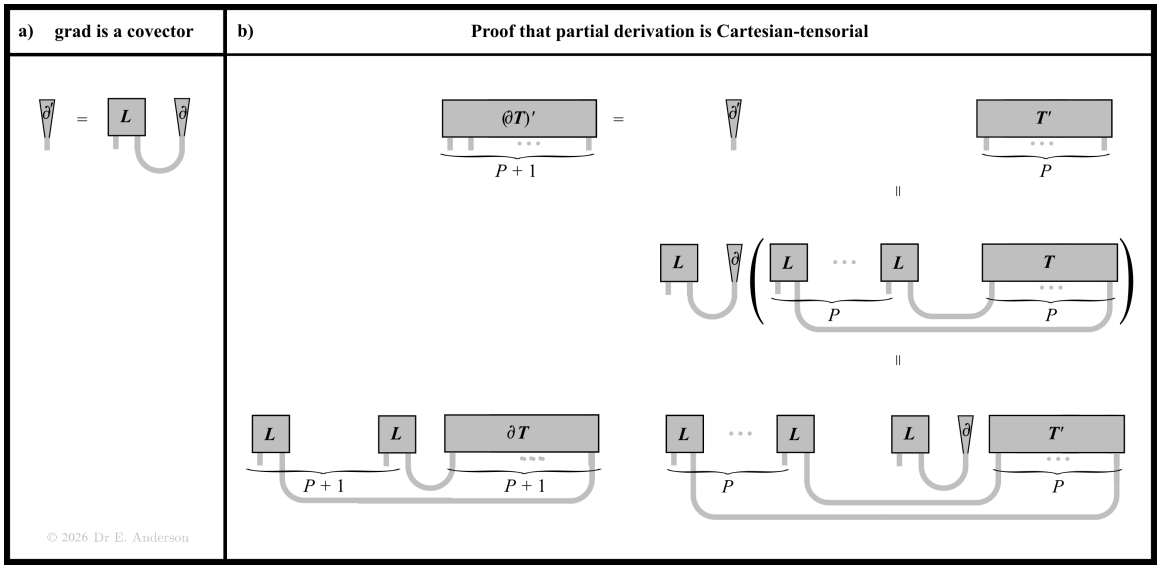


Figure 46:

8.5 Plain versus u.d.d.'s vertical variant

Remark 1 Appendix 8's diagrams, with plain legs – all downstairs – are rather more complicated and take more steps to draw accurately than the main body's vertical variant with downstairs–unpstairs distinction.

Remark 2 The plain case's 'athletics track lanes' become just the signature parallel vertical lines in the planar version. These can be handled piecewise with line segments and circular arcs, forming e.g. equally-spaced lanes.

Remark 3 Removing the crossing in the symmetry-antisymmetry figure is however more pronounced a departure in the vertical case. See Fig 47

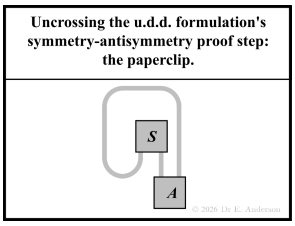


Figure 47:

Remark 4 Also the plain case's transformation matrices stack up in the horizontal direction. So as to produce expressions that are markedly more anisotropic in the page than the vertical variant's.

Remark 5 All in all, the vertical variant is considerably more efficient. Especially if rectilinear crosses are permissible even when they are not minimum-crossing [70] drawings.

9 Conclusion

9.1 The current Review's Combinatorics and arenas

Remark 1 Tensor rank's combinatorial tracking arena (CTA) forms Pascal's triangle. If upstairs and downstairs are distinguishable but meaningless labels, a common and natural C_2 quotient of Pascal's triangle is realized. We furthermore point to a map from these to the partitions with at most 2 parts (and their conjugates). It is rather more well-known that tensor symmetries realize partitions. For these index-permutation symmetries realize corresponding representations of the symmetric group. And partitions' Young diagrams provide a particularly convenient notation for these.

Remark 2 While dimension and rank's joint CTAs form various structural combinations of copies of Pascal's triangle. Cones, pencils, prisms, pinched prisms, windmills... And likewise for the above C_2 quotient.

Remark 3 The usual arenas are pointwise vector spaces. Globally forming tensor bundles: a simple and widely encountered subcase of fibre bundles. We however also place general bundles over the discrete dimension-and-rank arenas. And also contemplate a bundle structure with two base spaces: the model of Physical space $\mathbb{R}^D$ and the CTA.

9.2 Extending to other notions of tensor

Tensor transformation laws apply in further settings, just with more general transformation matrices as laid out in the Introduction. The upstairs-downstairs distinction does not however cover all further basic cases. For instance complex Differential Geometry carries a second independent barred versus unbarred ambiguity. This points to how we shall eventually need larger discrete Combinatorial tracking arenas for types of tensor rank.

Linearity and the tensor product also transcend to other notions of tensor. As does contraction, more generally using a metric $\boldsymbol{g}$ rather than just a $\boldsymbol{\delta}$-tensor. And so by composition (l-fold) inner product carries over as well. The tensor product supports furthermore the quotient theorem.

Symmetry considerations extend to each index type. Isotropy extends to polynomials of the metric tensor. With up to 1 further factor of $\boldsymbol{\epsilon}$'s suitable generalization. To the tensor density

$$\sqrt{\det \boldsymbol{g}}\, \boldsymbol{\epsilon} \ . \tag{114}$$

A The world of combinations

A.1 Combinations themselves

A.1.1 Definition and notation

Structure 1 The *combinations* [58, 57] are the number of ways of picking K objects from a bag of N distinguishable objects, without replacement. [33] establishes that these were already under consideration around 2400 years ago.

Notational Remark A The modern notation is

$$\binom{N}{K} \ . \tag{115}$$

This goes back to von Ettingshausen [42] in the 19th century. And has come to be pronounced N *choose* K .

Remark 1 Structure 1's description immediately leads to the following Algebraic formula.

$$\binom{N}{K} \ = \ \frac{N!}{(N \ - \ K)!\,K!} \ . \tag{116}$$

Lemma A.1 (in $=$ out identity)

$$\binom{N}{K} \ = \ \binom{N}{N \ - \ K} \ . \tag{117}$$

A.1.2 Various binomial applications

Structure 1 Combinations occur as the *binomial coefficients* featuring in the *binomial expansion*

$$(1 \ + \ x)^N \ \text{ for each } \ N \ \in \ \mathbb{N}_0 \ . \tag{118}$$

Historical Remark A.1 The combinations furthermore have applications in basic Probability: compare the names *binomial probability distribution* and binomial expansion. And indeed the 17th century Mathematician Blaise Pascal made foundational contributions to Probability, including in this way.

Remark 1 More eventually, the above binomial expansion and its $\mathbb{R}$-power generalization leads to the Calculus formula for the derivative of each power function. And tells us part of how to combine approximations.

A.2 Combination arenas

A.2.1 Binomial expansion's chains of combinations

Structure 1 The combinations supported by the bag of N distinguishable objects can be totally ordered by the number of choices K. These jointly feature in the Nth power's binomial expansion. Where the total order now corresponds to the coefficients' cofactors: the powers of K. These combinations' values consequently straightforwardly add up to 2^N.

This chain is furthermore symmetric [37] about its midpoint, by Lemma 1. Conferring a second name upon this Lemma: *symmetry identity*. In the chain arena, this is realized as reflection symmetry about the midline.

Lemma 2 For even N, the midpoint that our reflection is about is occupied by the combination

$$\binom{2\,K}{K} \ . \tag{119}$$

While for odd N it is empty.

A.2.2 Pascal's triangle of combination values

Remark 1 The above chains are furthermore the rows of *Pascal's triangle*. The first 5 rows of which are presented in Fig 48.a); these are the ones that the current Review needs.

Remark 2 Pascal's triangle can furthermore be viewed as the outcome of adding each row's entries into the next row's adjacent entries along whichever diagonals are realized. This is a process that can be continued forever (hence the dots in the Figure). This boils down to the following identity as viewed as a recurrence relation between combinations.

Lemma A.3 (Pascal's identity)

$$\binom{N\ +\ 1}{K} \quad = \quad \binom{N}{K\ +\ 1} \ + \ \binom{N}{K} \ . \tag{120}$$

Exercise A.1$^-$ Prove Lemmas A.1 and A.3, firstly Algebraically and secondly Combinatorially. Also interpret our name [71] for Lemma A.1. Finally prove Lemma A.2.

Remark 3 See Appendix B for another combinational identity, and e.g. [57, 72, 66, 64] for yet more.

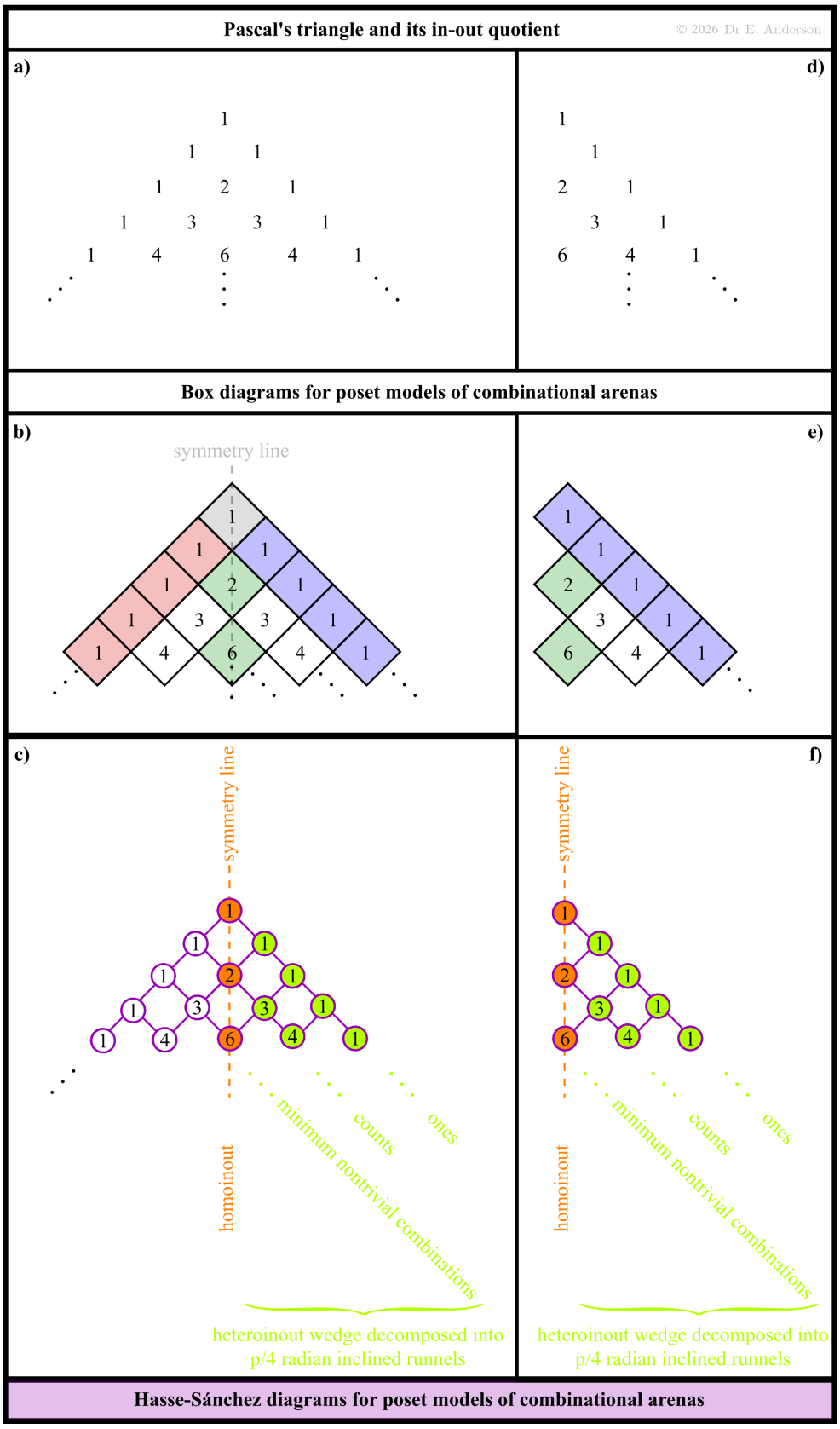

Pascal's triangle and its in-out quotient
© 2026 Dr E. Anderson
a)
d)
Box diagrams for poset models of combinational arenas
b)
symmetry line
e)
c)
symmetry line
homoinout
minimum nontrivial combinations
counts
ones
heteroinout wedge decomposed into
p/4 radian inclined runnels
f)
symmetry line
homoinout
minimum nontrivial combinations
counts
ones
heteroinout wedge decomposed into
p/4 radian inclined runnels
Hasse-Sánchez diagrams for poset models of combinational arenas

Figure 48:

A.2.3 Pascal's triangle as the arena of combinations, = $\mathfrak{C}$ombi

Naming and Notational Remark A.1 Pascal's triangle can be viewed as the *recurrence arena* [73] corresponding to the recurrence relation view of Pascal's identity. Let this confer the notation

$$\mathfrak{P}\text{ascal} \ .$$

It can furthermore be viewed as the *arena of combinations*

$$\mathfrak{C}\text{ombi} \ .$$

And as the *arena of* $\mathbb{N}_0$ *binomial coefficients*,

$$\mathfrak{B}\text{in} \ .$$

In each case again as ordered by Pascal's identity.

Remark 1 See Subfig b) for a grid presentation, which our main text then uses to model the arena of tensor ranks with u.d.d. And Subfig c) for a poset Hasse diagram [63] Visualization. The arrows strutting between adjacent-N floors correspond to the combinations inter-related by the above recurrence relation. Indeed, truer names for Pascal's identity include *combinations' generative recurrence relation* and *binomial coefficients' generative recurrence relation* [71].

Remark 2 Let us next distinguish between full arenas and cumulative arenas. In the current Article, this refers to discrete-parameter arenas that form nice basic Combinatorial structures. Our discrete parameters take values throughout $\mathbb{N}$, so the full structures are discrete and yet infinite as well. In contrast, cumulative-approximation arenas entertain all values of ≥ 1 parameter up to some finite value per parameter. This applies to all of the main body of the text (dimension-rank arenas) and to the nice basic Combinatorial structures we find therein. Which are then explained in their own terms in this and the next Appendix. To denote cumulative approximation up to N , we place a function dependence $[N]$ at the end of each symbol.

A.2.4 The corresponding count returns the triangular numbers

Remark 1 The total count of combinations up to N is

$$C[N] \ = \ |\,\mathfrak{C}\text{ombi}[N]\,| \ = \ \frac{(N+1)(N+2)}{2} \ = \ T(N+1) \ . \tag{121}$$

Where the last equality picks out a shifted version of the well-known *triangular numbers*.

A.2.5 Pascal's triangle as presented is already-centred

Remark 1 The usual presentation of Pascal's triangle is already centred [35]. This readily follows from the in = out identity, now over however many floors are required. The reflection symmetry is now realized about its midline.

Naming Remark A.1 by paragraph 3 of Appendix , this midline is populated solely by 1 entry on each even floor. Constituting the *homoinout* combinations [71]. 'Central binomial coefficients' is a passable alias... The remaining combinations are *heteroinout*.

Remark 2 The homoinout combinations count out as follows.

$$S[N] \ = \ 1 + \left\lfloor \frac{N}{2} \right\rfloor \ = \ \frac{1}{2} \begin{cases} N+2 & N \text{ even} \\ N+1 & N \text{ odd} \end{cases} \ . \tag{122}$$

Where the middle expression uses the *floor function* [59].

Leaving the heteroinout combinations counting out as follows. Cumulatively these count out as

$$E[N] \ = \ P[N] - S[N] \ = \ \frac{1}{2} \begin{cases} N(N+2) & N \text{ even} \\ (N+1)^2 & N \text{ odd} \end{cases} \ . \tag{123}$$

Notational Remark A.2 Let us cast homoinout combinations in tangerine. And heterconjugate pairs as a white copy and a lime-green copy (Subfig c).

Aside 1 For some context, many other Combinatorial arenas do not come already centred. Indeed Pascal's own diagram [40], currently displayed at [75], precedes centring. Paralleling our first uncentred figure for tensor rank arenas with u.d.d. (Fig 12).

A.2.6 Reflection-quotient arena

Structure 1 $C_{2\,\mathrm{in}\,\leftrightarrow\,\mathrm{out}}$ further acts as a reflection about the midline at the level of the centred arena. We thus arrive at the quotient

$$\widetilde{\mathfrak{C}\mathrm{ombi}} \;:=\; \frac{\mathfrak{C}\mathrm{ombi}}{C_{2\ \mathrm{ref}}} \,. \tag{124}$$

See Subfig d-f) for cumulative versions of this quotient matching each of the above styles. These are slightly larger than half of Pascal's triangle. For $C_{2\,\mathrm{in}\,\leftrightarrow\,\mathrm{out}}$ is inactive on heteroinout objects.

Remark 1 Cumulative counts are now

$$Q[N] \;:=\; \frac{P[N] \,+\, S[N]}{2} \;=\; \begin{cases} (\,N \,+\, 2\,)^2 & N \ \text{even} \\ (\,N \,+\, 1\,)(\,N \,+\, 3\,) & N \ \text{odd} \end{cases} \,. \tag{125}$$

Remark 2 The above quotient improves efficiency. For now only each power's distinct binomial coefficients are stored. It also corresponds to expanding the $N \,\in\, \mathbb{N}_0$ powers of

$$(\,x \,+\, y\,)^N \,.$$

From the point of view in which x and y are distinguishable and yet meaningless labels.

Notational Remark A.2 Let us next introduce $[c]$ to denote either a combination or its heteroinout. For homoinout combinations, $[c]$ collapses to just c . Given a $[c]$, one can pick either c or $c_{\mathrm{in}\,\leftrightarrow\,\mathrm{out}}$ as a representative. This admits a bundle section point of view, or similar. The current item involves quite a standard notation for quotients, with the sentence two sentences back being a specialization to C_2 quotients. We shall be encountering another instance of this specialization in Appendix B...

A.3 Graph and Order Theory continuation

A.3.1 For Pascal's triangle $=$ $\mathfrak{C}$ombi

Remark 1 This is the recurrence poset corresponding to Pascal's identity. I.e. 2 vertices are joined by an arrow if they both feature in the same copy of Pascal's identity. Alternatively, the arrow is 'add 1 power' in the binomial expansion conceptualization.

Historical and Naming Remark A.2 These are 'Zhu Shijie posets'. In the sense that this 13th and 14th century Chinese Mathematician may have been the first to draw [75, 40] what were subsequently pointed out to be poset Hasse diagrams for Pascal's triangle. Pascal's triangle itself precedes both Pascal and him, however, with contemporary Yang Hui [38] and 11th century predecessor Jia Xian [36] also having studied this triangle. Aside from Pascal and Jia Xian's rediscoveries, first known consideration goes back to Pingala [34] in 5th century B.C.E. India. Further historical names for Pascal's triangle are attached to this part of our bibliography. For sure, *combinational triangle* and *combinational poset* are truer names [71].

Remark 2 The full version is a tri-regular graph [73]. With the following 'stratified' regularity. All interior vertices are quartic. All side vertices are cubic. And just its corner (also its apex) is binary.

While the cumulative versions start as follows.

$N \,=\, 0$ exceptionally realizes just a degree-0 vertex.

$N = 1$ is minimum to realize 2 leaves at the bottom corners. And also minimum to realize the binary apex corner.

$N = 2$ is minimum to realize cubic side vertices, enjoying 2 of these. It is also minimum to realize a binary middle edge. While retaining the previous content – all persistent – of binary apex and 2 bottom-corner leaves.

$N = 3$ is minimum to realize an interior quartic vertex. Alongside the usual 2 leaves and binary apex. And now 2 binary bottom side elements. And 4 cubic side elements.

Exercise A.2^{-} Show that the Nth entry for $N \geq 1$ has 2 leaves. N binary vertices.

$$2(N - 1) \text{ cubic vertices .}$$

And

$$\frac{(N - 1)(N - 2)}{2} \text{ quartic vertices .}$$

What common type of Combinatorial number is the last of these closely related to? Also establish that $N = 3$ is minimum to have the subsequently persistent tetraregularity property.

Remark 3 This is also a regular stratified pattern, just now for a triangle instead of an infinite wedge. So that a third side, populated entirely by binary vertices, is in play. Alongside the 2 corners that this makes with the lateral sides, each of which is a leaf.

Our quotient maps between the following. From a reflection-symmetric wedge, as approximated by isosceles triangles that preserve this reflection symmetry. To a 'right-angled' wedge, as approximated by 'right angled' scalene triangles.

Remark 4 While one could insert the ungraph and then the unposet above $N = 0$, this would be somewhat artificial. In the sense that the Pascal identity's recurrence relation is not in play here, so such an arrow would require a separate significance. If one inserted this, then it would correspond to including the uncombination, or the empty coefficient of the binomial unpower...

A.3.2 Discussion

Naming Remark A.3 Some series that the usual modern centred Pascal's triangle separates out along runnels inclined at the Order Theorists' preferred Drawing and Visualization value of $\frac{\pi}{4}$ radians are as follows. The *unit series*, just consisting of 1's. The *counting series*, returning the row number N . And the *minimum combinationally nontrivial series*, returning

$$\binom{N}{2} .$$

We consider no more, since the current Review's application only goes up to $N = 4$. While the perpendicular runnels are the same as the ones described by the arena-level reflection symmetry realization of the in $\leftrightarrow$ out exchange symmetry...

Notational Remark A.2 These runnels are marked on Subfig c). Thus forming a '*Hasse–Sánchez diagram*' for a poset model of an arena.

$\mathfrak{C}$ombi sports a Combinatorial-Geometrically trivial runnelling. In the sense that no work needs to be done to cleanly separate out each series, and only that series, into a separate runnel. With furthermore a regular runnelling structure induced from the above systematically stratified graph regularity.

Pointer 1 The point of the previous paragraph is that for other posets which widely occur as arenas, however, there is no such regularity. With consequent Combinatorial-Geometric runnelling problems at the level of optimizing various Drawing and Visualization criteria. Such as how to pack

the runnels as close together as possible on the diamond grid of integers. That is with such as the smallest number of unoccupied integer runnels, or some constraint on how wide the largest gap between occupied integer runnels can be. The current Article's other salient case of partition poset arenas already leads to such problems...

A.3.3 ... and of its in-out quotient

Exercise A.3 [K. Everard] Work out our reflection quotient's version of Remark 3, Exercise A.2. and Remark 4.

A.3.4 Further Order Theory

Remark 1 Common direct heir and common direct ancestor serve as meet and join respectively. These place a further lattice [65, 67, 61] structure on the full Pascal arena and its reflection quotient.

Remark 2 The 1 at the summit, corresponding to the binomial expansion's zero power, is the bottom element. Unless one harmlessly appends the binomial expansion's unpower so as to precede this. There is however no unique bottom element in the cumulative case (for $N \geq 1$ unquotiented and $N \geq 2$ quotiented). By which the other cumulative approximands fail the greatest lower-bound condition for a lattice and end up only being semi-lattices.

Remark 3 The Pascal arena and its reflection quotient are furthermore distributive. That all the elementary cycles in our posets are squares is not a fatal regularity. But rather a symptom of a distributive structure: a structurally-interesting class [67].

A.4 Aside on naming some of the current Review's graphs

Naming Remark A.3 The main text's name ' Pascal-DI ' [71] follows from Pascal's triangle being the most basic setting producing this DI. The name 'Pascal' then follows in turn for one of our incipient posets and its underlying graph from resting on the preceding minimum distinctive structure. One could then use Pascal_N and Pascal-DI_N to denote the poset, graph and graph stemming from adding N further rows to Pascal's triangle.

Exercise A.4^{-} Check that the quotient of Pascal-DI's graph is however structurally standard. While showing that the quotient of Pascal-DI_1's graph is more structurally distinctive.

Thus it merits the notation $\widetilde{\text{Pascal-DI}}$. The corresponding order and graph $\widetilde{\text{Pascal}}$ shall subsequently show up in our Encyclopaedia for total rank $T = 5$.

B The world of partitions

B.1 Partitions

B.1.1 Definition and smallest examples

Definition 1 An (*integer*) *partition* $\mathrm{p}(N)$ of $N \in \mathbb{N}_0$ is as follows. A split of N into

$$N_i \in \mathbb{N}\,, \quad i = 1 \text{ to } k\,.$$

I.e. a split into additive parts,

$$\sum_{i=1}^{k} N_i = N\,.$$

Let us also use $p(N)$ to denote the number of ways in which N can be partitioned [76].

Example 0 We exceptionally also include the partitioning of 0 into no parts... I.e. the '*unpartition*'

$$\mathrm{un} = 0 := \emptyset\,.$$

Thus

$$p(0) = 1\,.$$

Example 1 1 object can be partitioned into a single piece, which we denote by 1. Since this is the unit of structure for subsequent use (Example 0 lacks any structure), let us also refer to this as

$$\text{atomos}\,.$$

Since there is no other way of partitioning one object,

$$p(1) = 1\,.$$

Example 2 and Notational Remark B.1 2 can be partitioned into a single piece of size 2. Or into $1+1$ using additive notation; another notation is $1\,|\,1$. Signifying 1 in one 'piece' and 1 in another such. Or just $1\,1$. Or use the rapidly more efficient notation 1^2, pronounced '1 with multiplicity 2'. Since there are no further ways of partitioning 2,

$$p(2) = 2\,.$$

$N = 2$ is minimum to distinguish between the following. The 1-*part partitions* and the *partitions all of whose parts are of size* 1. Respectively alias the *coarsest partition* and the *finest partition*, each for each fixed N.

Example 3 3 can be partitioned in any of the following ways. 3, $2\,1$ or 1^3. Thus

$$p(3) = 3\,.$$

Furthermore, $2\,1$ is the minimum partition to be neither coarsest nor finest. Coarsest and finest are Order-Theoretically top and bottom for each fixed N. Viewed in this way, $2\,1$ is the minimum nontrivial middle of a fixed-N poset of partitions.

Example 4 4 supports the following partitions. 4, $3\,1$, 2^2, $2\,1^2$ and 1^4. Thus

$$p(4) = 5\,.$$

Historical Remark B.1 Integer partitions were first posed by Leibniz [43], with Euler providing the first major theoretical advances [44].

B.1.2 Young diagrams

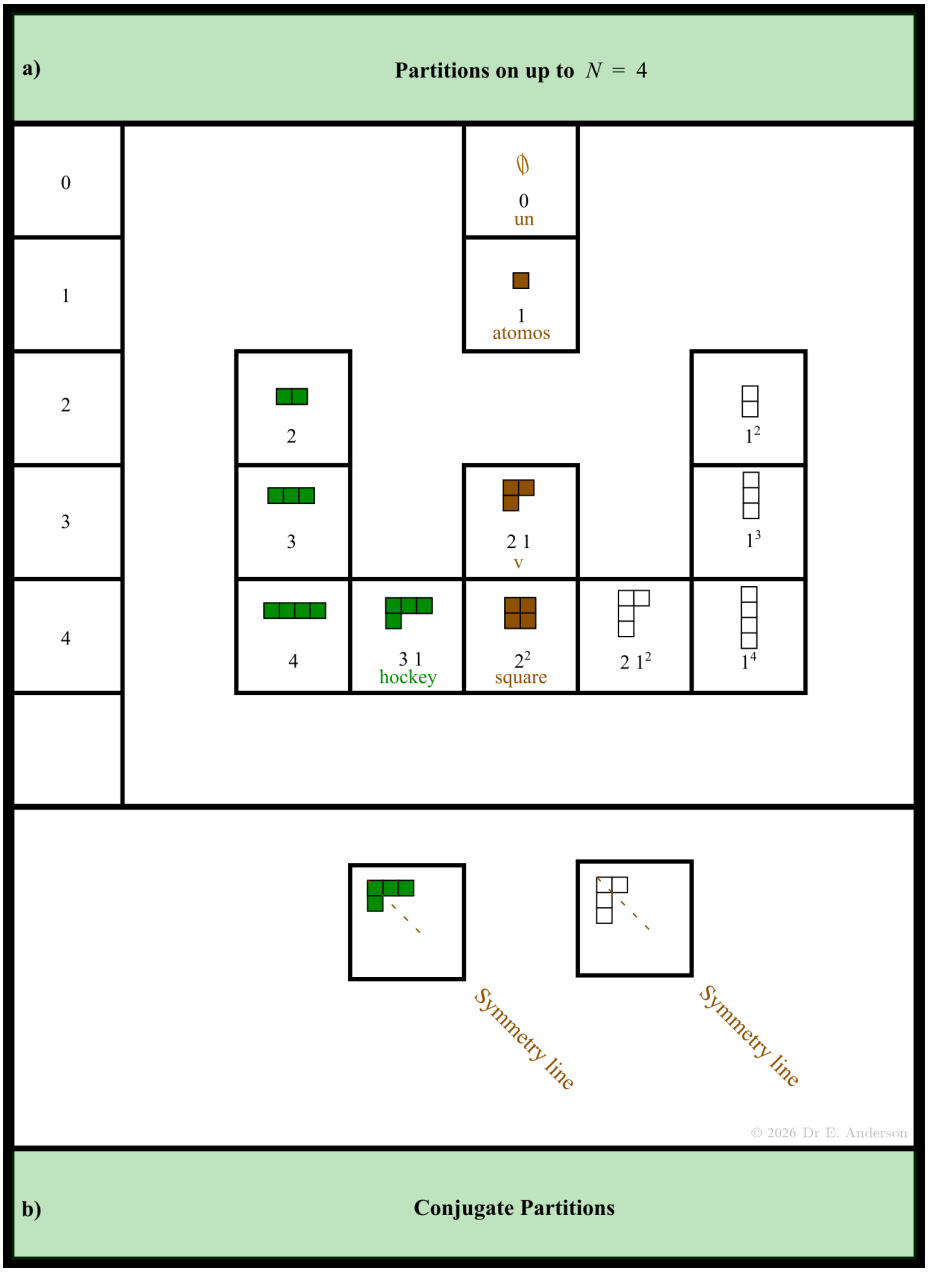


Figure 49:

Notational Remark B.2 Let p be the partition $N_1, \ldots, N_k$ with

$$N_1 \geq N_2 \geq \ldots \geq N_k \, . \tag{126}$$

Then its *Young diagram* [46, 51, 53, 73]. $y(\mathrm{p})$ is constructed as follows. Place N_1 squares into a first row. N_2 into a second row. That is aligned on the left with the first row, and so on.

Young diagrams [46, 51, 53, 54] are a useful Drawing and Visualization, and computational, tool for small partitions. In fact, this Combinatorial notation was preceded by the *Ferrers diagram* [45], using dots in place of boxes.

Examples 0 to 4 See Fig 49 for the $N = 0$ to 4 cases of partition box diagrams.

Naming Remark B.1 Let us also use

$$\text{v} \;:=\; 2\,1 \;.$$

$$\text{square} \;:=\; 2^2 \;.$$

$$\text{hockey} \;:=\; 3\,1$$

Which is short for 'hockey-stick'.

B.1.3 Conjugate partitions

Definition 1 The partition $\bar{\text{p}}$ is *conjugate to* the partition p if the following holds.

$$y\,(\bar{\text{p}}) \;\;=\;\; R\,y(\text{p}) \;. \tag{127}$$

Where R is the reflection about the Young diagram's leading diagonal; see Subfig b).

Remark 1 For each partition, conjugation always maps any given partition either to a unique distinct partner partition, or to itself

Definition 2 A partition p is *homoconjugate* if it is equal to its own conjugate,

$$\text{p} \;=\; \bar{\text{p}} \;. \tag{128}$$

Otherwise, it is *heteroconjugate*, with p, $\bar{\text{p}}$ constituting a *conjugate pair*

Notational Remark B.3 Let us colour homoconjugate partitions in brown. And heterconjugate pairs as a white copy and a christmas-tree-green copy.

Clearly un, atomos, v and square are homoconjugate. While each of the following are heteroconjugate. mono , i.e. [discrete] = discrete or indiscrete = 1-piece or part size $\leq$ 1 . And hockey .

Remark 2 Indeed, making conjugation symmetry manifest is one of the main benefits of Young (or Ferrers) diagrams...

Naming Remark C continued Also our naming stopped around half way down the partitions being introduced because the remaining partitions can be named via their conjugates!

Notational Remark B.4 Let us also use [p] to denote p and $\bar{\text{p}}$'s conjugacy class. Though if p is homoconjugate, we write just p in place of [p] . While for heteroconjugate [p] , whichever of p or $\bar{\text{p}}$ serves as a representative. This of course has its own bundle section interpretation, or similar...

B.1.4 k-part partitions

Structure 1 k-part partitions are indeed those which have precisely k parts. These are conjugation-dual to partitions with $\geq$ 1 part of size k and no part $> k$.

Structure 2 ($\leq k$)-part partitions have $\leq k$ parts. These are now conjugation-dual to partitions with no part $> k$.

Remark 1 ($\leq$ 2)-part partitions crop up in the current Article. These are readily identified by having $\leq$ 2 rows in their Young diagram. I.e. the number of upstairs indices and the number of downstairs indices.

Remark 2 Useful corresponding constrained counts are as follows.

$$p_{\,\leq\, 2}(N) \;\;=\;\; \lfloor\, \tfrac{N\,+\,2}{2}\,\rfloor \;. \tag{129}$$

While for the quotient,

$$\widetilde{p}_{\,\leq\, 2}(N) \;\;=\;\; \begin{cases} \lfloor\, \frac{N\,+\,2}{2}\,\rfloor & N \;\neq\; 2 \\ 1 & N \;=\; 2 \end{cases} \;. \tag{130}$$

B.1.5 Tensoring Young diagrams

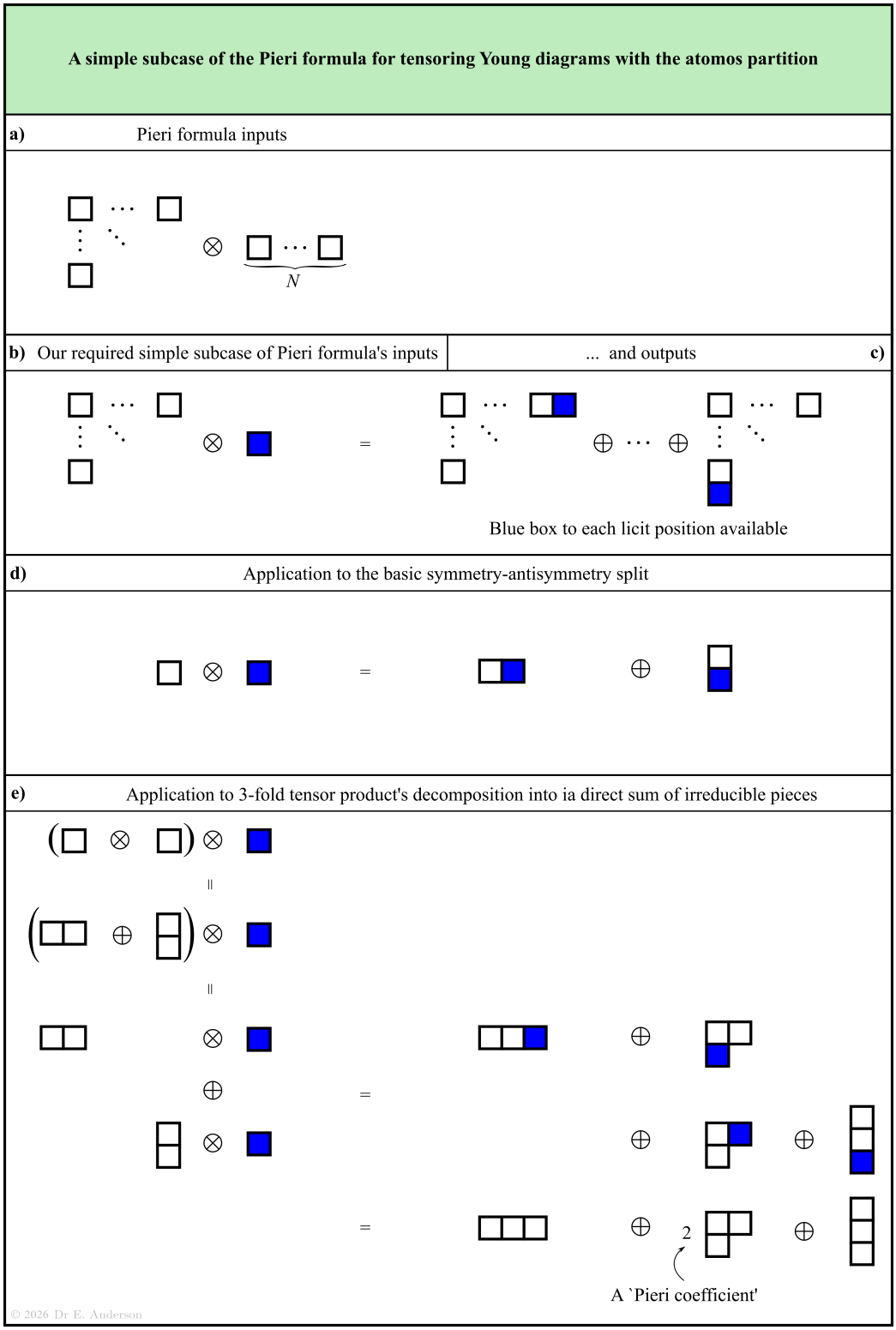


Figure 50:

Structure 1 *Pieri's formula* is for how the following breaks down into a sum of irreducibles. Tensoring an arbitrary Young diagram by a 1-part partition's Young diagram (Fig 50.a).

Pointer 1 This is a small subcase of the *Littlewood–Richardson formula* for the decomposition into

irreducibles of the tensoring of an arbitrary pair of Young diagrams. This and the previous item are of the '*branching rule*' conceptual type [55, 53].

Structure 1.1 The simple subcase of Pieri's formula that the current Article requires is as follows. Tensoring an arbitrary Young diagram by atomos (Subfig b). Here one places the new box – cast in blue – into each licit position available adjacent to the arbitrary-partition factor's Young diagram in turn. See Subfig c) for a schematic depiction.

Examples 2-3 Also see Subfig d) for application to the basic and well-known symmetry-antisymmetry split. And Subfig e) for its tensor triple product counterpart. Which the current Review applies to deriving the minimum example of mixed-symmetry tensors for rank 3 . Consisting of 1 type of mixed symmetry, occurring in the irreducible decomposition with multiplicity 2 . Let us also for now refer to the numbers featuring in the right-hand-sides of these formulae as 'Pieri coefficients'.

Exercise B.1 Establish that

$$\bigotimes_{I=1}^{4} \square \;=\; [\square\square\square\square] \;\oplus\; 3\,\Big[\begin{array}{ll}\square&\square\\\square&\end{array}\Big] \;\oplus\; 2\,\begin{array}{ll}\square&\square\\\square&\square\end{array}\,. \tag{131}$$

Simplifying for tensors in 3-D to

$$\bigotimes_{I=1}^{4} \square \;=\; \square\square\square\square \;\oplus\; 3\,\Big[\begin{array}{ll}\square&\square\\\square&\end{array}\Big] \;\oplus\; 2\,\begin{array}{ll}\square&\square\\\square&\square\end{array}\,. \tag{132}$$

And for tensors in 2-D to

$$\bigotimes_{I=1}^{4} \square \;=\; \square\square\square\square \;\oplus\; 3\,\begin{array}{ll}\square&\square\\\square&\end{array} \;\oplus\; 2\,\begin{array}{ll}\square&\square\\\square&\square\end{array}\,. \tag{133}$$

B.1.6 Hook lengths and box values

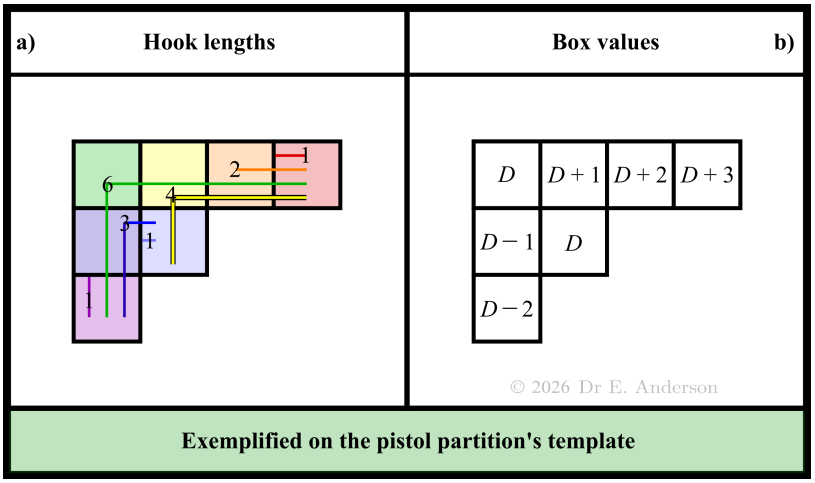


Figure 51:

Definition 1 *Hook lengths* h_c are assigned to each box c in a Young diagram as follows. One counts how many boxes there are vertically down and also horizontally right. Including our box itself once in the count.

Notational Remark B.5 In Fig 51.a), the squares are colour-coded in pastel. With each hook length marked in intense counterparts of these colours. So say the pastel red square's hook-length is marked in bright red. N.B. that this is a discrete path-length. I.e. counting the number of boxes involved. As opposed to exactly how Euclideanly-long each path is drawn in one's diagram...

Lemma B.1 The dimension of each S_N irreducible representation is given by [55]

$$\dim \rho \;=\; \frac{N!}{H}\,. \tag{134}$$

Where the *hook-length product*

$$H \;:=\; \prod_{c \text{ squares } \in y(\rho)} h_c \;.$$

Notational Remark B.5 continued For our pistol, c is a colour and the small arena $Y(\text{pistol})$ is a rainbow palette...

Definition 2 *Box values* b_c are assigned to a Young diagram's boxes c as follows. Place the value D in the origin-corner (top-right in our standard notational convention). Increase the value by 1 whenever moving 1 box to the right. And decrease the value by 1 whenever moving 1 box down.

Remark 1 This is more contextual than the hook length. Our context is that we have a real D-dimensional tensor. So we are assigning $\mathfrak{sl}(D)$, or $\mathfrak{gl}(D)$ Lie algebra representations to our Young diagram, upgrading it to a *Young tableau* [55, 53]. It is then possible to apply such a level of decor somewhat differently, so as to accommodate other Lie algebras or generalizations... Ease of fitting labels inside Young's boxes provides a reason to prefer them over Ferrers' dots...

The weaker contextuality of the hook length is that the role of N is played by the rank P of our tensor.

Lemma B.2 [72] The number of independent components of our tensor is

$$\#(\, y(\rho)\,) \;\;=\;\; \frac{B}{H} \;.$$

Where the *box-number product* is

$$B \;:=\; \prod_{c \text{ squares } \in y(\rho)} b_c \;.$$

B.1.7 Exercises

Exercise B.2 Find H , $\dim \rho$, B and $\#(\, y(\rho)\,)$ for each of the partitions within the current Review's scope. I.e. those whose Young diagrams are tabulated in Fig 49. Compare these with the symmetry counts in Sec 6.

Exercise B.3 $^{+}$ Use your consequent enhanced understanding of hockey to resolve $\left[\begin{array}{|c|c|c|}\hline \\ \hline \end{array}\right]$'s d.o.f. between its 2 conjugates. Including how many d.o.f. each word-type contributes. Also figure out what index symmetries are realized by each of these 2 conjugates. I.e. write out a complete set of symmetry relations for each of these in Bach brackets form. And then furthermore a tensor networks version for each.

Exercise B.4 Compare Exercise B.2's results with the 'Pieri coefficients' for the corresponding 0-through to $4-$fold tensor products.

Exercise B.5 Add fuel to the fire by computing Exercise B.2's 4 quantities for the following. pistol. driver $= 32$. And flag $= 32^2$.

(We shall need the driver and its conjugate in discussing the second Bianchi identity [13, 17] in 2027. While it is edifying make one's own illustration of hook lengths using a different take on the 'rainbow flag'. Here with 1 colour reserved for its flagpole component...)

Exercise B.6^{+} Prove that indeed every 'Pieri coefficient' coincides with the direct summand in question's $\dim \rho$.

[Though this only counts as a $+$ if you have not studied Representation Theory...]

Pointer 1 See e.g. [72, 73, 69] for further introductory material on partitions, and [47, 48, 49, 50, 52, 54, 56] for detailed reviews.

B.2 Partition arenas

B.2.1 Adapted variables for partitions

Structure 1 Let W and H parametrize the width and height of each Young diagram. Then pick

$$A \;=\; W \;-\; H$$

as a sign-choice for the signed conceptual of the anisotropy of Young diagrams. This serves furthermore as an adapted variable to centre homoconjugate partitions. With heteroconjugate partitions being realized in the arena of partitions by reflection about this midline.

Remark 1 To have isotropy in this context, we need zero departure

$$0 \;=\; A \;=\; W \;-\; H\;. \quad \text{I.e.} \quad H \;=\; W\;,$$

meaning that their Young diagram's 'hugging rectangle' [73] must be square. This is most obviously realized by the square family of partitions. It is however also realized by e.g. the v family, which incompletely box up their hugging square. Indeed, all homoconjugate partitions are isotropic in this sense.

R now acts as reflection about the midline.

B.2.2 The Young lattice arena of partitions

Structure 1 The *Young lattice* of partitions

$$\mathfrak{Y}\text{oung}$$

is the arena of all partitions

$$\mathfrak{p}\text{artition}$$

as equipped with the following operations.

The poset arrow operation is 'add box'. I.e. add a box to a given partition $p(N)$'s Young diagram in any licit way. In the sense that the output is itself a Young diagram, corresponding to some partition of $N \;+\; 1$.

2 partitions' meet and join is given by the intersection and the union of their Young diagrams.

Let the full version be for all N . And cumulative-approximation versions run from 0 to some fixed N . Also introduce the count

$$Y[N] \;:=\; |\,\mathfrak{Y}\text{oung}[N]\,| \;\;=\;\; |\,\mathfrak{p}\text{artition}[N]\,| \;\;=\;\; \sum_{J \,=\, 0}^{N} p(J)\;. \tag{135}$$

So

$$Y[0] \;=\; 1\;,\;\; Y[1] \;=\; 2\;,\;\; Y[2] \;=\; 4\;,\;\; Y[3] \;=\; 7\;,\;\; Y[4] \;=\; 12\;. \tag{136}$$

Whether in the full or the cumulative case, our order structure has a unique top element: un .

On the one hand, the full version has no bottom elements. On the other hand, cumulative approximations for each $N_0 \;\geq\; 2$ do not have unique bottom elements. Thus for these the greatest lower-bound condition for a lattice fails. So one has a $\mathbb{N}_2$-valued family of just semi-lattices. Alongside 2 trivial lattices: the point comprised solely of un , and the 2-chain consisiting merely of un and atomos . While for the full case, a lattice is attained: the Young lattice! In all cases, distributivity is furthermore satisfied.

Remark 1 The add-box ordering operation manages to respect partitions' $C_{2\,\mathrm{conj}}$ symmetry. With the Young lattice manifesting arena-level partition conjugation as reflection about the midline.

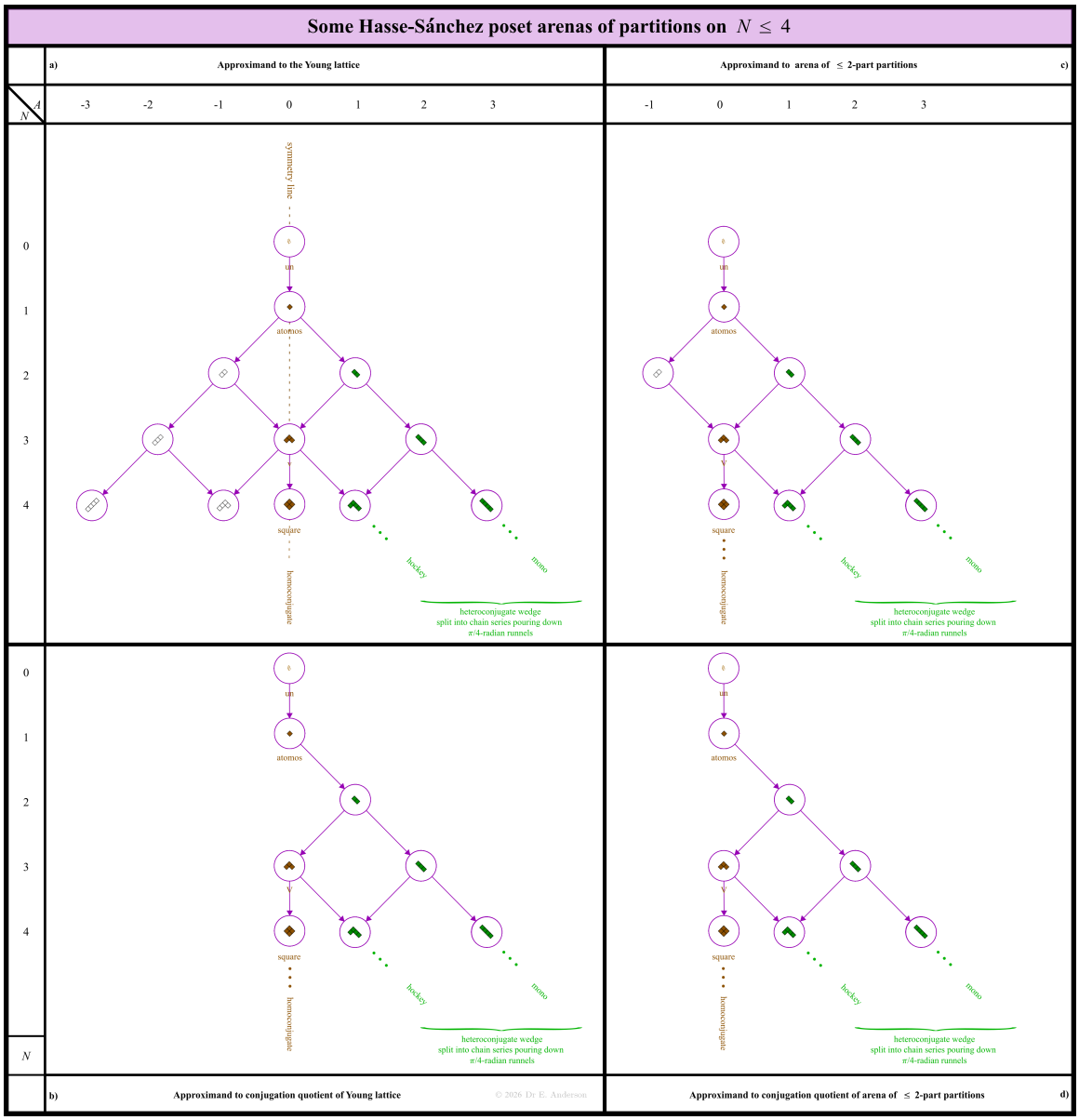


Figure 52:

Remark 2 Let us also benefit in our figure-drawing from how for $N \leq 5$, N and A serve as coordinates for [71] the cumulative Young semi-lattices.

B.2.3 Runnels and interesting irregularity

Remark 1 Up to $N = 4$, it is mono and hockey that separate out as runnels. See [78] for $N = 6$'s small extension.

Naming Remark B.2 Actually the combinations' own box visualization (Subfig A.b) already makes precisely this use of the name 'hockey stick'. In the form of the following Exercise's identity. Thus justifying [71] the name 'hockey-stick' in the further context of Young diagrams for partitions.

Exercise A.5 Prove both Algebraically and Combinatorially the combinations' further *hockey-stick identity* [39, 41]. I.e. that for $\mathbb{N}_0$ numbers $N \geq K \geq 0$,

$$\sum_{J=R}^{N} \binom{J}{K} = \binom{N+1}{K+1} . \tag{137}$$

Remark 2 In the Young lattice, the apex un and its proxime atomos are of fixed degrees 1 and 3 respectively: leaf and cubic. The lateral edges are also cubic.

So far, this is very comparable to ℭombi's regularity. Modulo the convention of appending an extra leaf upon its apex. Which would send its degree-2 apex to a cubic proxime below the now more artificial leaf vertex.

The Young lattice's interior, and its cumulative approximands' horizontal-slice side moreover exhibit interesting irregularity. For all that the current Review's application just running up to $N = 4$ is not particularly illustrative of this. See again [78] for $N = 6$'s small extension. This is one of the ways in which the Young lattice, and by extension the partitions, are more exhalted citizens of Kallista than the combinations and their arena. Even the count $p(N)$ is hard and interesting [76]. (Given this, $Y[N]$ is of more limited interest.) While all of the previous Appendix's Combinational counts are elementary.

B.2.4 The Young lattice's quotient by partition conjugation

Remark 1 The cumulative $N = 4$ case of this is presented in Subfig b).

B.2.5 Some other smallest arenas of partitions

Remark 1 See furthermore Subfig c) for the arena of partitions restricted to having at ≤ 2 parts.

And Subfig d) for the conjugation-quotient arena of this restriction.

Exercise B.7 Observe that for $N = 4$, the last 2 cumulative arenas coincide. Show that $N = 5$ is minimum for these not to coincide, and that this property persists for all subsequent N .

Remark 2 This gives a third reason to take our study as far as $T = 5$. In fact, we anticipate total rank $T = 5$ to 9 tensors entering our Encyclopaedia around the start of the 2030s. So the current Article's sequel shall extend all the way to $T = 9$.

Lemma B.1 (Pascal-2-part partition Lemma) [S. Sánchez 2018] [71]. This is formulated in Fig 53.

With a Corollary for tensor ranks specifically displayed in Fig 54.

Alongside the start of a proof in the form of a bijection of sets.

This admits the following extension to a poset isomorphism. For our simple partitions, add-box forms the same non-crossing zig-zag of immediate adjacency arrow pairs and singlets as the following. The combinations' defining recurrence relation. By both placing arrows running down at $\pm \frac{\pi}{4}$ from each vertex. Bar on the 'homo' subset, which only has room for the $-$ case.

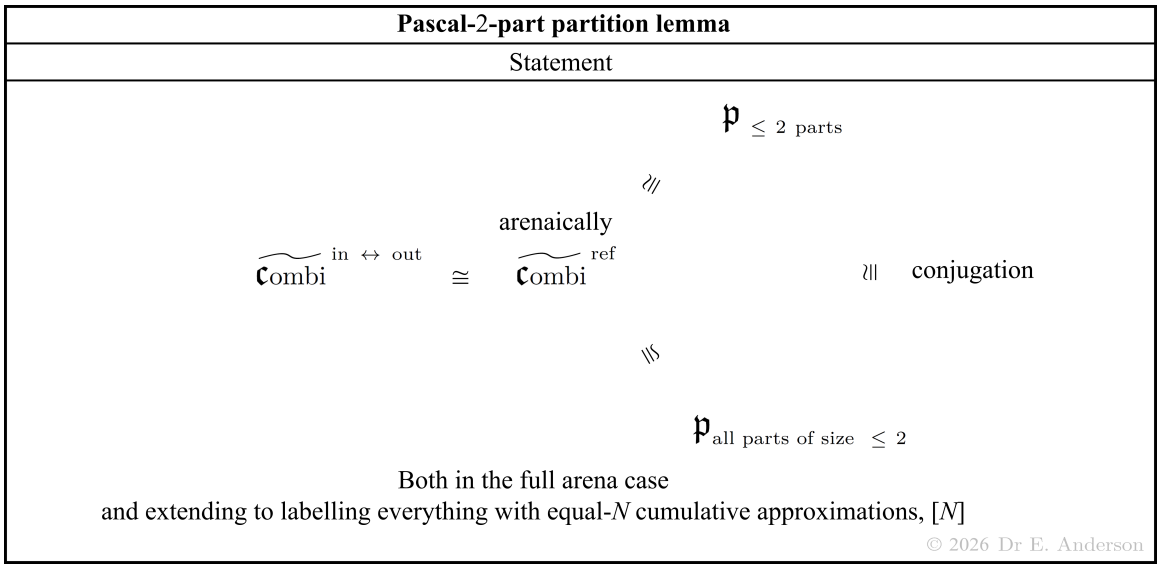


Figure 53:

The above common zig-zag furthermore aligns the following. Common direct heir and common direct ancestor meet and join for combinations. And Young diagram intersection and union. This finishes the establishing of a lattice isomorphism in the full case and the trivial $N \leq 1$ cumulative cases. And a semi-lattice one in the $N \geq 2$ cumulative cases. □

Remark 3 Incidentally, this means that we have shaved off enough partition irregularity to end up with a regular chunk of semi-lattice. Both in terms of degrees and in terms of the remaining runnels.

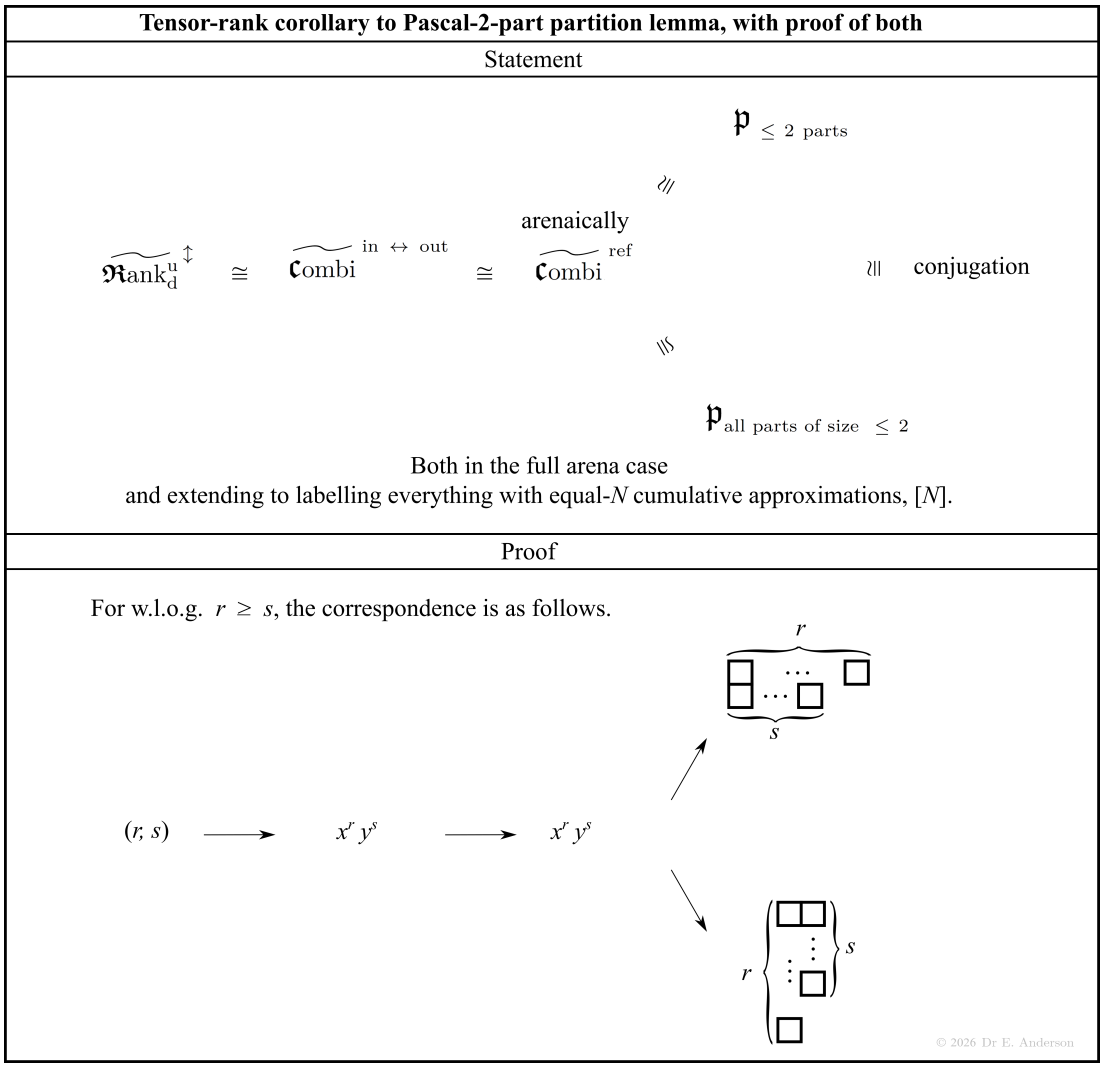


Figure 54:

End Remark The current Review makes two distinct applications of partitions.

On the one hand, a standard application to tensor symmetries. Which leads to an interestingly irregular arena – the Young lattice. Whose count is underpinned by a hard count: the number of partitions on N objects, $p(N)$. Underpinned, that is, in the manner of the corresponding cumulative sum.

On the other hand, a non-standard application to u.d.d. types. Which leads to the $\leq$ 2-part partitions' much simpler regular arena. Which in turn coincides with the in $\leftrightarrow$ out quotient that is realized as a reflection at the level of the arena of combinations Counts for which are systematic and basic.

Acknowledgements I thank S. Sánchez for teaching me Order Theory to partner my previous Graph Theory analyses. And for discussions on partitions, Drawing and Visualization, Arenization, writing scientific text, on the building figure-making factories, and a tour of some of the citizens of Kallista. A. Ford for being part of the middle four of these discussions. And for comments on tensor networks, trees, Theoretical Computer Science and padding. K. Everard, for discussions on vectors, matrices, tensors, eigentheory, partitions and applied Graph and Order Theory. Chris Isham for discussions, references and encouragement in the pursuit of bundles. And the Applied Combinatorics and Topology Discussion Group.

References

[1] J.J. Sylvester, "On the General Theory of associated Algebraical Forms" Cam. Dublin Math. J. **6** 289 (1851). This is reprinted in Vol 1. of his Collected Works (C.U.P., Cambridge 1904; A.M.S., Providence RI 2008).

[2] L.P. Eisenhart, *Riemannian Geometry* (P.U.P., Princeton 1926; 1949).

[3] H. Jeffreys, *Cartesian Tensors* (C.U.P., Cambridge, 1931).

[4] F.J. Murray and J. von Neumann, "Rings of Operators" Ann. Math. **37** 116 (1936).

[5] K. Yano, *Theory of Lie Derivatives and its Applications* (North-Holland, Amsterdam 1955).

[6] H. Jeffreys and Bertha Swirles Jeffreys, *Mathematical Physics* (C.U.P., Cambridge, 1956).

[7] M.R. Spiegel, *Vector Analysis (And An Introduction to Tensor Analysis)* (McGraw-Hill, New York 1959).

[8] B. Spain, *Tensor Calculus. A concise course.* (Oliver and Boyd, Edinburgh 1960; Dover, Mineola NY 2003).

[9] R. Arnowitt, S. Deser and C.W. Misner, "The Dynamics of General Relativity", in *Gravitation: An Introduction to Current Research* ed. L. Witten (Wiley, New York 1962), arXiv:gr-qc/0405109.

[10] R.C. Wrede, *Introduction to Vector and Tensor Analysis* (Self-published 1963; Dover, Mineola NY 1972).

[11] H. Flanders, *Differential Forms. With Applications to the Physical Sciences* (General Publishing Company, Toronto, Canada 1963; Dover, Mineola NY 1989).

[12] R.L. Bishop and S.I. Goldberg, *Tensor Analysis on Manifolds* (MacMillan, London 1968; Dover, Mineola NY 1980).

[13] C.W. Misner, K. Thorne and J.A Wheeler, *Gravitation* (Freedman, San Francisco 1973).

[14] R. Penrose "Angular Momentum: an approach to Combinatorial Spacetime," in *Quantum Theory and Beyond* ed. T. Bastin (C.U.P., Cambridge 1971); reprinted *Collected Works Vol 2. Works from 1968-1972* (O.U.P., N.Y. 2011).

[15] D. Lovelock and H. Rund, *Tensors, Differential Forms and Variational Principles* (Wiley, New York 1975; Dover, Mineola NY 1989).

[16] B. O'Neill, *Semi-Riemannian Geometry with applications to Relativity* (O.U.P., 1983).

[17] R.M. Wald, *General Relativity* (U. Chicago P., Chicago 1984).

[18] J.M. Stewart, *Advanced General Relativity* (C.U.P., Cambridge 1991).

[19] W. Rindler, *Relativity. Special, General and Cosmological* (O.U.P., Oxford 2001).

[20] B. O'Neill, *Elementary Differential Geometry* 2nd ed. (Elsevier, Amsterdam 2002; 1st ed. 1966).

[21] P. Cvitanović, *Group Theory, Birdtracks, Lie's, and Exceptional Groups* (P.U.P., Princeton NJ 2008).

[22] D.C. Kay, *Tensor Calculus* (McGraw–Hill, N.Y. 2011).

[23] T.W. Körner, *Vectors, Pure and Applied: A General Introduction to Linear Algebra* (C.U.P., Cambridge 2012).

[24] M. Abadi et al, "TensorFlow: a System for Large-Scale Machine Learning", arxiv:1605:08965.

[25] J.M. Lee, *Introduction to Riemannian Manifolds* (Springer, New York 2018).

[26] C. Roberts et al, "TensorNetwork: A Library for Physics and Machine Learning", arxiv:1905:01330.

[27] *Online Encyclopaedia of Tensors and other Multi-Linear Arrays*, institute-theory-stem.org/online-encyclopaedia-of-tensors-and-other-multi-linear-arrays/ .

[28] E. Anderson, "What is a Tensor?" Sample Chapter from *Linear Mathematics*, freely available at conceptsofshape.space/linear-mathematics-sample-chapter-tensors-1/ (2026).

[29] "Tensor Networks throughout the Basic Theory of STEM. I. Vector Algebra", Online Encyclopaedia of Tensors and other Multi-Linear Arrays, institute-theory-stem.org/oetoma-tensor-networks-in-vector-algebra/ (2026).

[30] "II. Vector Calculus", institute-theory-stem.org/oetoma-tensor-networks-in-vector-calculus/ (2026).

[31] E. Anderson and K. Everard (she/her), "III. Linear Algebra", institute-theory-stem.org/oetoma-tensor-networks-in-linear-algebra/ (2026).

[32] *Linear Mathematics*, forthcoming (2028); for now, see its Homepage conceptsofshape.space/linear-mathematics/ for news and further samples.

The above are the primary references: on tensors

History of combinations and their arena

[33] That we know of, individual combinations were first calculated in the 4th Century B.C.E., both in Ancient Greece, by Xenocrates of Chalcedon, and in India.

[34] Pingala, *Chandahs̄āstra* ('*Science of meters*') (3rd Century B.C.E. India). This poetic meter classification goes as far as introducing the binary numbers, 'Fibonacci numbers' and 'Pascal's triangle' alias the arena of combinations. His name for this triangle was, phonetically, *meru prastāra*, meaning 'staircase of Meru', where Meru is a particular mountain.

[35] Halāyudha (10th Century C.E. India) detailed how to centre Pingala's staircase.

[36] Jia Xian rediscovered and studied Pingala's staircase in 11th Century China.

[37] Rabbi Abraham ibn Ezra derived symmetry at the level of integer-power binomial coefficients in the 12th century.

[38] Yang Hui expanded this study around the turn of the 14th century.

[39] To the extent that Zhu Shijie referred to 'Yang Hui's triangle' in his work containing both of the following. What we referred to as the 'hockey stick identity'. And what would one day be identified as the first poset Hasse diagram depiction of our object.

A formal reference where this poset figure can be found is J. Needham, *Science and Civilization in China*, vol. 3 (C.U.P., New York 1959).

[40] B. Pascal rediscovered Pingala's staircase alias Yang Hui's triangle, writing down a study of its properties in "Traité du Triangle Arithmetique" (1654), published by Chez Guillaume Desprez in 1665. It consequently widely came to be known as Pascal's triangle. He proved what we refer to as Pascal's identity, for all that the elementary pattern that it describes was well-known to previous investigators.

[41] P. Fermat rediscovered the 'hockey stick identity' in the 17th Century.

[42] A. von Ettingshausen, "Die Combinatorische Analysis als Vorbereitungslehre zum Studium der Theoretischen höhern Mathematik" ("Combinatorial Analysis as preparatory instruction for the study of Theoretical higher Mathematics") (Wallishauffer, Vienna 1826).

History and reviews of integer partitions and their arena

[43] G. Leibniz, posed the problem of the integer partitions, under a name translating to 'divulsions', in a letter to Johann Bernoulli in 1674.

[44] By Euler's day, the name 'partition' had already appeared. The first major theoretical breakthough in the study of partitions was L. Euler, *Partitio Numerorum*, Novi Commentarii Acad. Sci. Petropol. (1753).

[45] Modern Combinatorial notation for partitions started with J.J. Sylvester, "On Mr Cayley's impromptu demonstration of the Rule for determining at sight the Degree of any Symmetrical Function of the Roots of an Equation expressed in terms of the Coefficients", Lond, and Edin. Phil. Mag. (1853), crediting N.L. Ferrers as the originator. This is also reprinted in Vol. 1 of Sylvester's Collected works...

[46] A. Young, "On Quantitative Substitutional Analysis", Proc. London Math. Soc. Series 1 **33** 97 (1901); **35** 361 (1902). This firstly provides a more useful Combinatorial notation for partitions, due to its amenability to further decor, from diagrams to tableaux encoding additional Lie-Theoretic information. Secondly, its principal purpose is developing an advanced version of the Representation Theory of the symmetric group. Thirdly, detailed study of the arena of partitions – the Young lattice – started from here...

[47] P.A. MacMahon, *Combinatory Analysis* Vols I and II (C.U.P., Cambridge 1915-1916). This is an excellent source of previous references.

[48] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers* (O.U.P., New York 2009; 1st ed. 1938).

[49] H.L. Alder, "Partition Identities - from Euler to the present", Amer. Math. Monthly **76** 733 (1969).

[50] G.E. Andrews, "Partition Identities", Adv. Math. **9** 10 (1972).

[51] D. Stanton and D. White, *Constructive Combinatorics* (Springer-Verlag, New York 1986).

[52] I.G. MacDonald, *Symmetric Functions and Hall polynomials* (O.U.P., Oxford 1995).

[53] W. Fulton, *Young Tableaux, with Applications to Representation Theory and Geometry* (C.U.P., Cambridge 1997).

[54] G.E. Andrews, "The Theory of Partitions", *Encyclopedia of Mathematics and its Applications* **2** (1998).

[55] W. Fulton and J. Harris, *Representation Theory. A First Course* (Springer, New York 2004).

[56] G.E. Andrews and K. Eriksson, *Integer Partitions* (C.U.P., Cambridge 2004).

Some useful modern Combinatorics references

[57] P.J. Cameron, Combinatorics, (C.U.P., Cambridge 1994).

[58] V.K. Balakrishnan, Combinatorics (McGraw-Hill, New York 1995).

[59] D.E. Knuth, *The Art of Computer Programming. Vol 1. Fundamental Algorithms* 3rd ed. (Addison-Wesley, Reading, Ma. 1997).

[60] J. Neggers and H.S. Kim, *Basic Posets* (World Scientific, 1998).

[61] G. Grätzer, *General Lattice Theory* 2nd ed. (Birkhäuser Verlag, Basel 1998).

[62] D.B. West, *Introduction to Graph Theory* (Prentice–Hall, Upper Saddle River N.J. 2001).

[63] B.A. Davey and Hilary A. Priestley, *Introduction to Lattices and Order* (C.U.P, Cambridge 2002).

[64] C.A. Charalambides, *Enumerative Combinatorics* (Chapman and Hall, Boca Raton Fl. 2002). **IP**

[65] G. Grätzer, *General Lattice Theory. I. The Foundation* (Birkhauser, Boston 2003).

[66] K.P. Bogart, *Combinatorics Through Guided Discovery*, completed posthumously by M.T. Keller, O. Levin, and K.E. Morrison, (Createspace Independent Publishing Platform, U.S.A 2017), also freely available as https://bogart.openmathbooks.org/pdf/bogart.pdf . Most of this material – K.P. Bogart's own contributions – dates back to 2004. **P**

[67] G. Grätzer, *Lattice Theory: First Concepts and Distributive Lattices* (Courier, 2009).

[68] R.J. Wilson, *Introduction to Graph Theory* (Longman, Edinburgh 2010). 1st ed. 1972.

[69] R.P. Stanley, *Enumerative Combinatorics* Vol. 1 2nd ed. (C.U.P, Cambridge 2010). 1st ed. 1997; use 2nd ed. for technical reference.

[70] *Handbook of Graph Drawing and Visualization* ed. R. Tamassia (Taylor & Francis, Boca Raton, Fl. 2014).

[71] Combinatorial discussions between S. Sánchez and E.A (2018-2020).

[72] D.B. West, *Combinatorial Mathematics* (C.U.P., Cambridge 2021).

[73] E. Anderson, *Applied Combinatorics*, Widely-Applicable Mathematics Series. A. Improving understanding of everything with a pinch of Combinatorics. **0**, (2022). Made freely available in response to the pandemic here: conceptsofshape.space/applied-combinatorics/ .

[74] E. Anderson and A. Ford (she/her), "Simple Graphs' 8 Double-Irreducibility Classes", institute-theory-stem.org/oeagot-graph-double-irreducibility-classes/ (2026).

[75] See e.g. "Pascal's triangle", https://en.wikipedia.org/wiki/Pascal%27s_triangle (09-2026 version).

[76] "The Number of Partitions of n ", https://oeis.org/A000041 (09-2026 version).

[77] "Sum_{$k = 0..n$} $p(k)$ where $p(k)$ = number of partitions of k ", https://oeis.org/A000070 (09-2026 version).

[78] The cover-figure of [79] debuts an advanced Drawing and Visualization style poset arenas using the Young lattice of partitions as an example.

[79] Online Encyclopaedia of Applied Graph and Order Theory, institute-theory-stem.org/online-encyclopaedia-of-graphs-and-orders/ .

References we use in motivating Arenization

[80] P. Halmos, *Finite-Dimensional Vector Spaces*, (Springer-Verlag, New York 1942).

[81] C. Lanczos, *The Variational Principles of Mechanics* (U. Toronto P., Toronto 1949).

[82] J.-P. Serre, *Lie Algebras and Lie Groups* (Benjamin, New York 1965).

[83] M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities* (Prindle, Boston 1969; Dover, Mineola NY 1992).

[84] S. Smale "Topology and Mechanics. II. The Planar N-Body Problem", Invent. Math. **11** 45 (1970).

[85] D.E. Knuth, *The Art of Computer Programming* Vol 3. *Sorting and Searching* (Addison-Wesley, 1973; 1998).

[86] V.I. Arnol'd, *Mathematical Methods of Classical Mechanics* (Springer, New York 1978).

[87] M. Nakahara, *Geometry, Topology and Physics* (I.O.P., London 1990).

[88] D. Husemoller, *Fibre Bundles* (Springer, New York 1994).

[89] C.J. Isham, *Modern Differential Geometry for Physicists* (World Scientific, Singapore 1999).

[90] D.G. Kendall, D. Barden, T.K. Carne and H. Le, *Shape and Shape Theory* (Wiley, Chichester 1999).

[91] J.B. Conway, *A Course in Functional Analysis* (Springer, 2007).

[92] R. Ghrist, *Elementary Applied Topology* (2014).

[93] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States. An Introduction to Quantum Entanglement* 2nd ed. (C.U.P., Cambridge 2017).

[94] G. Carlsson and M. Vejdemo-Johansson, *Topological Data Analysis with Applications* (C.U.P., Cambridge 2022).

[95] R. Montgomery, *Four Open Questions for the N-Body Problem* (C.U.P., Cambridge 2025).

[96] E. Anderson and K. Everard (she/her) "Eigentheory of $\mathbb{R}$-Symmetric Matrices from a Graph and Order Theory point of view, for [27] forthcoming (2026).

[97] E. Anderson, *The Structure of Flat Geometry*, forthcoming (2027); for now, see its Homepage conceptsofshape.space/geometry/ .

[73, 79, 78, 27, 32] are also references in this sense.

Other supporting technical references

[98] H.S.M. Coxeter, *Introduction to Geometry* (Wiley, New York 1961; 1989).

[99] R. Courant and D. Hilbert, *Methods of Mathematical Physics* Vol. 2 (Interscience, New York 1962; Wiley, Chichester 1989).

[100] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*. (Wiley, New York 1974; Dover, Mineola NY 2002).

[101] See Part I of J-P. Serre, *Linear Representations of Finite Groups* (Springer–Verlag, New York, 1977).

[102] G.E. Martin, *Transformation Geometry: An Introduction to Symmetry*, (Springer, 1982; 1996).

[103] Yvonne Choquet-Bruhat, Cécile DeWitt–Morette and Margaret Dillard-Bleik, *Analysis, Manifolds and Physics* (Elsevier, Amsterdam 1982).

[104] M.A. Armstrong, *Basic Topology* (Springer-Verlag, New York 1983).

[105] A.F. Beardon *Algebra and Geometry* (C.U.P., Cambridge 2005).

[106] J. Stillwell, *The Four Pillars of Geometry* (Springer, New York 2005).

[107] *Naive Lie Theory* (Springer, N.Y. 2010).

[108] J.M. Lee, *Introduction to Smooth Manifolds* (Springer, New York 2013).

[109] C. Laurent-Gengoux, Anne Pichaereau and P. Vanhaecke, *Poisson Structures* (Springer-Verlag, Berlin 2013).

[110] E. Anderson, "Hopf's Little Mathematics 32 times in Physics and Geometry", forthcoming (2026).

[97] is also a reference in this sense.