Tensor Networks throughout the Basic Theory of STEM. IV. Cartesian Tensors

E. Anderson*

Abstract

We continue our program of applying tensor networks throughout the basic theory of STEM, today with a Review on Cartesian tensors. While basic accounts often treat these 1) plainly rather than 2) with upstairs-downstairs distinction, we entertain and compare both. 2) extends to vastly more general notions of tensor. We cover tensor transformation laws, Tensor Algebra, tensor symmetries, and isotropic tensors, all in tensor networks notation.

We also cover arenas of tensors, i.e. the totalities of various type of tensor. Not only A) tensors' (co)vector space products of fixed rank and dimension and their tensor field generalizations in fibre bundle form. But also the arenas of B) ranks, C) dimensions, and D) both together. And then the bigger arenas placing A) over B-D), now as general bundles. In adapted variables, B) can be viewed in terms of Pascal's triangle and 2-part partitions, while D) is somewhat more intricate. Various meaningful quotients of these arenas are also considered. Such extensive focus on arenas is part of Modern Applied Topology, with the current Article benefiting from simpler Combinatorial and Linear instances.

Arenizing tensor symmetries returns the full Young lattice arena of partitions. While arenizing isotropic tensors gives a rather interesting application of the joint arena of dimensions and ranks. Aside from the current Review's overall distinctively Combinatorial point of view arising standardly from symmetries and novelly from arenization, more pseudotensor, quotient-theorem and isotropic-tensor variants than usual are included.

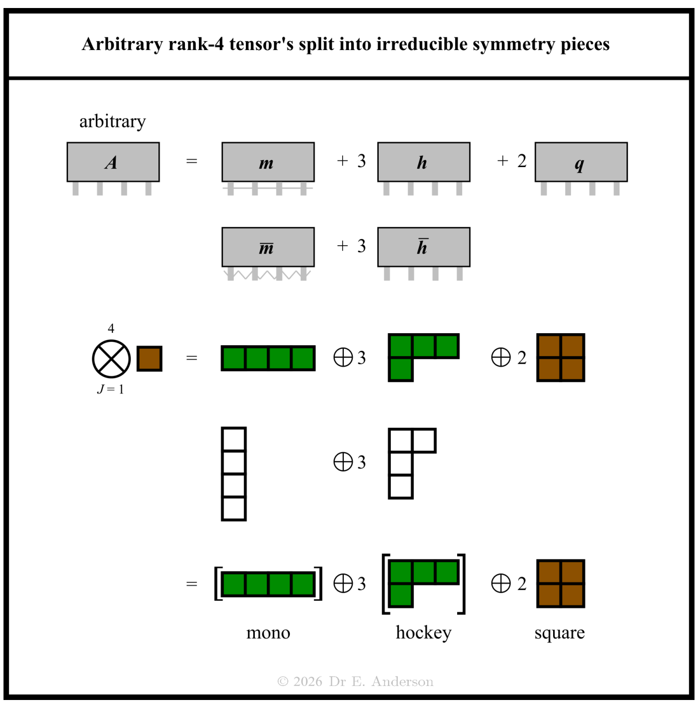

Figure 1:

This Review is ⟨3⟩: accessible to third-year undergraduates.

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* Dr.E.Anderson.Maths.Physics *at* protonmail.com . Institute for the Theory of STEM

Contents

1 Introduction

1.1 The underlying transformations

Structure -1 The current Article works with frames in

$$\mathbb{R}^D \ .$$

With a shared origin, and a globally-valid set of axes at right angles to each other. These are *Cartesian frames*, corresponding to *Cartesian coordinate systems*. For all that these italicized terms continue to make sense without the shared origin.

Structure 0 The following transformations

$$\boldsymbol{L}$$

[26, 27, 8] preserve these frames, shared origin included. *Rotations* (Fig 2.a) about an axis passing through the origin. And *reflections* (Subfig b) about a hyperplane $\mathbb{R}^d$ though the origin. Where

$$d \ := \ D \ - \ 1 \ . \tag{1}$$

In 2-D, this is just a line through the origin, and in 3-D, just a plane through the origin.

For whichever fixed D, the $\boldsymbol{L}$ form the *orthogonal group*

$$O(D)$$

[26, 29, 15]. These are the homogeneous-linear transformations that furthermore preserve the usual (*Euclidean*) *scalar product*, and thus both *Euclidean lengths* and *Euclidean angles*. Schematically, using the habitual primed and unprimed frame notation,

$$\underline{\boldsymbol{x}} \ \longrightarrow \ \underline{\boldsymbol{x}}' \ = \ \underline{\underline{\boldsymbol{L}}} \cdot \underline{\boldsymbol{x}} \ . \tag{2}$$

Two equivalent phrasings to preserving the Euclidean scalar product are preserving the *Euclidean inner product*, and preserving the *Euclidean metric*

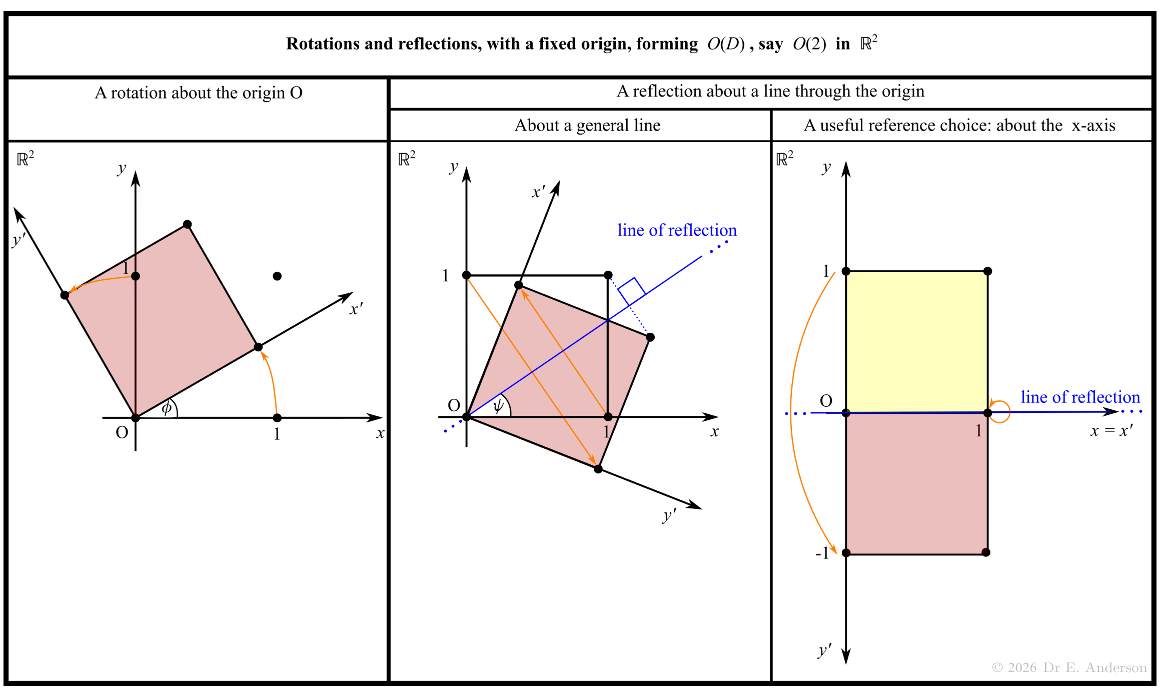

Figure 2:

Structure 0′ We often also drop the discrete requirement of preservation under reflections. The transformations left are then purely rotations, forming the group

$$Rot(D) \quad = \quad SO(D) \; .$$

The ensuing tensors are often referred to as (*Cartesian*) *pseudotensors* in introductory and Physics texts.

Remark 1 First courses on Cartesian tensors [3, 4, 31, 10, 30] often continue at this point by presenting some preliminary identities afforded by orthogonality. The current Review is however a second course, and so benefits from its intended audience having seen this before. Furthermore, the current Review is part of an online Encyclopaedia [37] that concentrates on reformulating the basic theory of STEM in terms of tensor networks. And is the Encyclopaedia's fourth Article. So by this point we have already presented tensor networks formulations of Vector Algebra [39], Vector Calculus [40] and Linear Algebra [41]. As such, our audience is already familiar with what tensor networks are.[1]

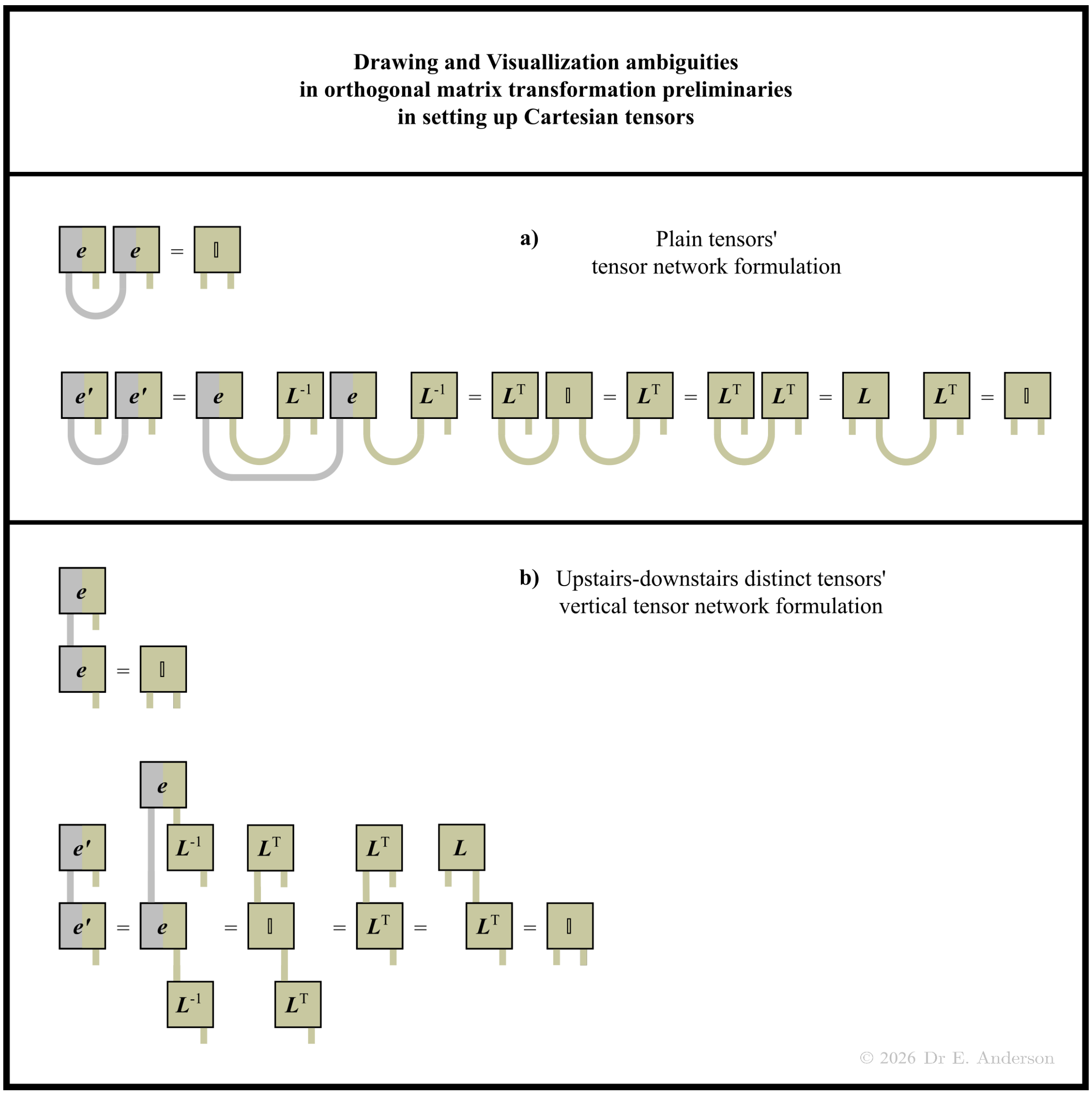

Figure 3:

[1] If not, consult [39]...

Exercise 0 And so we do not present the abovementioned identities in a first course's habitual component notation. Instead, we present them in firstly one tensor network notation, and then another![2] And expect the Reader to figure out what these equations (Fig 3) look like in components. And thus to identify them with equations that featured in your first course on Cartesian tensors... What basic Mathematical move does each step in the figure correspond to? Where $\boldsymbol{e}$ and $\boldsymbol{e}'$ are an unprimed and primed pair of orthonormal linear bases...

1.2 Aside: coordinate transformations more generally, leading to further kinds of tensor

1.2.1 The other linear transformations

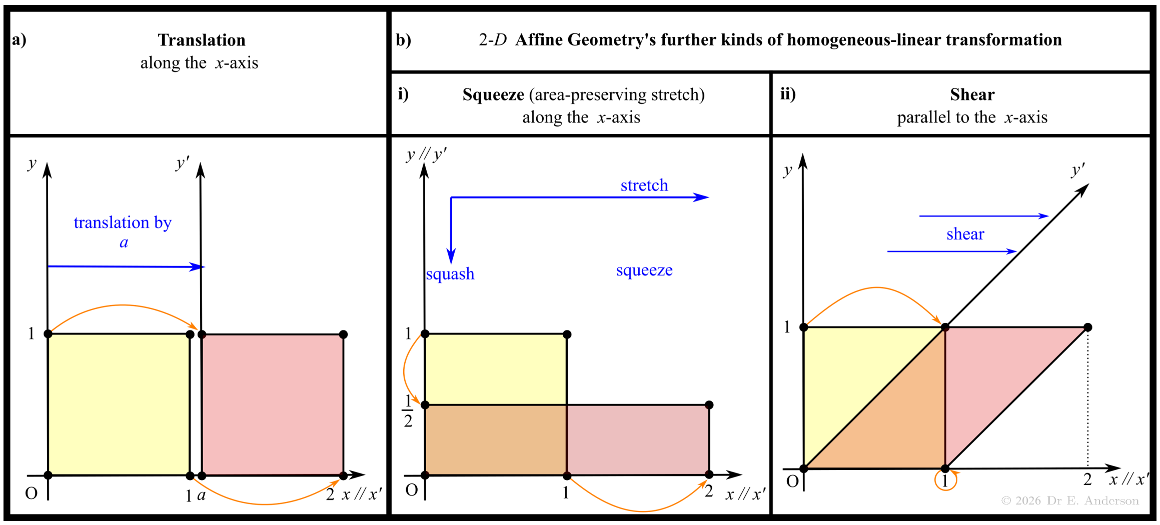

Figure 4:

Remark 1 Since this is a second course, comparison with other coordinate transformations that you have seen elsewhere is reasonable. Leading in various cases to notions of tensor that you have seen in other courses, or shall be studying soon...

Structure I Were the origin to be free to shift under transformations, *translations* (Fig 4.a) would also be in play. Then the whole of the *Euclidean group* $Eucl(D)$ [19] of transformations would be involved. Corresponding to Euclidean Geometry. Which is very widely taught in the case of the Euclidean plane $\mathbb{R}^2$ as equipped with the Euclidean metric (or norm $||\ ||$ or inner product). Though some of you will have also specifically studied Geometry in Euclidean space $\mathbb{R}^3$. And all of you will have used $\langle \mathbb{R}^3, ||\ || \rangle$ as an excellent model for Physical 3-space....

In this context, we pass from (2) to its inhomogeneous-linear counterpart,

$$\overline{\boldsymbol{x}} \longrightarrow \overline{\boldsymbol{x}}' = \underline{\overline{\boldsymbol{L}}} \cdot \overline{\boldsymbol{x}} + \overline{\boldsymbol{a}} \, . \tag{3}$$

Structure II An alternative generalization is from Euclidean length preserving homogeneous-linear transformations to merely invertible homogeneous-linear transformations. This enlarges $O(D)$ to the *general linear group*

$$GL(D) \, .$$

At the level of matrices, passing from $GL(D)$ to its $O(D)$ subgroup enforces that the transposed matrices are to serve as the inverses. And consequently that the determinant is ± 1 . With $+1$ characterizing the special orthogonal group. And -1 the unique group coset of other orthogonal transformations. Which can be taken to comprise $SO(D)$ multiplied by whichever reflection.

[2] See [41] for this bicoloured notation, alongside motivation for multicoloured or 'rainbow' tensor networks...

There are however other determinant ± 1 general linear transformations. I.e. the $+1$ determinant special linear transformations

$$SL(D) \ .$$

These come in 2 conceptual types, 2-D representatives of which are drawn in Subfig b). And then their group coset upon multiplying by some reflection, covering the -1 determinant case. The independent further transformation in the general-linear case is the overall *dilation*, alias change of scale.

Structure III Both of the above generalization can be applied at once, giving the inhomogeneous general-linear transformations,

$$\overline{\boldsymbol{x}} \longrightarrow \overline{\boldsymbol{x}}' \ = \ \underline{\overline{\boldsymbol{G}}} \cdot \overline{\boldsymbol{x}} \ + \ \overline{\boldsymbol{a}} \ . \tag{4}$$

Which form the *Affine group*

$$Aff(D)$$

[19, 44] in the $\mathbb{R}^D$ setting.

Remark 2 Tensors can also be defined for $SL(d)$ and $GL(d)$. With or without reflections. Which indicates part of why '*Cartesian* pseudotensors' is a more precise description of part of what the current Review covers than just 'pseudotensors'.

1.2.2 Special Relativity's inertial frames

Structure IV Those Readers who have already studied Special Relativity will have seen a close parallel of the above discussion there. With in particular the Lorentz group in place of the orthogonal group, and the Poincaré group in place of the Euclidean group. Living on Minkowski spacetime $\mathbb{M}^{D\,+\,1}$ in place of $\mathbb{R}^D$. And leading to a theory of Lorentzian tensors that closely parallels that of Cartesian tensors. Now modelling Physics in Special Relativity's [24] notion of inertial frames.

1.2.3 General curvilinear transformations

Structure V Let us next entertain *general curvilinear coordinate transformations* in $\mathbb{R}^D$. Now with not some constant transformation matrix but some coordinate-dependent matrix

$$\overline{\boldsymbol{x}} \longrightarrow \overline{\boldsymbol{x}}'(\overline{\boldsymbol{x}}) \ . \tag{5}$$

With corresponding transformation matrix with components

$$J_i{}^{j'} \ := \ \frac{\partial x'^{j'}}{\partial x^i} \ . \tag{6}$$

A first coordinate-free notation for this is

$$\underline{\overline{\boldsymbol{J}}} \ := \ \frac{\partial \overline{\boldsymbol{x}}'}{\partial \overline{\boldsymbol{x}}} \ . \tag{7}$$

This is the *Jacobian matrix*. We leave it to the Reader to cast this definition in tensor networks form.

Remark 1 From here on it will not do to transform just our little yellow square. Since now distinct small approximate squares will in general transform distinctly from each other. Thus we need a *coordinate grid* containing at least a few small approximate squares both to see what is going on and to indicate this variability with location. See Fig 5 for a first simple example.

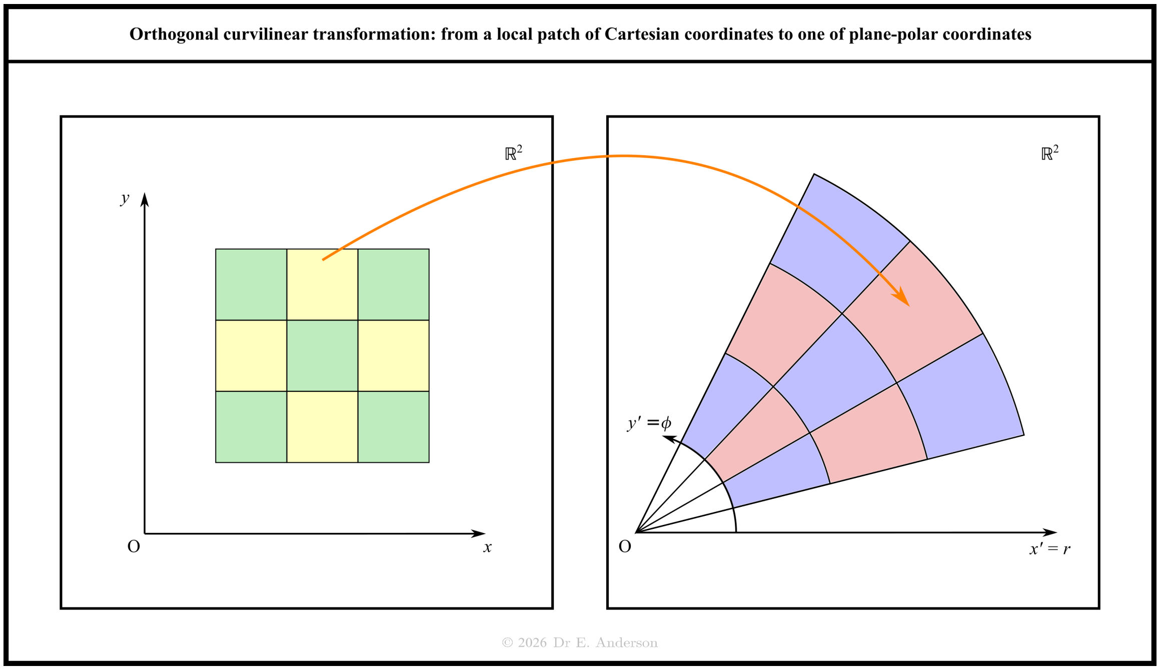

Figure 5:

A theory of tensors is then supported using the general-curvilinear $\boldsymbol{J}$ in place of the Cartesian tensors' orthogonal $\boldsymbol{L}$. The corresponding group of transformations is the *diffeomorphism group*[3]

$$Diff\left(\mathbb{R}^D\right) .$$

Everything in the current SSSec can also be carried out in $\mathbb{M}^{D\,+\,1}$. Corresponding to modelling not just Special-Relativistic inertial frames but also accelerated frames as well. For all that this is not usually included in basic courses.[4]

1.2.4 Differentiable manifolds and General Relativity

Structure 6 Even more generally, Jacobians can be locally defined in differentiable manifolds [32] $\mathfrak{m}^D$. To provide some context, if these are given their own Affine structure [23, 35], then a notion of curvature is supported. And so our modelling scope includes in general curved spaces, as considered in Differential Geometry [25]. If these are furthermore given a metric level of structure, then we arrive at Riemannian Geometry for positive-definite metrics. And at semi-Riemannian Geometry [17] for the $-\,+\,...\,+$ subcase of indefinite-signature metrics. Part of whose modelling scope is the in-general curved spacetimes of General Relativity [22, 13].

The way this generalization works is as follows. Differentible manifolds are guaranteed to support *local coordinate charts* (Fig 6.a),

$$\phi \;:\; \mathfrak{U} \longrightarrow \mathbb{R}^D .$$

For $\mathfrak{U}$ some open subset within $\mathfrak{m}^D$, amounting to a first use of the above 'local'.

The second use is to consider pairs of coordinate charts ϕ, ϕ' whose respective $\mathfrak{U}, \mathfrak{U}'$ overlap (Subfig b). So then one can form the Jacobian matrix from ϕ 's unprimed local coordinates to ϕ 's primed ones. As enabled by now one locally working with a map

$$\mathbb{R}^D \longrightarrow \mathbb{R}^D .$$

[3]There are multiple specific such, depending on which sufficiently tractable function space [81] we wish to work with.

[4]If interested in this, carefully peruse [24]. Whose text is not only an introduction to the subject but also interwoven with more advanced comments on this particular subtlety.

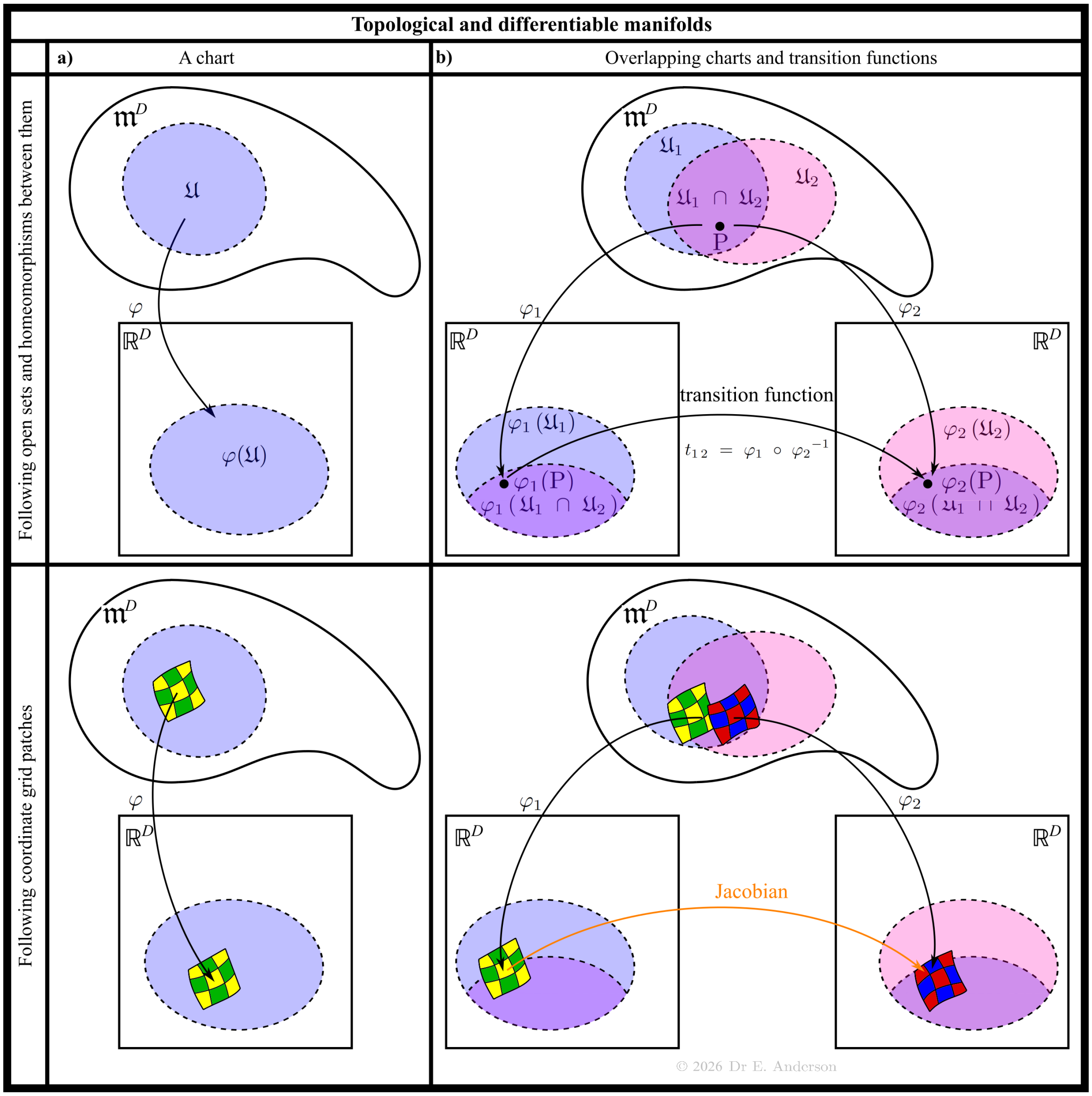

Figure 6:

So that the usual Calculus is available. By which the Jacobian's derivatives are meaningfully defined, at least for sufficiently well-behaved coordinate functions.

And so the flat-space general-curvilinear Tensor Calculus' Jacobian coordinate transformations extend to use on Differentiable manifolds. Providing us with a general-curvilinear Tensor Calculus on differentiable manifolds. Whose transformation group is now[1]

$$Diff\left(\mathfrak{m}^D\right) .$$

For use in Differentiable Geometry and its applications, including General Relativity.

1.2.5 The upstairs–downstairs distinction (u.d.d.)

Naming Remark 1 On the one hand, the above formula for the Jacobian matrix displays an *upstairs–downstairs distinction* of indices.

On the other hand, Cartesian tensors are usually presented without such a distinction, which we refer to as *plain indices*.

Aliases for u.d. indices include *raised* and *lowered* indices. U.d. is a more straightforward name. After where the indices are are located. As opposed to with reference to some raising or lowering action...

The alias *contra-* and *co-* indices is also widespread, following Sylvester [1]. 'Co' here refers to transforming like the frame, and 'contra' to transforming oppositely. This conveys a further piece of information relative to the above aliases. *Covariant* and *contravariant* are the general words of this kind. Another commonly encountered name is *covector*, short for *covariant vector*; *contravariant vectors* are usually just called vectors. In turn, *dual vector* and 1-*form* [11, 13, 20, 33], are further common aliases for covectors. While *column vector* and *row vector* are simple instances of vectors and covectors respectively.

At least the downstairs-upstairs and co-contra names continue to make sense in the context of index-free notation, of which the tensor networks notation [14, 28, 34, 36] that the current Encyclopaedia is written in is an example. With reference to the usual tensorial mantra of 'define and conceptualize index-free, for all that one usually evokes indices if calculations are required...

Notational Remark 1 Three common styles of index-free notation are as follows. Use empty slots in place of indices. In the u.d.d. formulation, upstairs and downstairs slots need to be distinguished. Or use underlines in the plain case, to underlines and overlines in the u.d.d. formulation. Or use tensor networks, which have unlabelled legs in place of indices. Emanating out of the top of the tensor's box in place of overlines, or out of its bottom in place of underlines. This is the current Encyclopaedia's preferred choice of notation.

Remark 1 A piece of folklore, in particular among Physics majors, is that u.d.d. is a curved space(time) complication. This is incorrect.

Most obviously because it already features in curvilinear coordinates on flat space(time).

But trying to work with plain indices in the context of Special Relativity's inertial frames also incurs mistakes. Via say missing the sign change incurred by

$$\eta_{t\,t} \, V^t \;\; = \;\; - \, V_t \; .$$

Where $\boldsymbol{\eta}$ is the Minkowski metric and $\boldsymbol{V}$ is whatever 4-vector.

Let us leave it to the Reader to find a mistake incurred by neglecting to use u.d.d. in plane polar coordinates (**Exercise 1**).

All in all *each of curvilinear coordinates in flat space, and indefiniteness, also require making the u.d.d.*

Remark 3 In turn, this means that a second piece of folklore – that the theory of Lorentzian tensors is a very close copy of the theory of Cartesian tensors – comes with limitations. At least from the point of view that one usually learned Cartesian tensors in the plain formulation.

There is however another point of view. The above dismissal of the first piece of folklore leads to Cartesian tensors being highly-isolated in supporting a plain-index formulation. While Cartesian tensors support an equivalent u.d.d. formulation. So my chosen course of action is to give a u.d.d.

formulation of Cartesian tensors. This is motivated by its much wider applicability. Upon which the second piece of folklore is rendered rather more true, provided that it refers to u.d.d. formulations of Cartesian tensors...

Pedagogical Remark 1 U.d.d. formulations might not be presented in a course for first or second years. By choosing immediate use, simplicity and tradition. Over what happens when a proportion of the students more eventually encounter curvilinear, Special-Relativistic, Differential-Geometric and Generally-Relativistic material in some order.[5] The current Review furthermore specifically covers much of what could occur if a second course on Cartesian tensors were to be made available to third or fourth year students. As such, the u.d.d. formulation of Cartesian tensors makes for a very natural component of such a course.[6]

Pedagogical Remark 2 On the one hand, degree courses seldom offer second courses in Cartesian tensors. On the other hand, many graduate school programs make use of symmetries and representations. And another of the current Review's components concerns how tensors (not necessarily Cartesian) provide an interesting application of some of this material.

Furthermore, some graduate school programs make heavy use of tensor networks in various specialized areas. This is almost always done without drawing on how many core undergraduate subjects – by this stage very familiar material to the students – can be recast in tensor networks. Graduate school programs also rather more frequently introduce coordinate-free notation for conceptualization. In graduate-level courses, without drawing on how this can already be done for core undergraduate courses. Additionally such ventures done without tensor networks are quite rapidly vulnerable to ugly or broken coordinate-free notation.

Ergo more graduate-level programs should use tensor networks to mediate their wishes to be coordinate-free. While those students starting graduate school programs that do make heavy use of tensor networks would do well to see what happens when tensor networks are applied to the basic undergraduate courses. Which graduate schools do not provide. Providing one motivation for the current Encyclopaedia containing, and indeed starting just with, tensor networks in core basic courses.

Notational Remark 2 A separate benefit of u.d.d. is that it affords a more elegant tensor networks notation. Along the lines of Penrose's vertical variant [14]. Due to this, the current Encyclopaedia [37] has so far presented even Vector Algebra [39] Vector Algebra [40] and Linear Algebra [41] – subjects arguably simpler than Cartesian tensors – in u.d. notation. Without even displaying any plain-index tensor networks counterpart. In the current Review, however, we do for the first time present plain-index tensor networks, and furthermore compare them with the vertical variant.

[5]Plane, cylindrical and spherical polar coordinates are very useful throughout STEM and happen to be orthogonal-curvilinear. As such, students will often have already encountered orthogonally-curvilinear material prior to a first course in Cartesian tensors. Whether or not this material made active use of orthogonal-curvilinear features, or called them such, or abstracted to orthogonal-curvilinearity more generally.

[6]For this would be taught concurrently with Differential Geometry and General Relativity. And with advanced Special Relativity, or possibly after this.

1.3 Outline of contents

1.3.1 The main body

In Sec 3, we define rank for Cartesan tensors, in both plain and u.d.d. formulations. Let us refer to the case without as *plain tensors*. While using *total rank* to mean the sum of upstairs and downstairs ranks.

Cartesian tensor transformation laws are explained and tabulated in tensor networks form in Sec 2. Basic Tensor Algebra results are given and proven in tensor networks form in Sec 4. See [38, 45] for material at a comparable level to the current Review, the first on Cartesian tensors and the second on both these and tensor networks notation for them. Also see [24, 7, 6, 30, 12, 13, 16, 22, 2] for literature on further kinds of tensor, along the previous SSec's lines. Which the current Encyclopaedia shall be recasting in tensor networks notation over the next three years.
Both of the above Sections are mounted upon parts of the following arenaic framework.

In Sec 4, we consider A) standard arenas of tensors for fixed dimension and rank. Where an *arena* is the Mathematical space formed by the totality of some particular kind of Mathematical object, as motivated in SSSec 3.3.3. We furthermore entertain B) partly new Combinatorial tracking arenas (CTA), motivated in SSSec 3.7. Concentrating on all specific examples for total rank ≤ 4. Rank gives Pascal's triangle and 2-part partition mathematics. Various meaningful quotients, and basic Graph-Theoretic compositions for rank-and-dimension, are also contemplated. We finally introduce new discrete-base continuous-attached-object general bundles that jointly model A) and B).

Index-exchange symmetries supported by rank-≤ 4 tensors are listed in Sec 5. These symmetries are classified by partitions. We also cover the decomposition supported by these symmetries and the trace-tracefree decomposition. Both of these carry Repreresentation-Theoretic significance. We also include many more symmetry-enabled corollaries of the quotient theorem for tensors than is usual.

Finally, classification of isotropic Cartesian tensors with rank ≤ 4 is provided in Sec 6 with supporting tensor network diagrams.

The next issue of the current Periodically-Updating Review shall include a further Section on tensor fields and the simple Cartesian tensor Calculus that these support.

The Appendices and Conclusion

On the one hand, Cartesian tensors are taken to be plain in most previous literature. On the other hand, tensor networks are both more usually, and more simply and elegantly, presented using u.d.d. Indeed, the Penrose-type [14] vertical tensor networks that the body of the current Review uses, are a subcase of u.d.d. formulation. The above contrast motivates Appendix A providing plain tensor network diagrams for the main body's topics. And our Conclusion consisting of comparing these two as tensor network notations.

Some Order-Theoretic and Graph-Theoretic structure of Pascal's triangle and its quotient under reflection [67, 69] are exhibited in the brief Appendix B. Both Sec 3's tensor rank arenas and Sec 5's tensor symmetries are substantially underpinned by partitions [66, 48, 57] and their arenas [71]. Motivating Appendix C to provide a unified account of these. With their own highly-developed Drawing and Visualization formalism: *Young diagrams* [46, 51, 55, 69]. And with a cut-off in its specific small examples at the partitions of the number 4. So as to match the current Article's modelling limitation of covering only up to plain- or total-rank 4 tensors.

1.3.2 Conceptual underpinning for the current Review's range of Mathematics

A useful organizational observation at this point is as follows.

On the one hand, tensor networks provide a Combinatorial notation for our Cartesian tensor objects. And Young diagrams (partition Combinatorial notation) for their symmetries. With some support from the tensor product and direct-sum operations (Linear Algebra). As enabled by symmetry groups supporting representations via a well-known map from Group Theory to Linear Algebra: Representation Theory [18, 62].

But arenas seldom belong to the same 'area of Mathematics' as their objects. So it is rather likely that arena-level Combinatorial structure, conceptualization, and diagrammatic notation will be different for that for their objects.

Indeed, tensors' arenization for fixed dimension-and-rank is handled by Linear Algebra's vector spaces, duals and tensor products. And then Differential Geometry's fibre bundles for tensor fields. Order Theory sitting upon Graph Theory at the level of CTAs, revealing Pascal's triangle and further partition Mathematics alongside various basic graph-and-order compositions. And general bundles when tensors over nontrivial CTAs are in play. The symmetries' partitions form the standard Young lattice arena, to a rank arena picking out a simple conjugate pair of subarenas of this. Finally isotropy leans heavily on the CTA, placing interesting heat maps and subarena specifications thereupon.

And so, in an Encyclopaedia covering one 'easy end' of Modern Applied Topology, that the titular 'tensor and other multilinear array' objects are systematically handled using tensor networks does not cover the arenaic half of the subject. Which can however fortunately be handled by a fairly complete orchestra of basic Combinatorics [69]. Alongside several definitions concerning bundles (no details or proofs involving bundles required). Leaving the arenaic extension of the theory of Cartesian Tensors accessible to third-year undergraduates. And so here we are!

2 The Cartesian tensor transformation laws

2.1 $O(D)$-tensors

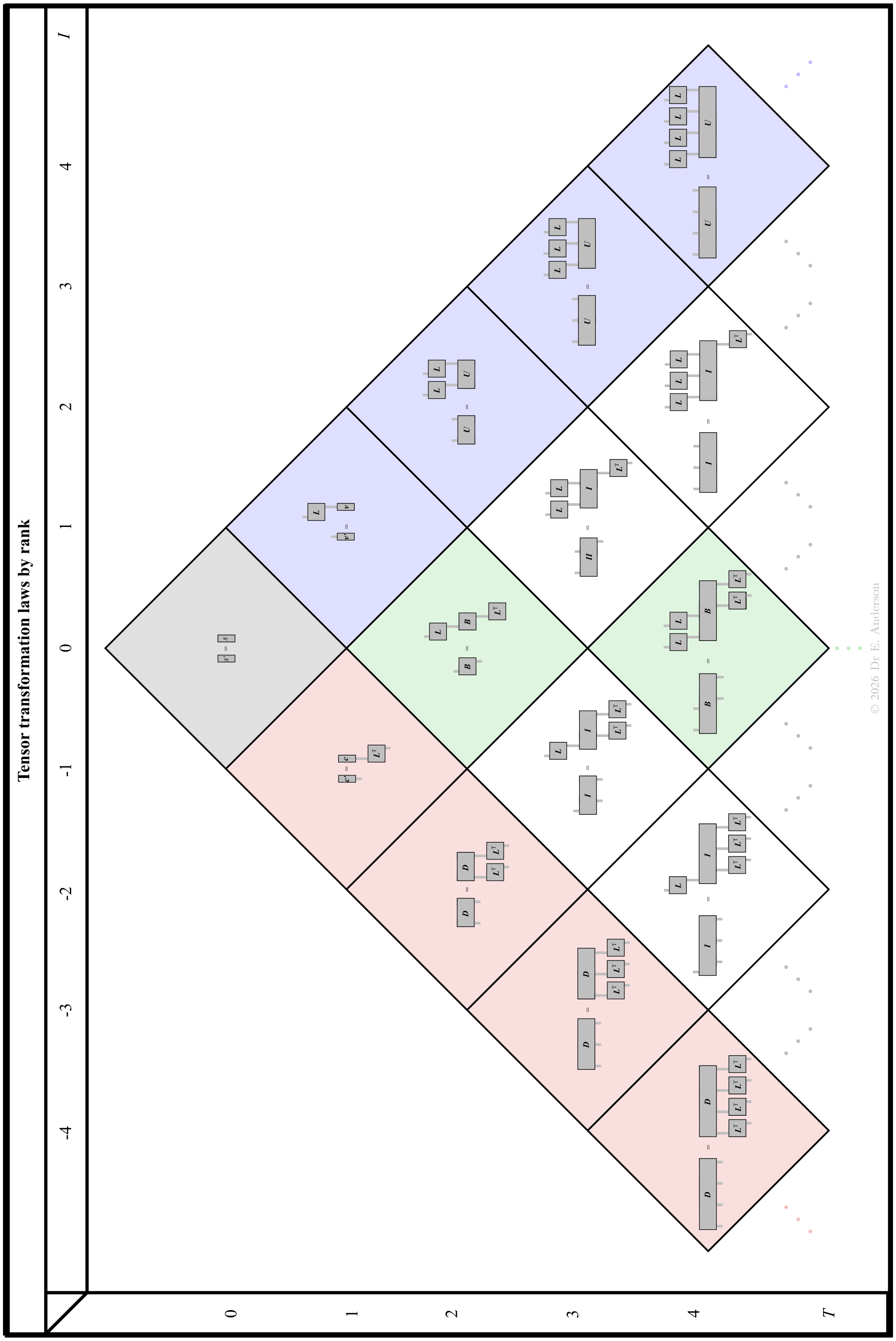

Figure 7:

Remark 1 Let us first use the transformation group $O(D)$, comprising both reflections and rotations. We also refer to the corresponding Cartesian tensors as $O(D)$-*tensors*.

Remark 2 Fig 7 presents the u.d.d. tensor transformation laws from $T = 0$ to 4 inclusive. Making u.d.d. In a Penrose-type vertical variant tensor networks notation. And displayed upon the arena of tensor transformation laws.

Remark 3 See also Fig 8 for a schematic depiction of the general-rank tensor transformation law, for subsequent use.

Remark 4 See instead Appendix A.1 for the smaller repertoire of plain index tensor transformation laws. Using a type of plain tensor network. Upon the simpler arena of plain tensor ranks.

Remark 5 We explain both of these arenas in the next Section, via their being isomorphic to the corresponding arenas of tensor ranks.

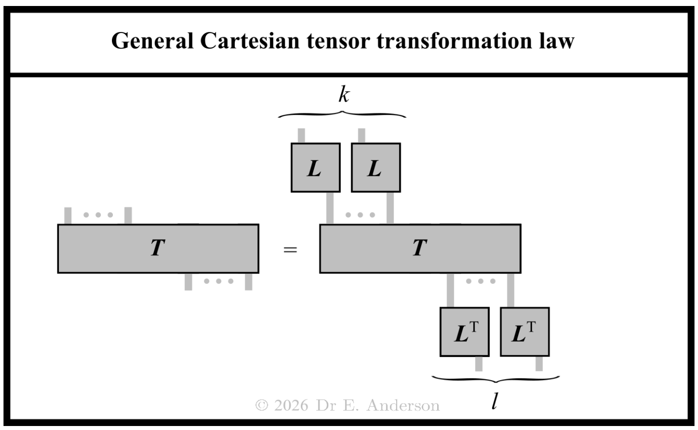

Figure 8:

2.2 $SO(D)$-tensors alias pseudotensors

Structure 1 Let us next drop the reflections, leaving us with the special-orthogonal group $SO(D)$. The corresponding tensors are Cartesian $SO(D)$-*tensors*. Alias Cartesian *pseudotensors*, from the point of view of still considering reflections while permitting the tensor transformation law to generalize as follows. The *sign of the transformation*

$$\sigma(\boldsymbol{L})$$

is to feature as an extra factor in the transformation law. This sign is $+1$ for rotations, -1 for reflections, and multiplicative:

$$\sigma\,(\,\boldsymbol{L}_1\,\boldsymbol{L}_2\,) \quad = \quad \sigma\,(\boldsymbol{L}_1)\,\sigma\,(\boldsymbol{L}_2)\ . \tag{8}$$

So it is $+1$ for the product of whatever number of rotations (for all that this is clearly still a rotation). $+1$ for the product of 2 reflections, which the Reader is invited to check that this is indeed also a rotation. And thus $\sigma = +1$ for the product of an even number of reflections, versus $\sigma = -1$ for the product of an odd number of reflections.

And finally and most pertinently, $\sigma = -1$ for the product of a reflection and a rotation. Which comprises the half of $O(D)$ that is not in $SO(D)$. Forming a second disconnected 'copy' of $SO(D)$. Albeit not now containing the identity element, and thus not constituting a subgroup. The Reader is also encouraged to check that this 'copy' is indeed instead a group coset [26].

Remark 1 Since this is the same modification in each case, let us not re-display this for all the smallest ranks. Rather illustrating it just for the pseudoscalar, pseudovector, and arbitrary rank $(\,q,\,p\,)$-pseudotensor in Fig 9. The $SO(D)$ and 'pseudo' points of view are compatible since for rotations the sign is always $+1$ and thus need not be included as an extra factor...

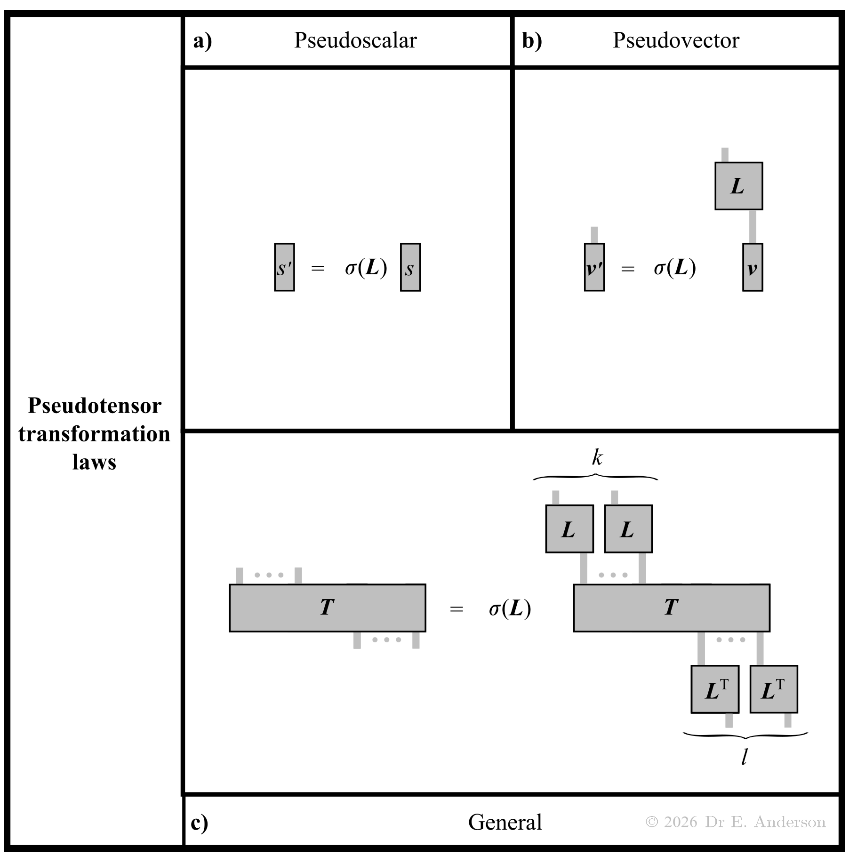

Figure 9:

3 Tensor arenas

3.1 Arenas of tensors for fixed dimension and rank

Structure 1 The plain Cartesian tensors' fixed dimension and rank arenas are as follows.

$$\mathfrak{T}\text{ensor}(D,\, P) \;:=\; \mathfrak{T}\text{ensor}\left(\mathbb{R}^D,\, P\right) \;:=\; \prod_{i\,=\,1}^{P} \mathbb{R}^D \;\;\cong\;\; \mathbb{R}^{D\,P}\;. \tag{9}$$

There are various ways in which the last simplification above is undesirable. One is that it is still $O(D)$ which acts. As opposed to the rapidly larger $O(P\,D)$ that would a priori be more natural on $\mathbb{R}^{P\,D}$.

Another is that a large and widely applicable generalization consists of replacing $\mathbb{R}^D$ with some other vector space

$$\mathfrak{V}\;.$$

Then the first two blockwise expressions continue to hold. While the remaining expressions may not pick out particularly meaningful objects.

Structure 2 For mixed-rank tensors, we strictly require

$$\mathfrak{T}\text{ensor}\left(\mathbb{R}^D,\, q,\, p\right) \;:=\; \mathfrak{T}\text{ensor}(\,D,\, q,\, p\,) \;:=\; \left(\prod_{i\,=\,1}^{q} \mathbb{R}^D\right) \prod_{i\,=\,1}^{p} \mathbb{R}^{D\,*} \;\;=\;\; \mathbb{R}^{D\,q}\,\left(\mathbb{R}^{D\,*}\right)^p\;. \tag{10}$$

Where the $*$ stands for dual space. Which enters since covectors are dual vectors. It would here be more tenuous to mix the starred and unstarred copies of $\mathbb{R}^D$. For all that for D finite, these are indeed isomorphic as vector spaces. Tenuous because each plays a different Geometrical role. For starters, the vectors belonging to the one and the covectors belonging to the other transform oppositely under coordinate transformations.

Structure 3 On differentiable manifolds,

$$\mathfrak{V}^D \;=\; \mathfrak{T}\text{gt}_{\mathrm{m}}\left(\mathfrak{m}^D\right)\;. \tag{11}$$

I.e. the tangent space at the point m in question. Then

$$\mathfrak{T}\text{ensor}\left(\mathfrak{m}^D,\, q,\, p\right) \;:=\; \left(\prod_{i\,=\,1}^{q} \mathfrak{T}\text{gt}_{\mathrm{m}}\left(\mathfrak{m}^D\right)\right) \prod_{i\,=\,1}^{p} \mathfrak{T}\text{gt}^*_{\mathrm{m}}\left(\mathfrak{m}^D\right) \tag{12}$$

does not further simplify.

In the sense that

$$\mathfrak{T}\text{gt}_{\mathrm{m}}\left(\mathfrak{m}^D\right) \;\times\; \mathfrak{T}\text{gt}_{\mathrm{m}}\left(\mathfrak{m}^D\right)$$

is specifically the square of the tangent space rather than some further unified object of dimension $2\,D$.

3.2 Combinatorial tracking arenas (CTAs)

3.2.1 Introduction

Structure 1 In the current Article, the role of *Combinatorial tracking arena* (*CTA*) is played by the totality of one or both of dimensions and tensor ranks. Or of some meaningful quotient thereof. That these are Combinatorial is preliminarily-enabled by dimension and rank taking discrete values. And embodied by the resulting arenas exhibiting some patterns familiar from basic Combinatorics.

3.2.2 The arena of dimensions

Remark 1 The notion of dimension in the current Article is a conflation of the Linear- Algebraic and Flat-Geometric notions. The second of these notions extends to the notion of manifold dimension via the chart construct.

Structure 1 For continua, the *arena of dimensions* is

$$\mathfrak{D}\mathrm{im}_1(d) \;=\; \mathbb{N} \; . \tag{13}$$

If the discrete point is included – the connected case of dimension 0 – then the arena of dimensions is as follows.

$$\mathfrak{D}\mathrm{im}(d) \;:=\; \mathfrak{D}\mathrm{im}_0(d) \;=\; \mathbb{N} \; . \tag{14}$$

Finally, if the empty set is included as well, then a schematic appending scheme is afforded by

$$\mathfrak{D}\mathrm{im}_{-1}(d) \;=\; \mathbb{N}_{-1} \; . \tag{15}$$

Where $\mathbb{N}_{-1}$ starts at -1 just like $\mathbb{N}_0$ starts at 0 . But the -1 value – from depicting the arena with the same spacing from 0 as 1 has – is more truly the following. A more loosely attached appendage of the empty set in its aspect as the *undimension*.

Of course, all of these arenas are isomorphic as sets to $\mathbb{N}$. By shifting the initial number up or down by 1 .

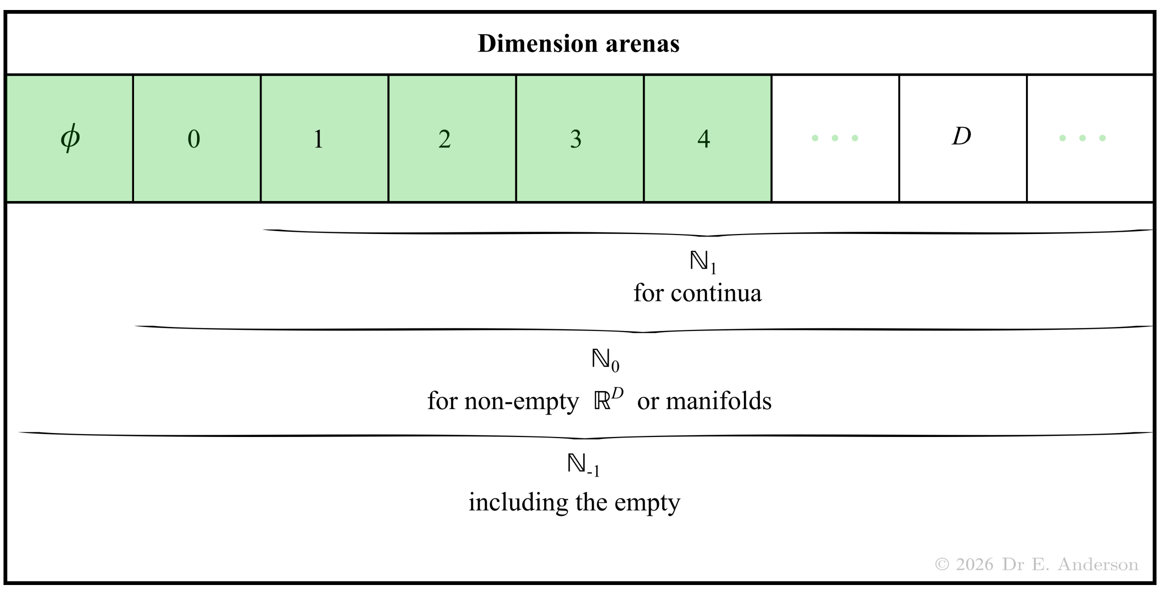

Figure 10:

Arena of plain tensor ranks $\mathfrak{R}$ank								
0	1	2	3	4		P		$\mathbb{N}_0$
scalar s	vector	2-tensor	3-tensor	4-tensor	$\cdots$	P-tensor	$\cdots$	

Figure 11:

3.2.3 Plain-rank arena

Structure 1 The *arena of plain tensor ranks* – is just

$$\mathfrak{R}\text{ank} \;=\; \mathbb{N}_0$$

(Fig 11). Starting with the rank-0 scalar and the rank-1 vector.

3.2.4 U.d.d. rank arena

Structure 3 The *arena of u.d.d. tensor ranks* is

$$\mathfrak{R}\mathrm{ank}^d_u \; = \; {\mathbb{N}_0}^2$$

(Subfig b). With the vector now taken to be the object with 1 upstairs index. Versus a distinct notion of covector, the object with 1 downstairs index. The general case is a rank-(q, p) tensor. I.e. with p downstairs and q upstairs indices.[7]

Arena of upstairs-downstairs tensor ranks $\mathfrak{R}\mathrm{ank}^d_u$								
	0	1	2	3	4		q	$\mathbb{N}_0$
0	scalar s	vector	(2, 0)-tensor	(3, 0)-tensor	(4, 0)-tensor	$\cdots$	(q, 0)-tensor	
1	covector	***B***	***I***	***I***	***I***		Imbalanced mixed-rank tensors: upstairs-dominant	
2	(0, 2)-tensor	***I***	***B***	***I***	***I***			
3	(0, 3)-tensor	***I***	***I***	***B***	***I***			
4	(0, 4)-tensor	***I***	***I***	***I***	***B***			
	$\vdots$					$\ddots$		
p	(0, p)-tensor	Imbalanced mixed-rank tensors: downstairs-dominant					Balanced mixed-rank tensors	
$\mathbb{N}_0$	© 2026 Dr E. Anderson							

Figure 12:

[7] This notation is consistent with the common practise of referring to totally-antisymmetric tensors with all-lowered indices as p-forms.

3.2.5 Adapted variables

Remark 1 Let us next introduce the following adapted variables [67, 69, 44]. The *total rank*

$$T \;:=\; q \;+\; p \;. \tag{16}$$

And the (*signed*) *rank imbalance*

$$I \;:=\; q \;-\; p \;. \tag{17}$$

Examples 2-4

$T \;=\; 2$ is minimum to possess a mixed tensor rank. This is balanced: rank $(1, 1)$.

$T \;=\; 3$ is minimum to possess imbalanced mixed tensors. Here there are 2 types: rank $(2, 1)$ and rank $(1, 2)$.

$T \;=\; 4$ is minimum to exhibit both balanced and imbalanced mixed-rank tensors. The balanced case is of rank $(2, 2)$. While the imbalanced cases are of rank $(3, 1)$ and $(1, 3)$.

Remark 2 Imbalanced ranks always come in pairs.

Remark 3 Rotating the arena clockwise through $\frac{\pi}{4}$ radians forms horizontal lines of constant T. Centred about the balanced-rank cases, scalars included at the summit. The purely upstairs and purely downstairs tensors then map into each other under reflection symmetry. Which also exchanges each imbalanced-rank pair. Fig 13 picks out the 2 pure lines and the balanced line using S. Sánchez' [67] runnel notation for chain series. Alongside the remaining region of imbalanced mixed-tensor ranks: in 2 disconnected equal wedges. Our 3 privileged lines are concurrent at the scalar summit.

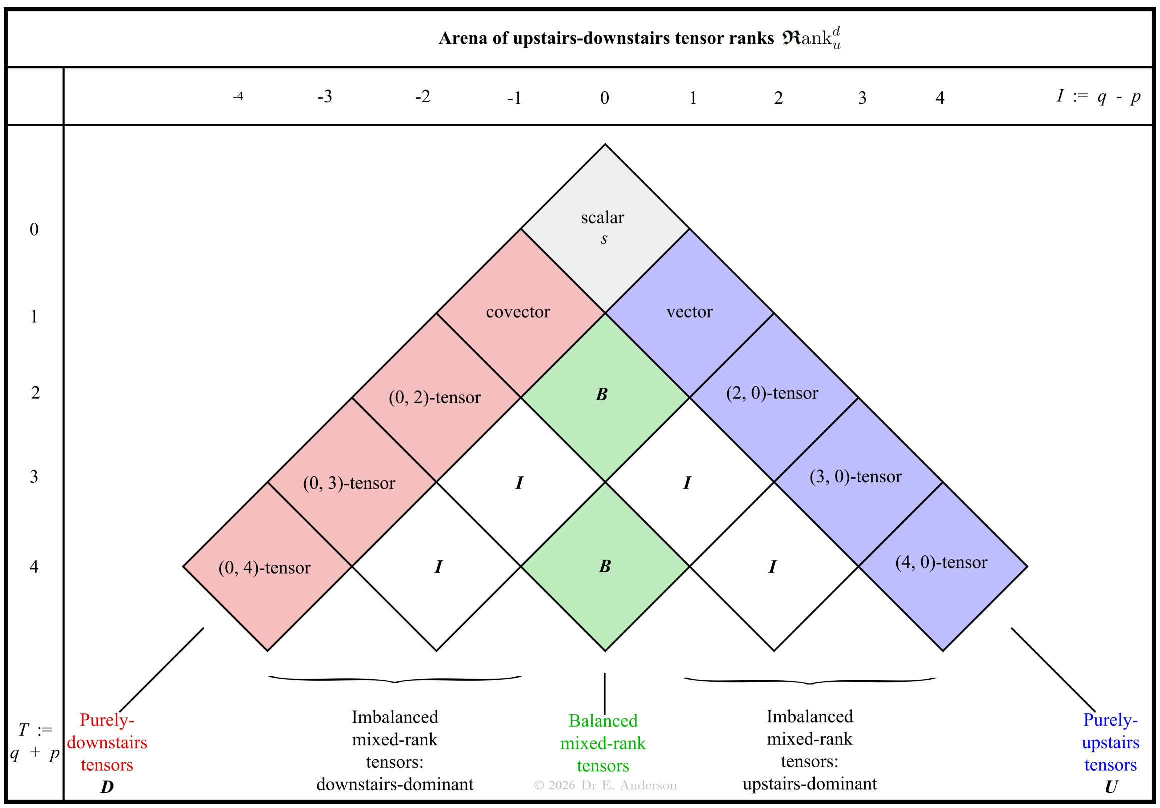

Figure 13:

3.2.6 0- and 1-D collapses

Remark 1 In 0- and 1-D , each higher-rank tensor is just a copy of the scalar. Thus identifying all of these copies,

$$\mathfrak{R}\mathrm{ank} \;=\; \mathrm{point} \;=\; \{0\} \;:$$

with the scalar rank as a natural label for it.

Remark 2 The undimension does not support any tensors at all. This could be accorded a single point, modelling the untensor arena interpretation of the empty set...

3.2.7 The u.d. distinguishable and yet meaningless quotient arena

Remark 1 The $C_{2\updownarrow}$ quotient

$$\widetilde{\mathfrak{R}\mathrm{ank}^u_d} \;=\; \frac{\mathfrak{R}\mathrm{ank}^u_d}{C_{2\updownarrow}} \tag{18}$$

corresponds to treating upstairs and downstairs as *distinguishable and yet meaningless labels* [69]. We exhibit this arena in Fig 14.

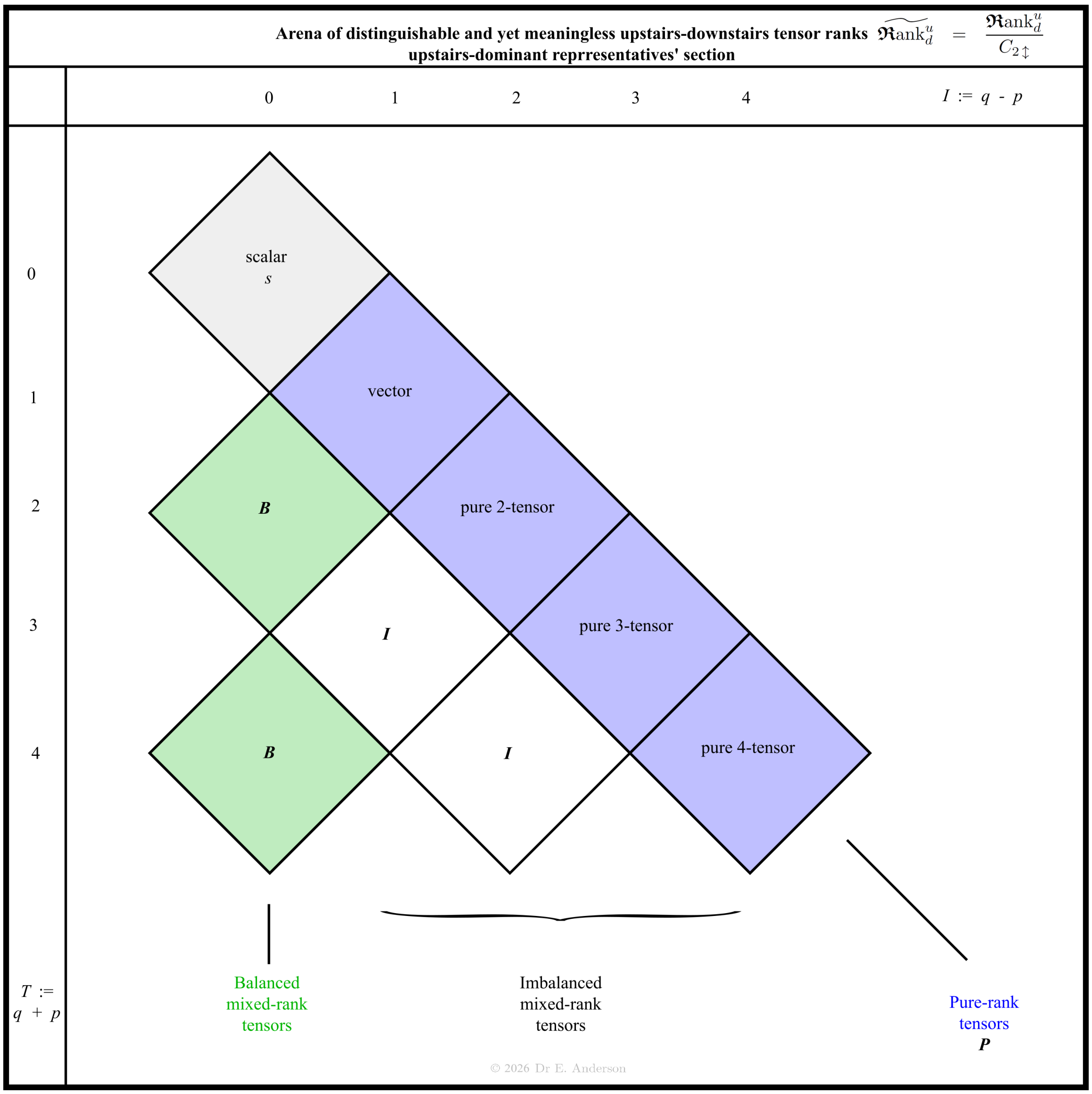

Figure 14:

2 privileged lines persist: pure and balanced. With now just the 1 connected wedge between these of imbalanced mixed-rank cases.

Motivation 1 Some Readers might quibble at this point that co- transforms with the frame and contra- oppositely. There is however also a notion of coframe. And now it is contra- which transforms with this to co- transforming oppositely. And in some contexts, which is the frame and which is the coframe is arbitrary...

Notational Remark 3 Exchange symmetries are usually denoted by $\leftrightarrow$. We however take licence to introduce $\updownarrow$ in this case, since the exchanged entities firstly are located upstairs and downstairs. Secondly, they are called upstairs and downstairs. And thirdly they are denoted by superscript u and subscript d on our rank arena symbol!

3.2.8 Aside: manifolds case

Remark 1 Here we have a problem in that 1 interpretation of bigger arena involves all manifolds in all dimensions. Easy classifications give us 0-, 1- and 2-D. But 3- and 4-D are wild and unknown. Thus this arena is too big for Humanity so far, or in the foreseeable future.

Remark 2 A sop is to consider all the tensors on all the spheres. Or on all of the $\mathbb{T}^D$, $\mathbb{RP}^D$ or $\mathbb{CP}^K$ (with complex dimension K corresponding to real dimension $2\,D$). Each of these is analogous to the $\mathbb{R}^D$ case, in fact with exactly the same CTAs.

3.3 Order-Theoretic and Graph-Theoretic analysis of dimension and rank separately

3.3.1 The trivial cases

Remark 1 For dimension's $\mathbb{N}_0$ (give or take 1), we can assign the arrow of dimensional increase by 1.

While for plain rank's $\mathbb{N}_0$ specifically, we can assign the arrow of rank increase by 1. I.e. of appending 1 further index. Which Sec 4 shall further describe as tensoring in a vector. And Sec 5 as tensoring in a single Young box.

For dimensions, the cumulative arenas – $\leq P$ are chains, of length $P + 1$:

$$\mathfrak{R}\text{ank}[P] = \text{Chain}_{P+1} .$$

With underlying graphs P_{P+1}: the path of length $P + 1$. The full arenas are the semi-infinite chain and the corresponding semi-infinite path.

While for ranks, the cumulative arenas are chains of length $D + 1$. (Or D for just continua. Or $D + 2$ if including the undimension).

Chains are Order-Theoretically trivial, in the sense of being total orders rather than partial orders (posets).

3.3.2 Full u.d.d. version

Figure 15:

Structure 1 We again assign the arrow of rank increase by 1. With now distinction between appending 1 upstairs index or 1 downstairs index.

This now produces a poset. See Fig 15 for the corresponding cumulative posets in its first column of subfigures. Whose edges are arrows encoding the add-index operation.

These posets are furthermore *semi-lattices* and *distributive* [69, 60, 61]. While if we prolong to infinite rank, a distributive lattice ensues. The meet operation is index-intersection, while the join operation is index-union. It is this join which fails, resulting in but a semi-lattice. Unless the poset is compensatingly semi-infinite. Which wipes out the lack of a unique lower bound in the cumulative approximations past just the point case.

Structure 2 Viewing $\mathfrak{R}\mathrm{ank}_u^d$ furthermore as a poset, [56, 60], T furthermore serves as a *depth function*. The *level lines*[8]

$$T = \text{constant}$$

then supply cut-offs for our cumulative arenas.

Remark 1 For $T = 1$ to 4, a distinct double-irreducible (*DI*) poset is pinned under the actual poset. In each case, this is arrived at by defoliation: cutting off all leaves. This suffices for every case in the current Article (the so-called DI class P [70]). More generally one would need to serially-defoliate to eliminate all degree-1 vertices. And to pohomeomorph – carry out all licit removals of degree-2 vertices – so as to arrive at the DI. For the 'double' refers to [67] serial pohomeomorphing and serial defoliation in whichever order.

Remark 2 The corresponding underlying graphs are provided in the second column of subfigures. Starting with the point Pt, the 3-path P_3 and Buckle.

Remark 3 Getting to the DI now involves homeomorphs as well as defoliations, leading to a wider range of DI classes being exhibited. I.e. classes C and C_4 in addition to P and $T = 0$'s class D that says that the point is already a double-irreducible. Observe that this graph-level structural analysis works out differently from its poset-level counterpart. This is since the poset axioms forbid triangles. For our particular small arenas, every homeomorph turns out not to be a licit pohomoeomporph for this reason. It may also be useful to point out at this stage that graph homeomorphs are very standard [69, 59]. In contrast, poset homeomorphs – *pohomeomorphs*! – are a relatively new and unexplored analogue [67].

[8] Compare level curves, level surfaces and level sets [32].

3.3.3 Motivational aside on arenas

Motivation 1 The bigger picture is that given any type of Mathematical object, asking what the totality formed by objects of that type is a meaningful question. This is the *arenization question* of Modern Applied Topology [82, 84, 69, 44, 45].

'Early birds' – topics in which arenas were already being investigated before the 2010s advent of Modern Applied Topology – include the following. Physics and Dynamics' *configuration space* [74] and *phase space* [79]. Linear Algebra's *vector spaces*, *eigenspectra* and *eigenspaces* [73]. Functional Analysis' *function spaces* [81]. Matrix arenas that so happen to be (chunks of) Lie groups [75]. Or polygons forming complex-Projective arenas, be it in a Dynamical [77, 85], Statistical [80] or purely Flat-Geometric [44] context.

Motivation 2 Since Theoretical Computer Science has become the pre-eminent field of study of our times, let us double the previous paragraph with examples of arenas from this field. Arenas of trees [69], especially binary and/or rooted, play a key role in the traditional Computer Science of search-and sort [78]. The arena of doubly-stochastic matrices is the convex hull of the arena of permutation matrices [76, 83]. Rather analogously, the arena of Quantum mixed states is a convex set on the complex-Projective space of q-n-its [83]. A sizeable part of 'Big Data' involves ascribing Topological properties to large data sets. [84]'s model examples are arenas of barcodes. Some major aspects of Machine Learning can be summarized as (Multi-)Linear Algebra leading to the solution of large parameter-number variational problems. Consequently, (Multi-)Linear Algebra arenas play a role here, as do arenas of varied curves that are being extremized. See [34, 36] for tensor networks in the context of Machine Learning.

Motivation 3 The above allusion to Topology follows from how arenas are not just sets, but rather carry further levels of Mathematical structure. Rendering in particular what natural Topologies are carried by arenas to always be a pertinent question. More occasionally, arenas furthermore carry Differential-Geometric structure [80], or Order-Theoretic structure [69, 42], among other possibilities.

Motivation 4 On the one hand, arenas very often belong to a 'different branch of' Mathematics to the original objects... On the other hand, firstly many basic Combinatorial objects' arenas turn out to themselves be Combinatorial. Well-known ones include power sets – the totality of subsets of a finite set – and the Young lattice of partitions ([71] and Appendix B.2). Secondly, many finite Linear Algebra arenas turn out to be Combinatorial [42, 45, 37]. In particular, this accounts for the current Article on Cartesian tensors containing various incidences of orders, graphs and partitions, all of which can be viewed as basic types of Combinatorics. More generally, it accounts for us being in the middle of a large-scale sweep to arenize all of basic Combinatorics [69, 72] and (Multi-)Linear Algebra [42, 45, 37] .

3.3.4 A first two Mathematical analogues of u.d.d. tensor rank arenas

Remark 1 The above families of posets and graphs are familiar from elsewhere in basic Mathematics. For instance, as arenas corresponding to Pascal's triangle and its quotient [69]. Or in the arena of eigenvalue signs for nondegenerate square matrices [42].

3.3.5 Corresponding quotient arenas

Remark 1 We now follow through with the above-mentioned quotient $C_{2\updownarrow}$ at the level of arenas. This is slightly over half the size of the unquotiented version. For rank-(p, p) tensors constitute fixed points. The cumulative distributive semi-lattices in the third column of our Figure ensue. Their underlying graphs are provided in the fourth column. Now starting with the point Pt, the 2-path P, Claw alias 3-star S_3, and the A-graph.

Remark 2 This quotient has a meaningful analogue both for Pascal's triangle: store only each row's distinct entries. And for matrix eigenvalue signs: regard $+$ and $-$ as distinguishable but meaningless. Corresponding for instance to the $+++-$ and $---+$ formulations of the Minkowski spacetime of Special Relativity being taken to only differ by convention...

3.3.6 Analogy 3: arena of partitions with ≤ 2 parts

Structure P More primarily, the above applications all concern *partitions with at most* 2 *parts* [69]. With partition conjugation playing the role of quotienting operation where applicable. See [42] for more about the corresponding arenas, including subsequent cumulative cases.

Modelling Assumption 1 For us for now, cutting off at $T = 4$ suffices, since we are not going further than the elasticity tensor or the Electromagnetic hexadecapole this year. Nor any further than the Riemann and Weyl curvature tensors over the subsequent two years...

Remark 1 Basic counts for these are as follows.

$$p_{\leq 2}(T) = \left\lfloor \frac{T+2}{2} \right\rfloor . \tag{19}$$

While for the quotient,

$$\widetilde{p}_{\leq 2}(T) = \begin{cases} \left\lfloor \frac{T+2}{2} \right\rfloor & T \neq 2 \\ 1 & T = 2 \end{cases} . \tag{20}$$

Remark 2 Our mirror symmetry can be interpreted as a vestige of the partition conjugation operation (see Appendix B.1).

Remark 3 See Appendix C for $\mathfrak{R}$ank's relation to Pascal and to various subarenas of partitions.

Remark 4 Firstly excising $D = 0$ and 1, this is the infinite starlike graph with the scalars serving as star point.

3.4 Dimension-and-rank arenas, including Order- and Graph- Theoretic analysis

3.4.1 Full plain version

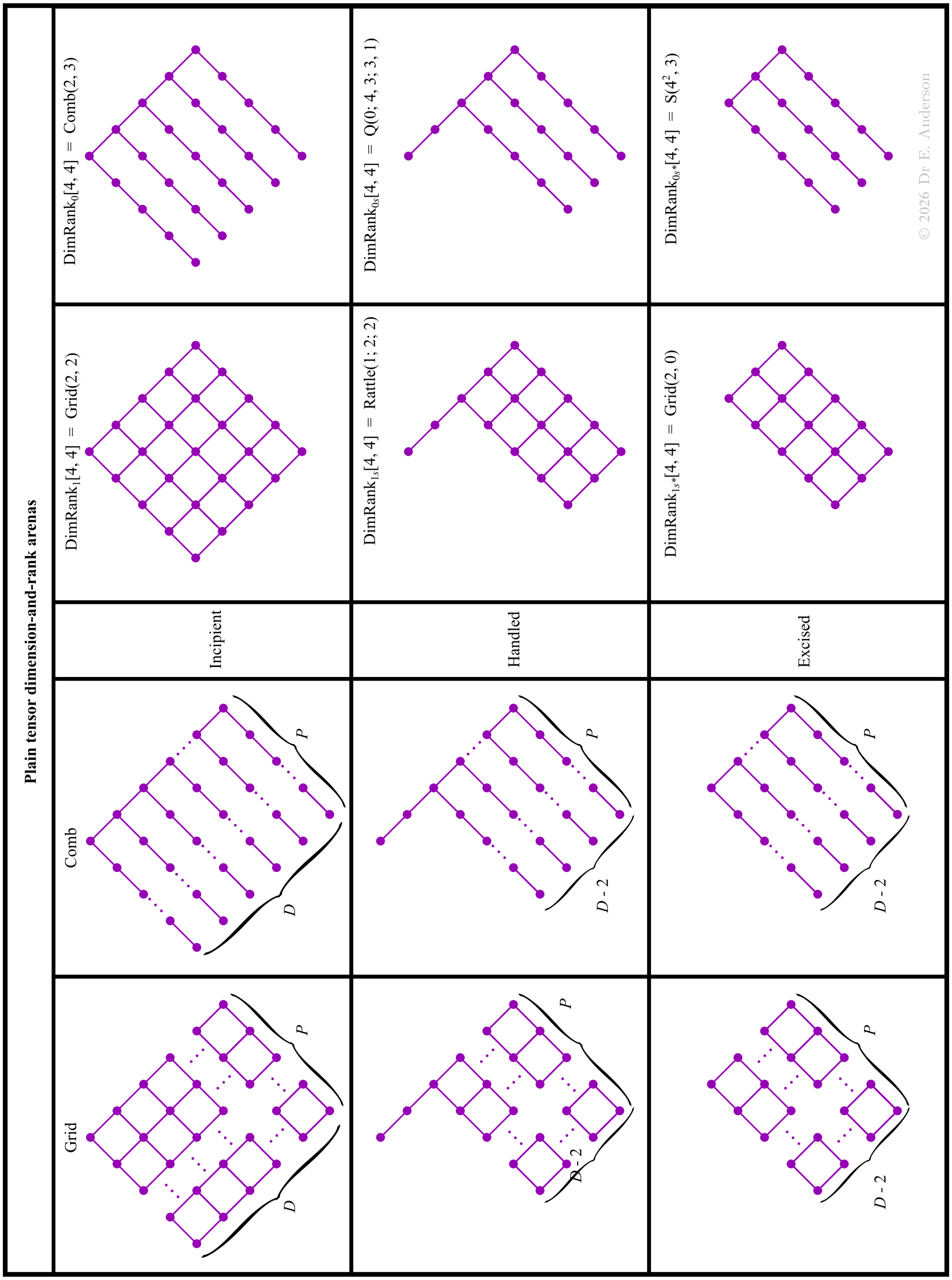

Figure 16:

Model 1 (Product) Suppose that the joint CTA is modelled by according to the arrows add-index and extend dimension by 1. Then we arrive at

$$\mathfrak{D}\text{im}\mathfrak{R}\text{ank} \;=\; {\mathbb{N}_0}^2 \,.$$

Whose cumulative approximands are the diagonal-grid posets (Fig 16.1). Which are furthermore distributive lattices. And whose underlying graphs are the grid graphs,

$$\mathfrak{D}\text{im}\mathfrak{R}\text{ank}[\,D,\,P\,] \;=\; \text{Grid}_{D\,+\,1,\,P\,+\,1} \;=\; \text{Grid}(\,D\;-\;2,\,P\;-\;2\,)\,. \tag{21}$$

Where the last parameter is *excess box size*, by which convention 2×2 boxes is the minimum grid,

$$\text{Grid} \;=\; \text{Square}\,. \tag{22}$$

Model 0 (Comb of paths) Here the same arrow types enter, but with the add-dimension arrow applied in a sparser generative manner: to scalars only. This returns the comb-of-paths posets (Subfig 0), which are furthermore at-most binary rooted trees. Whose cumulative approximands are

$$\mathfrak{D}\text{im}\mathfrak{R}\text{ank}_0[\,D,\,P\,] \;=\; \text{Comb}(\,D\;-\;2,\,P\,)\,. \tag{23}$$

With excessive-variable calibration that there are to be ≥ 3 rows of teeth to have more than just a bent path. And where $P - 1$ is the *excess tooth length*.

Within the current Article's $(\,D,\,P\,) \leq (\,4,\,4\,)$ remit, these are confined to the following types. Paths, starlike, quadrupeds and scorpions. I.e. homeomorphs of P, $\text{S} = \text{Claw}$, P-of-Claws and P_3-of-Claws [69].

Model 0.c Here the 0-D and 1-D tensors are separately conflated. Giving a leading P handle, preceding the equal-length teeth of the comb (Subfig 0.c). It is somewhat less wasteful to introduce the unscalars at this point, by passing to a P_3 handle. As opposed to uniformly extending model 0 to include a priori distinct untensors for each rank... Here the cumulative approximands are

$$\mathfrak{D}\text{im}\mathfrak{R}\text{ank}_0[\,D,\,P\,] \;=\; \text{HandledComb}(\,H,\,D\;-\;4,\,P\,)\,. \tag{24}$$

With this modelling, the current Article's remit now only gets as far as some homeomorphs of Quadruped.

Model 1.c) Now the grid picks up a P (or P_3) handle. Thus forming the *rattle* family of Subfig 1.c). Whose cumulative approximands are

$$\mathfrak{D}\text{im}\mathfrak{R}\text{ank}_{1.\text{c}}[\,D,\,P\,] \;=\; \text{Rattle}(\,H,\,D\;-\;4,\,P\;-\;2\,)\,. \tag{25}$$

Where the *excess handle length* parameter is 1 for P as handle. And 2 for P_3 as handle.

Model 0.c$_*$) Now we excise the handled comb's handle. This of course sends us back to just a comb, albeit with shifted tooth-number, as follows.

$$\mathfrak{D}\text{im}\mathfrak{R}\text{ank}_{0.\text{c}_*}[\,D,\,P\,] \;=\; \text{Comb}(\,D\;-\;4,\,P\,)\,. \tag{26}$$

See Subfig 0.c$_*$). With this modelling, the current Article's remit now only takes us as far as some starlike trees.

Model 1.c$_*$) Next we matchingly excise the rattle's handle. Sending us of course back to just a grid, albeit with modified parameter, as follows (Subfig 1.c$_*$).

$$\mathfrak{D}\text{im}\mathfrak{R}\text{ank}_{1.\text{c}_*}[\,D,\,P\,] \;=\; \text{Grid}(\,D\;-\;4,\,P\;-\;2\,)\,. \tag{27}$$

Using this modelling, the current Article's remit now only takes us as far as some starlike trees.

Remark 1 The current Article's top grid, comb, handled comb, rattle, excised comb and excised grid are presented in the right-hand-side windowpane of subfigures.

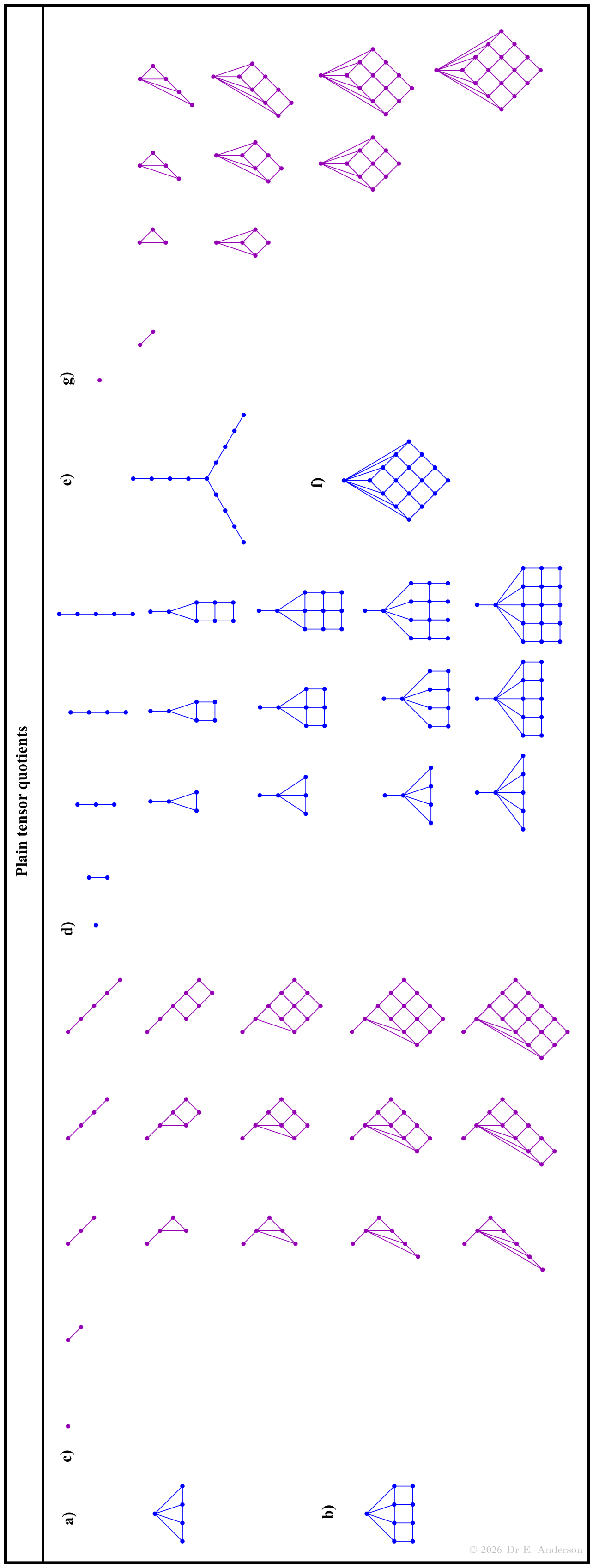

Figure 17:

3.4.2 Plain quotients

Model 1.q (Fans and Pencils) Suppose that the 1-D tensor ranks are identified in the grid model. And that the arrow from 1-D to 2-D P-tensors is preserved. Then 1-D 's identified point is linked to every 2-D rank. If no further dimensions are in play, then this returns the Fan subcase of cone in the Graph-Theoretic sense (17.a). While if subsequent dimensions are included, a direct-product pencil shaft grows away from the fan (Subfig b).

The cumulative approximands here are

$$\mathfrak{D}\text{im}\mathfrak{R}\text{ank}_u^d \quad 1.\text{q}[\, D,\, P\,] \quad = \quad \text{Pencil}[\, D \;-\; 2,\, P\,]\;. \tag{28}$$

Where the first parameter is the *excess pencil shaft length*.

Sticking to including 0-D but not the untensor, the current Article's cumulative approximands are tabulated in Subfig c) as orders. Yielding the underlying graphs in Subfig d).

Exercise 2 Work out their DI irreducibles and DI classes.

Remark 1 If the unobject is furthermore included, then our pencil is sharp: exposed lead P . While if instead we restrict attention to continua, our pencil is blunt: no exposed lead at all.

Structure 1 (additivity quotient) Let us also entertain quotienting down to collections of tensors within each of which addition is supported. This pinches all dimensions' scalars in the CTA to a single point.

Model 0.a (uniform starlike graphs) If this is applied to the comb model, a uniform starlike graph ensues (Subfig e). The cumulative approximands here are

$$\mathfrak{D}\text{im}\mathfrak{R}\text{ank}_u^d \quad 1.\text{a}[\, D,\, I\,] \quad = \quad S_I(\, I \;-\; 2\,)\;. \tag{29}$$

I.e. the I-pointed stars whose radiances are of excess length $I \;-\; 2$.

Model 1.a (Trowel grids) But if applied to the grid model, Subfig f) ensues. These trowel grids are just parametrized in the manner of grids,

$$\text{TrowelGrid}(\, D \;-\; 4,\, I \;-\; 3\,)\;. \tag{30}$$

Shifted by 2 in the first argument since $D \;=\; 0$ and 1 are no longer part of the grid. And by 1 in the second since $I \;=\; 0$'s scalars are not in the grid.

3.4.3 Full u.d.d. version

Model 1 (Pascal's prism) Now as per Fig 18.1),

$$\mathfrak{D}\text{im}\mathfrak{R}\text{ank}_{u1}^d \quad = \quad \mathbb{N}_0 \;\times\; \text{Pascal}\;. \tag{31}$$

With cumulative approximands

$$\mathfrak{D}\text{im}\mathfrak{R}\text{ank}_{u1}^d[\, D,\, I\,] \quad = \quad [\, 0,\, D\,] \;\times\; \text{Pascal}[I]\;. \tag{32}$$

Model 0 (Pascal's comb) While

$$\mathfrak{D}\text{im}\mathfrak{R}\text{ank}_{u0}^d \quad = \quad \text{Pascal-Bunting}\;, \tag{33}$$

in the sense of Subfig 0). With cumulative approximands

$$\mathfrak{D}\text{im}\mathfrak{R}\text{ank}_{u0}^d[\, D,\, I\,] \quad = \quad \text{PascalBunting}[\, D \;-\; 3,\, I\,]\;. \tag{34}$$

Where the first parameter is the excess number of Pascal's triangles mounted onto the comb; 2 would just be a Bridge .

Model 0.s (Handled comb-of-Pascals) Here tensor-rank indistinguishability collapses the first 2 Pascal triangles down to points. Returning the handled comb of Pascals depicted in Subfig 0.s).

Model 1.s (Handled prism-of-Pascals) Here the same collapse returns Subfig 1.s).

Remark 1 Excising 0- and 1-D , one once again returns to the original structures – prism and comb – with values of D shifted down by 2 .

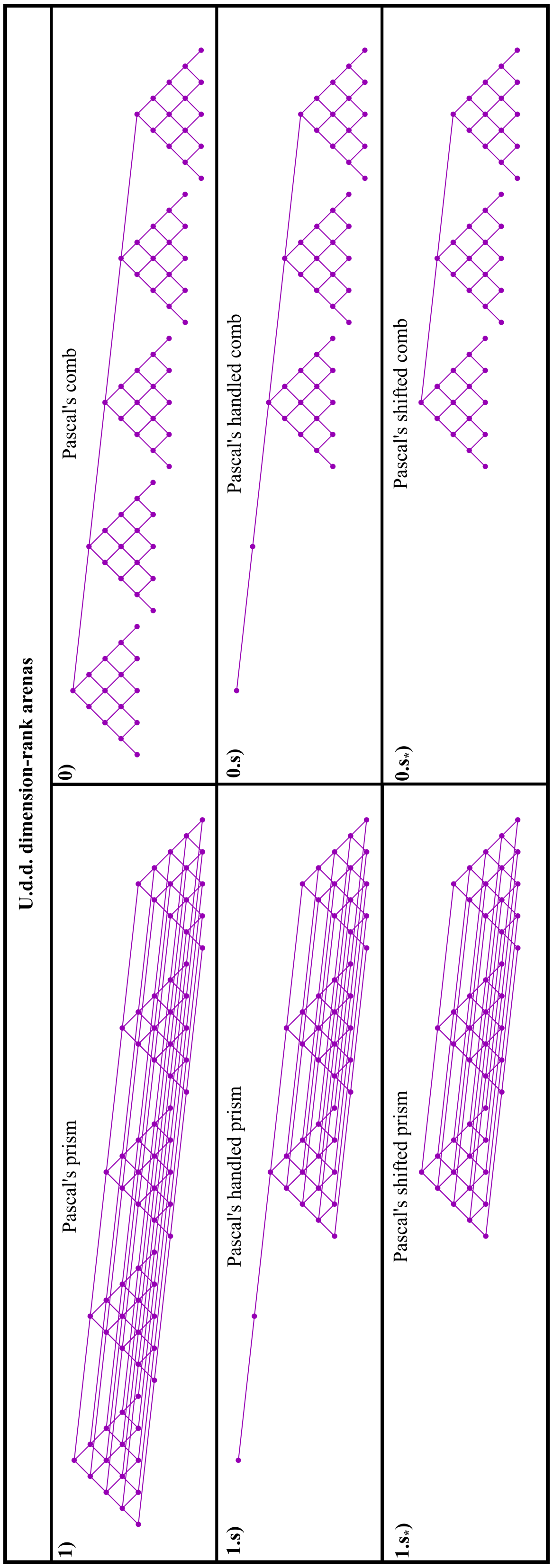
U.d.d. dimension-rank arenas
1)
Pascal's prism
0)
Pascal's comb
1.s)
Pascal's handled prism
0.s)
Pascal's handled comb
1.s*)
Pascal's shifted prism
0.s*)
Pascal's shifted comb

Figure 18:

3.4.4 Analogues of the preceding quotients

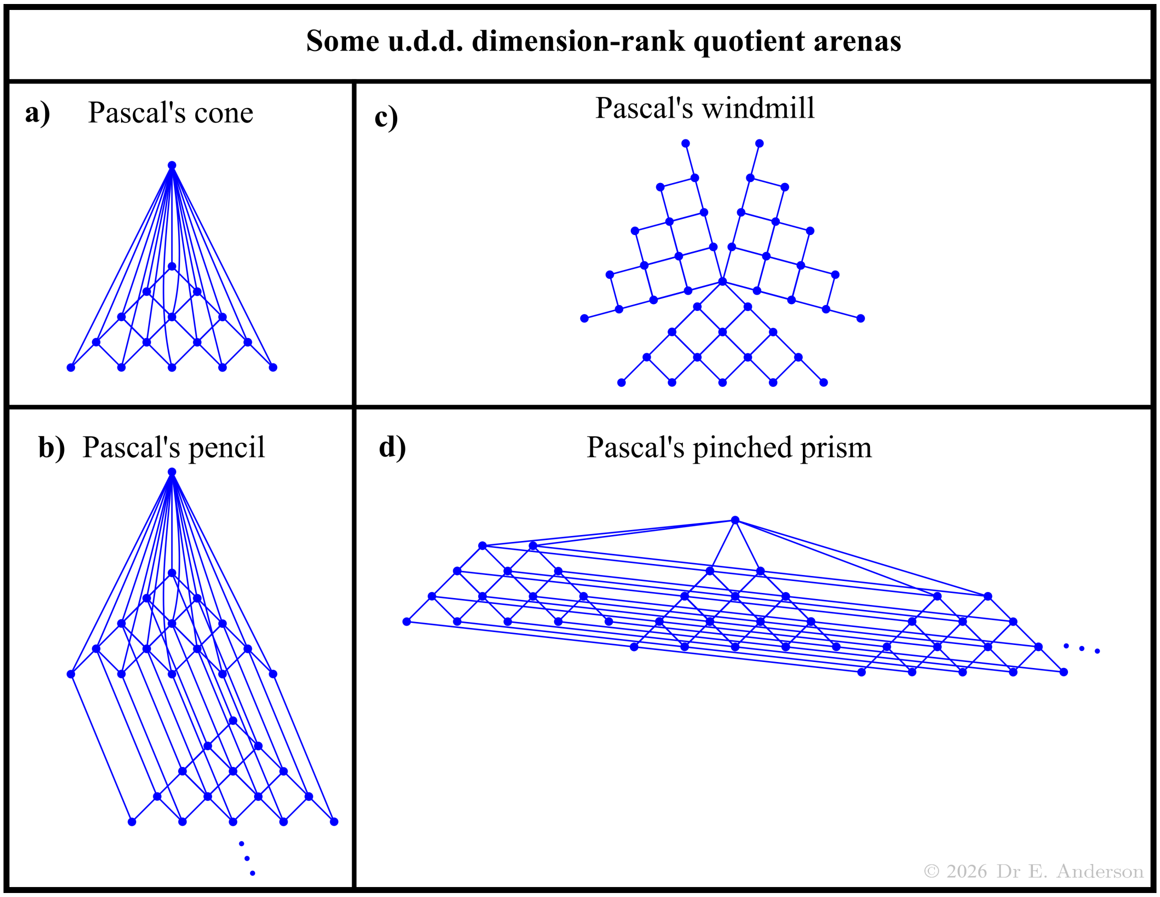

Figure 19:

Model 1.q (Pascal's cones and pencils) The cumulative approximands here are firstly cones over Pascal's triangle. Down from which, a shaft of Pascal's prism is built, forming a pencil (Fig 19.a),

$$\mathfrak{Dim}\mathfrak{Rank}^d_u \quad 1.\mathrm{q}[\,D,\, I\,] \quad = \quad \mathrm{PascalPencil}[\,D \;-\; 2,\, I\,]\;. \tag{35}$$

Model 1.a (Pascal's windmills) The cumulative approximands here are windmills with $D - 1$ blades intersecting at the unqualified scalars' nexus point,

$$\mathfrak{Dim}\mathfrak{Rank}^d_u \quad 1.\mathrm{a}[\,D,\, I\,] \quad = \quad \mathrm{PascalWindmill}[\,D \;-\; 1,\, I\,]\;. \tag{36}$$

See Subfig b).

Model 1.a (Pascal's pinched prism) See Subfig c).

3.4.5 Interplay with $\updownarrow$ quotient

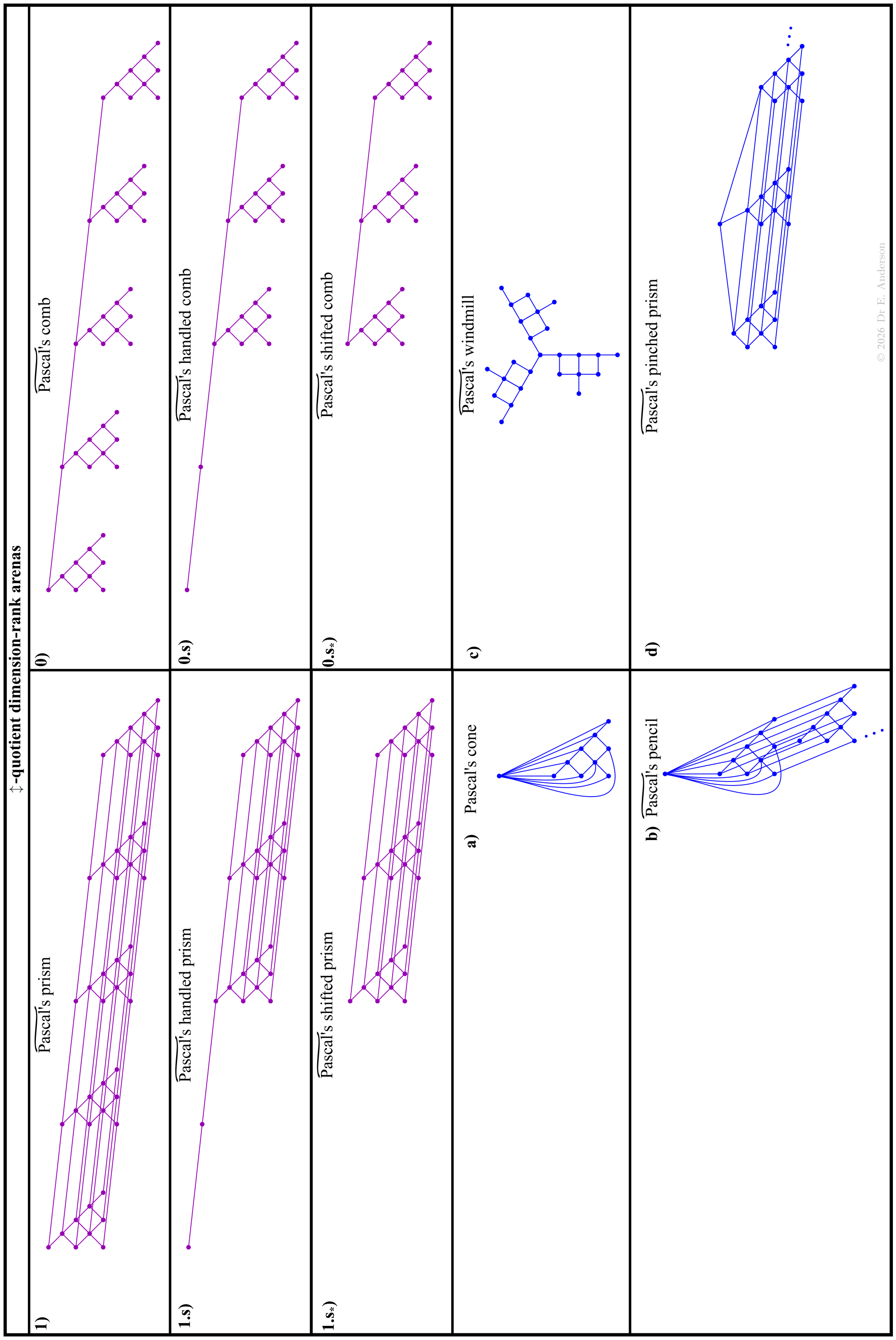

Figure 20:

Remark 1 This gives the $\widetilde{\text{Pascal}}$ counterparts of the previous two SSSecs, as per Fig 20.

Exercise 3$^-$ Which of these pictures is the only one that can be assigned poset structure?

3.5 Bigger arenas of tensors. i. Definitions and structures

3.5.1 Introduction

Structure 1 One bigger arena of plain Cartesian tensors is

$$\mathfrak{T}\mathrm{ensor}(D)\ .$$

Of tensors of all ranks on a fixed $\mathbb{R}^D$.

Another is

$$\mathfrak{T}\mathrm{ensor}(R)\ :$$

the rank-R Cartesian tensors in no matter what dimension.

Finally we have the arena in Fig 21.a). I.e. the totality of ranks of Cartesian tensor throughout the totality of dimensions.

The current Article's grand arenas (as yet not fully structured)

a) plain

$$\bigcup_{D \in \mathbb{N}} \mathfrak{T}\mathrm{ensor}(D)$$

$$\mathfrak{T}\mathrm{ensor} \qquad \bigcup_{D \in \mathbb{N}} \ \bigcup_{P \in \mathbb{N}} \mathfrak{T}\mathrm{ensor}(\,D,\, P\,)$$

$$\bigcup_{P \in \mathbb{N}} \mathfrak{T}\mathrm{ensor}(P)$$

b) u.d.d.

$$\bigcup_{D \in \mathbb{N}} \mathfrak{T}\mathrm{ensor}^d_u(D)$$

$$\mathfrak{T}\mathrm{ensor}^d_u \qquad \bigcup_{D \in \mathbb{N}} \ \bigcup_{q \in \mathbb{N}} \ \bigcup_{p \in \mathbb{N}} \mathfrak{T}\mathrm{ensor}(\,D,\, q,\, p\,)$$

$$\bigcup_{q \in \mathbb{N}} \ \bigcup_{p \in \mathbb{N}} \mathfrak{T}\mathrm{ensor}(\,q,\, p\,)$$

Figure 21:

Structure 2 One big arena of Cartesian u.d.d. tensors is

$$\mathfrak{Tensor}^u_d(D) \ := \ \bigcup_{q \,\in\, \mathbb{N}} \ \bigcup_{p \,\in\, \mathbb{N}} \mathfrak{Tensor}(\,D,\, q,\, p\,)\ .$$

Of tensors of all ranks on a fixed $\mathbb{R}^D$.

Another is

$$\mathfrak{Tensor}^d_u(\,q,\, p\,) \ := \ \bigcup_{D \,\in\, \mathbb{N}} \mathfrak{Tensor}(\,D,\, q,\, p\,) \ :$$

the rank-P Cartesian tensors in no matter what dimension.

Finally we have the arena in Subfig b). I.e. the totality of ranks of Cartesian tensor throughout the totality of dimensions.

3.5.2 Fibre bundles

* A standard short description shall go here (Fig *.a).

3.5.3 General bundles

* A standard short description shall go here (Subfig b).

* Both fibre and general bundles admit a notion of section.

3.5.4 Discretized base spaces

Remark 1 Some kinds of simplicial space [21] are conceptually similar to manifolds. Graphs are among these. And placing orders on these is not particularly disruptive [67]. Thus we can contemplate bundles over discrete spaces without and with orders.

3.6 ii. Plain examples

3.6.1 The full plain $\mathfrak{Tensor}(D)$

Structure 1 Here we place each fixed-rank tensor arena $\mathfrak{Tensor}(\,D,\, P\,)$. Above the corresponding dimension, that themselves form the chain $\mathfrak{Dim}(D) \ = \ \mathbb{N}_0$. These attached objects are entirely distinct. Except for scalars, for all the constant or pointwise $\mathfrak{Tensor}(\,D,\, 0\,)$ are identical copies. So this exception gives a fibre bundle while the other normative cases are all general bundles. With vectors minimal in realizing realize the persistent generic general-bundle structure. See Subfigs c-e) for the scalar, vector and general case.

3.6.2 The full plain $\mathfrak{Tensor}(R)$

Structure 2 Now instead we position each fixed-dimension tensor arena $\mathfrak{Tensor}(\,D,\, N\,)$ Above the corresponding rank, that themselves form the chain $\mathfrak{Rank}(P) \ = \ \mathbb{N}_0$. These attached objects are entirely distinct. Except for dimension 0 and 1 , for which they are all identical. Because all higher rank tensors collapse to scalars. So we have 2 identical fibre bundles, and then a heap of distinct general bundles

Finally, 2-D is minimum to realize the persistent generic general-bundle structure.

3.6.3 Quotients so far

Remark 1 It furthermore makes sense to quotient each of the previous four fibre bundles down to a single scalar arena, each over a single point. For higher tensors are not distinguishably defined in 0- or 1-D . The base point then furthermore ceases to be a useful identifier. So we return to the scalar arena by itself.

In the case of the scalars in arbitrary-D , there often still is Physical distinction. Say between a scalar realized on a curve, a surface, or the whole space. Say total line charge versus total surface charge versus total charge enclosed in a volume element. If this is relevant, then one would not quotient.

Remark 2 One context for quotienting is to group together all terms that can be added, or, more generally, linearly combined.

Scalars can be added irrespective of the dimension each is defined on. This underpins how a particle Lagrangian can just be added to a background field Lagrangian. We correspondingly quotient $\mathfrak{T}\text{ensor}(D, 0)$ to a single copy over a single point. And similarly for $\mathfrak{T}\text{ensor}$ and for u.d.d. counterparts.

3.7 Motivation for CTAs

3.7.1 $\mathfrak{T}$**ensor**(R)

Example R.1 In the next Section, we shall see that contraction, and products of tensors, interrelate tensors of distinct rank.
Structure R Some tensor splits decompose larger ranks into smaller ranks.

Example R.2 The *trace-tracefree split* that we shall cover in Sec 5.

Example R.3 Partial derivation sends a rank-R Cartesian tensor to a rank-$(R + 1)$ one.

3.7.2 $\mathfrak{T}$**ensor**(D)

Structure D While other tensor splits decompose higher-D tensors into lower-D tensors.

Example D.1 One of the moves that can be used in setting up Hopf's little map [43] can be phrased as splitting a 4-D position vector into 2 2-D position vectors.

Example D.2 Hypersurface Geometry [25, 22, 37] underlies many examples. For instance in the local splitting of a manifold into a codimension-1 surface and a *normal-direction* vector field.

3.7.3 $\mathfrak{T}$**ensor**

Example RD.1 *Duality* in dimension D maps rank-R to rank $D - R$.

Example RD.2 Physically interpreting Special Relativity provides numerous examples concurrently involving dimensional rank splits.

Splitting the *event* 4-vector into a 3-D position vector and a 1-D time scalar.

The rank-2 4-D Minkowski spacetime metric into the following. The rank-2 3-D Euclidean metric. A 1-D time scalar. And the zero 3-vector of space-time cross-terms.

The *energy-momentum* 4-vector into the 3-D momentum vector and 1-D energy scalar.

The rank-2 *energy-momentum-stress* 4-tensor into the following. The rank-2 3-D stress tensor. The 3-D momentum vector. And 1-D energy scalar.

Or the rank-2 *electromagnetic field strength* 4-tensor into the following 3-D objects. The electric field vector and the magnetic field pseudovector.

Example RD.3 The further split of a manifold's metric as induced by example D.1. From a rank-2 D-dimensional metric to the following. A rank-2 hypersurface metric of dimension $d := D - 1$

. A normal-displacement scalar. And a strutting d-vector for how what were the hypersurface's coordinates are dragged around upon moving away from the hypersurface.

Example RD.4 The $3 + 1$ split of General Relativity in effect combines both of the previous examples. Interpreting the manifold as an in-general curved spacetime. The hypersurface as a spatial slice. And the normal-displacement to it as a time coordinate. Then the normal displacement scalar is called the *lapse*. And the strutting 3-vector is called the *shift*. In each case in Arnowitt, Deser and Misner's parlance [9].

Let us leave the GR-coupled $3 + 1$ split versions of energy-momentum-stress and of Electromagnetism for a further occasion!

End Remark All in all, we need the CTAs and the big and grand arenas as the natural place in which to view these splits. And generalizations and yet further splits, which pervade the more advanced theory of tensors [37].

4 Cartesian tensor algebra

4.1 The full Cartesian case: $O(D)$

4.1.1 Linearity

Proposition 1 Let $\boldsymbol{T}$ and $\boldsymbol{S}$ be Cartesian tensors of the same rank. Then so are the following.
S)

$$\lambda \boldsymbol{T} \ . \tag{37}$$

A)

$$\boldsymbol{S} + \boldsymbol{T} \ . \tag{38}$$

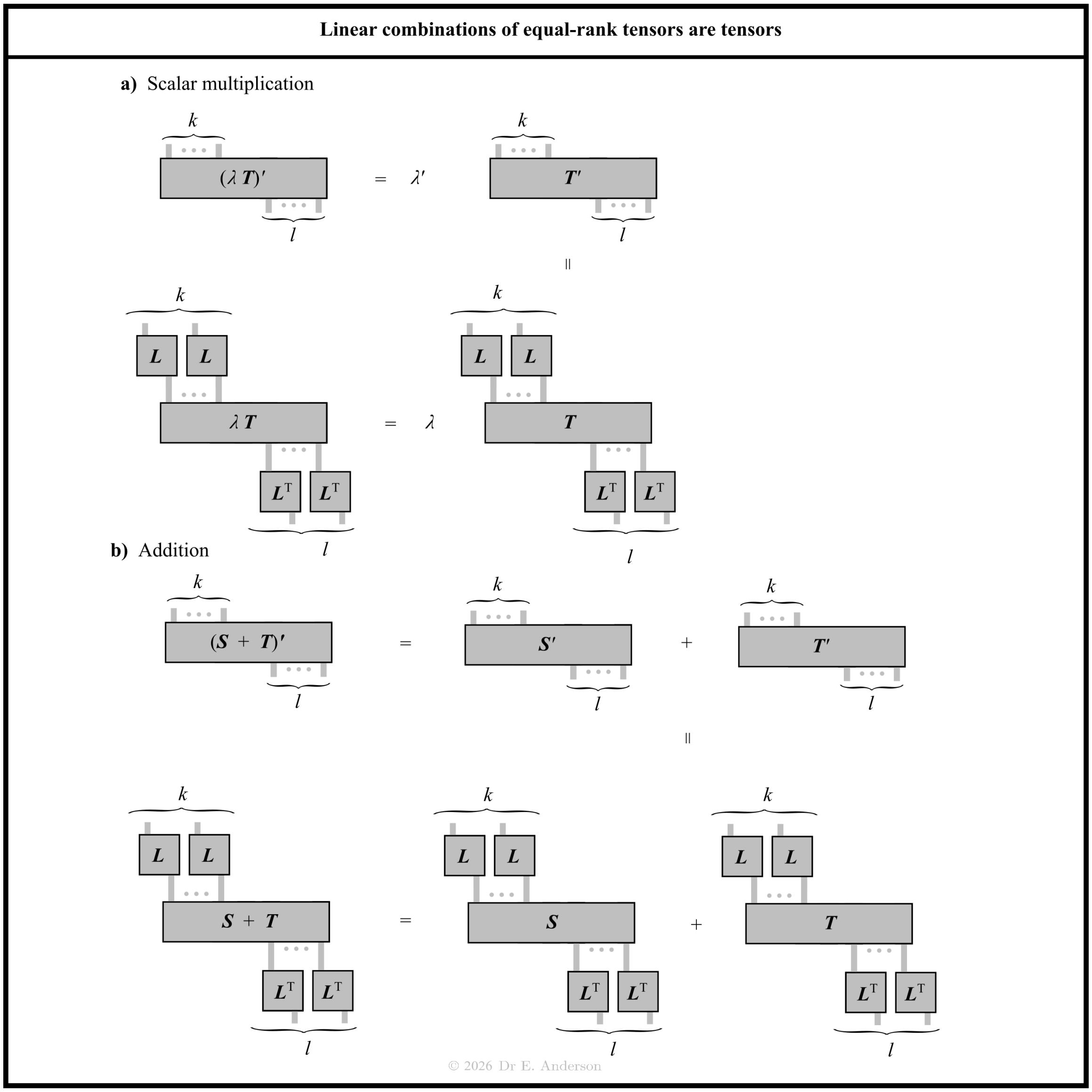

Figure 22:

Proof See Fig 22 for a tensor networks rendition.

S) In the first step, we unpackage the scalar multiple in the primed frame. In the second step, we apply the 2 corresponding tensor transformation laws. In the final step, we reform the scalar multiple in the unprimed frame.

A) In the first step, we unpackage the sum in the primed frame. In the second step, we apply the 2 corresponding tensor transformation laws. In the final step, we reform the sum in the unprimed frame. □

Corollary 1 L) Let $\boldsymbol{S}_{(i)}$ $(i) = (1)$ to (k) be Cartesian tensors of the same rank. Then so is the linear combination

$$\sum_{(i)\,=\,(1)}^{(k)} \lambda_{(i)}\, \boldsymbol{S}_{(i)} \ . \tag{39}$$

D) This includes the difference of any 2 equal-rank tensors,

$$\boldsymbol{S} \ - \ \boldsymbol{T} \ . \tag{40}$$

Remark 1 An alternative statement of Proposition 1 is that tensoriality is linear: scalar-multiplicative and additive. Corollary 1 then collects the implications that tensoriality is linearly-combinational and thus for instance subtractive.

Remark 2 Proposition 1 and the existence of the zero tensor of each dimension and rank establish the following. That $\mathfrak{T}\text{ensor}(\,D,\,P\,)$ and $\mathfrak{T}\text{ensor}^d_u(\,D,\,p,\,q\,)$ are vector spaces.

4.1.2 The outer product

Definition 1 The *outer product* of 2 tensors $\boldsymbol{T}$ and $\boldsymbol{U}$ is as follows. Their multiplicative concatenation without sharing any indices, $\boldsymbol{T}\boldsymbol{U}$. In say the u.d.d. case, this is a map

$$\mathfrak{T}\text{ensor}\,(\,D,\,q_1,\,p_1\,) \ \times \ \mathfrak{T}\text{ensor}\,(\,D,\,q_2,\,p_2\,) \ \longrightarrow \ \mathfrak{T}\text{ensor}\,(\,D,\,q_1 \ + \ q_2,\,p_1 \ + \ p_2\,) \ .$$

Proposition 2 Let $\boldsymbol{U}$ and $\boldsymbol{T}$ be tensors. Then so is their outer product $\boldsymbol{T}\boldsymbol{U}$.

Proof See Fig 23 for a tensor networks rendition. Where the first step unpackages the outer product in the primed frame. Each factor is prescribed to be a tensor, enabling step 2's tensor transformation laws. Finally we reform the outer product in the unprimed frame. □

Notational Remark 5 von Neumann's [5] *tensor product symbol* $\otimes$ has come to often be used to denote outer products in an more unambiguous manner. Now one writes not $\boldsymbol{T}\boldsymbol{U}$ but

$$\boldsymbol{T} \ \otimes \ \boldsymbol{U} \ .$$

See below for its further use. It has various further Algebraic and Categorical connotations, such as being dual to Linear Algebra's direct-sum operation $\oplus$.

Corollary 2 Let $\boldsymbol{T}_{(i)}$ $(i) = (1)$ to (k) be Cartesian tensors. Then so is

$$\prod_{(i)\,=\,(1)}^{(k)} \boldsymbol{T}_{(i)} \ = \ \bigotimes_{(i)\,=\,(1)}^{(k)} \boldsymbol{T}_{(i)} \ . \tag{41}$$

Remark 1 In other words, tensoriality is outer-productive. Binarily by Proposition 2 and finitely by Corollary 2.

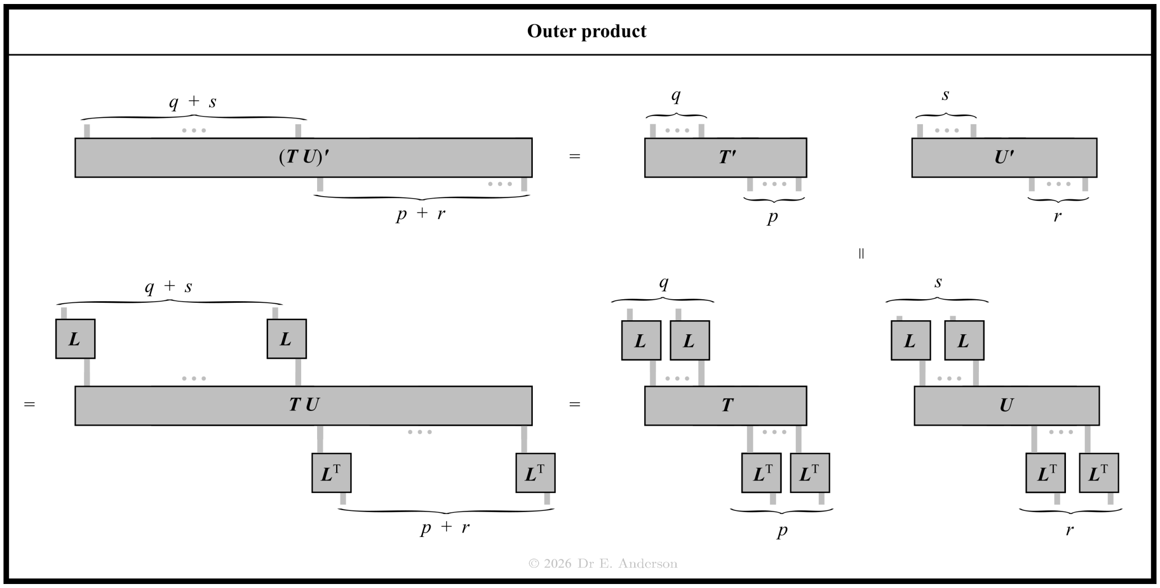

Figure 23:

4.1.3 Contraction and inner product. i. Definitions and notation

Structure 1 A *contraction* of a pair of indices involves summing over all values taken by each member of that pair.

E.g. in the u.d.d. case, this is overall a map

$$\mathfrak{T}\text{ensor}\,(\,D,\,q,\,p\,)\;\;\longrightarrow\;\;\mathfrak{T}\text{ensor}\,(\,D,\,q\;-\;1,\,p\;-\;1\,)\;\;.$$

Notational Remark 6 In longhand, this is just written as a sum. For instance

$$\sum_{j\,=\,1}^{d} T_{i\,j}\,U^{j\,k}\;\;.$$

A first shorthand is *Einstein's summation convention.* By which any repeated pair of indices in a tensor, or a product of tensors, is taken to carry such a sum. E.g.

$$T_{i\,j}\,U^{i\,k}\;\;.$$

A second shorthand is to join the pair of indices' legs in whichever tensor networks formulation. See for instance the top-right diagram in Fig 24.

A *dummy index change* is a common step in working with indices, consisting of the following. That changing any summed-over index pair's letter for any other at that point unused letter is a licit move. Clearly the tensor networks notation has no use for this move, since each joined leg in effect carries an arbitrary such index pair. In a manner that, when multiple joined legs are present, these are distinguishable from by which slots on which tensors each joins.

Structure 2 A (*singly-contracted*)*inner product* of tensors can be envisaged as an outer product followed by a single contraction. E.g. in the u.d.d. case, this is overall a map

$$\mathfrak{T}\text{ensor}\,(\,D,\,q_1,\,p_1\,)\;\;\times\;\;\mathfrak{T}\text{ensor}\,(\,D,\,q_2,\,p_2\,)\;\;\longrightarrow\;\;\mathfrak{T}\text{ensor}\,(\,D,\,q_1\;+\;q_2\;-\;1,\,p_1\;+\;p_2\;-\;1\,)\;\;.$$

Notational Remark 7 A standard notation is

$$\boldsymbol{T}\,\cdot\,\boldsymbol{U}\;\;.$$

For all that one may well need to take care to point out which index on $\boldsymbol{T}$ is contracted with which index on $\boldsymbol{U}$. A standard convention is that last-to-first is the default, requiring no further comment. Of course, using tensor networks notation, it is automatically clear which leg on $\boldsymbol{T}$ is being joined to which leg on $\boldsymbol{U}$.

Structure 3 An *l-fold-contracted inner product* can be envisaged as an outer product followed by an l-fold contraction. Which in turn can be viewed as l single contractions.
E.g. in the u.d.d. case, this is now overall a map

$$\mathfrak{T}\text{ensor}\,(D, q_1, p_1) \times \mathfrak{T}\text{ensor}\,(D, q_2, p_2) \longrightarrow \mathfrak{T}\text{ensor}\,(D, q_1 + q_2 - l, p_1 + p_2 - l)\ .$$

Notational Remark 8 A fairly standard notation for a 2-fold contracted inner product is

$$\boldsymbol{T} \bigcirc \boldsymbol{U}\ .$$

In some sources, the interior of this circle is shaded to form a bullet:

$$\boldsymbol{T} \bullet \boldsymbol{U}\ .$$

Higher l-fold versions feature considerably less in the literature. So notation for these is rather more down to each Author entertaining any such.

$$\boldsymbol{T} \text{ ⓛ } \boldsymbol{U}$$

is ours in this style.

The above matter of care multiplies with the number of indices getting contracted. The standard convention extends to last-l-into-first-l. The tensor network notation, however automatically renders clear which legs are being contracted with which others. Rendering this style increasingly superior with increasing l.

4.1.4 ii. Inner-product tensoriality

Proposition 3 Let $\boldsymbol{T}$ and $\boldsymbol{U}$ be tensors. Then so is any of their singly-contracted inner products $\boldsymbol{T} \cdot \boldsymbol{U}$.

Proof We consider this in Fig 25 in the case in which both input tensors have all-upstairs indices followed by all-downstairs indices. And the single contraction is between the last downstairs on the first and the first upstairs on the second.

The first step unpackages the inner product in the primed frame. The second applies the tensor transformation law to the uncontracted object. The third and fourth hit with orthogonality and the unit property respectively. The final step reforms the inner product in the unprimed frame. □

Corollary 3 Let $\boldsymbol{T}$ and $\boldsymbol{U}$ be tensors. Then so is any of their l-fold contracted inner products

$$\boldsymbol{T} \text{ ⓛ } \boldsymbol{U}\ .$$

Up to whichever value of l that their ranks support.

Remark 1 In other words, tensoriality is (l-fold) inner-productive. Binarily by Proposition 3 and Corollary 3. And also readily extendible to finitely, by connecting up legs across (k) objects.

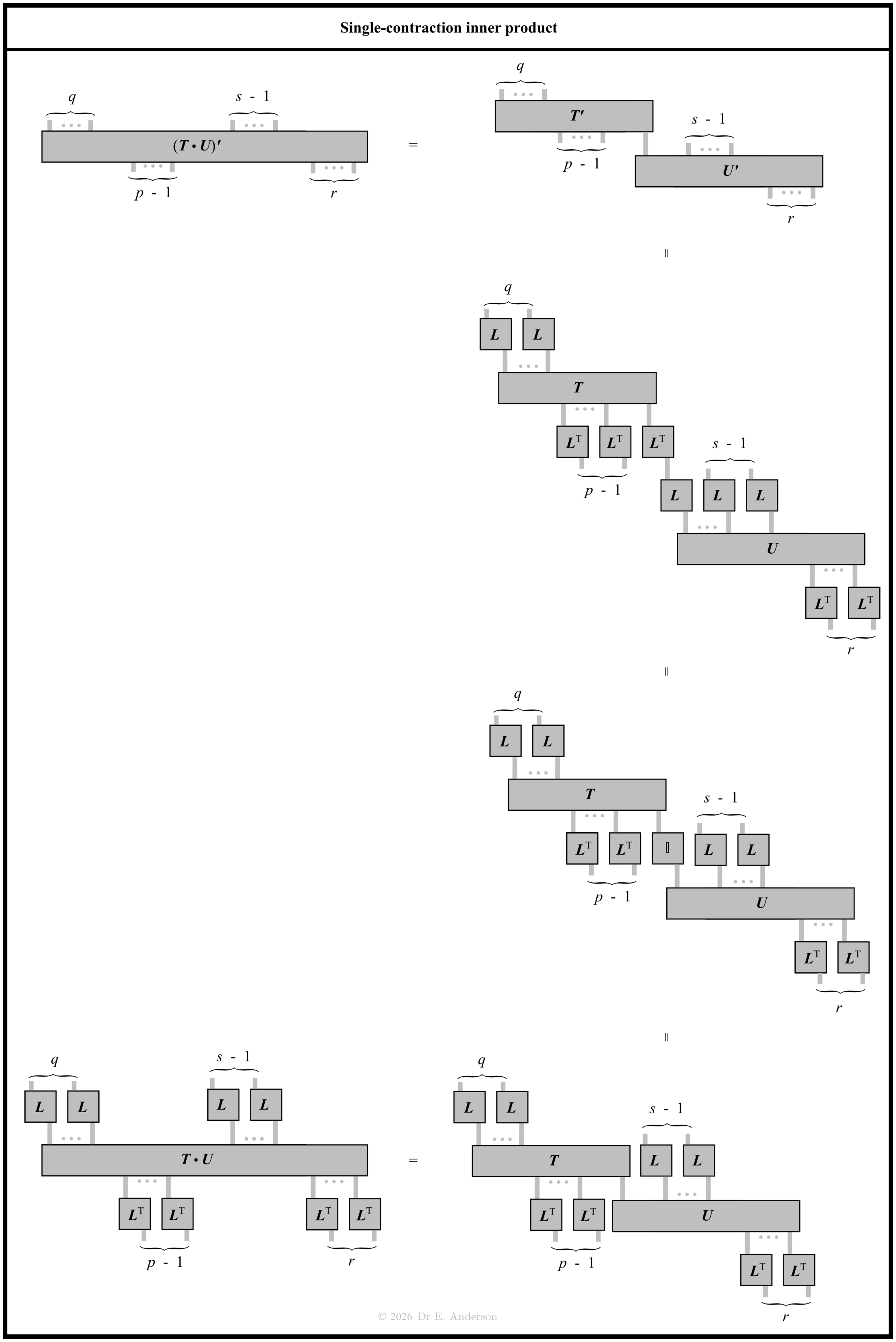
Single-contraction inner product
(T • U)′
T′
U′
T
U
T • U
L
Lᵀ
q
s - 1
p - 1
r
=
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Figure 24:

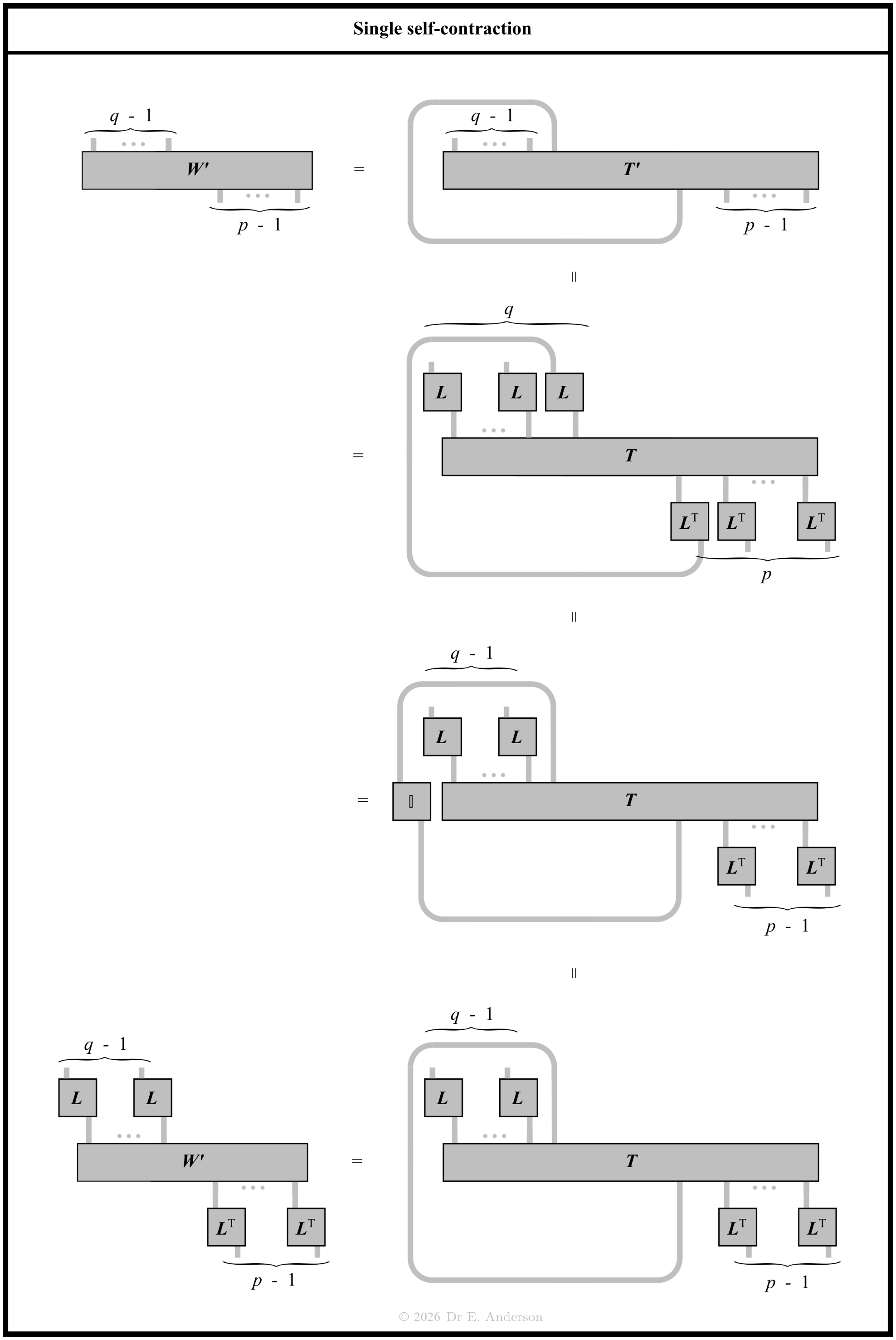


Figure 25:

4.1.5 iii. Self-contraction tensoriality

Proposition 4 The single self-contraction of a Cartesian tensor is itself a Cartesian tensor.

Proof See Fig 24 for a tensor networks rendition. Where the first step unravels the self-contraction in the primed frame. The second applies the corresponding tensor transformation laws to each input tensor. The third and fourth hit with orthogonality and the unit property respectively. While the last step reforms the self-contraction. □

Corollary 4 The l-fold self-contraction of a Cartesian tensor is itself a Cartesian tensor. Up to whichever value of l that its ranks support.

I.e. tensoriality is (l-fold) self-contractive.

Exercise 4 a)$^-$ Prove Corollaries 1 to 4.

b)$^-$ Explain how the scalar product of 2 vectors occurs as a subcase.

c) Prove Proposition 3 and Corollary 3 via use of Structures 1 and l.

Remark 1 In other words, tensoriality is self-contractive. One can also readily extend, whenever one requires, to the following. An l-fold contraction distributed in whichever manner over an n-ary outer product.

4.1.6 The quotient theorem

Theorem Q (Quotient theorem for Cartesian tensors) Suppose that a Cartesian tensor $\boldsymbol{T}$ is the outer product of an arbitrary Cartesian tensor $\boldsymbol{A}$ and a candidate object $\boldsymbol{C}$. Then $\boldsymbol{C}$ is itself a Cartesian tensor.

Proof See Fig 26 for a tensor networks rendition. The first step realizes the outer product in the primed frame. Since $\boldsymbol{C}$ has not yet been established to be a tensor, let us denote its factor in the primed frame not by $\boldsymbol{C}'$ but by some $\boldsymbol{C}^*$ to be determined. Since $\boldsymbol{T}$ is a tensor, it obeys the tensor transformation law as per step 2. We undo the outer product in the unprimed frame in step 3.

In step 4, we equate the first and last of the above, factorizing $\boldsymbol{A}$ out. Since $\boldsymbol{A}$ is arbitrary, we can cancel out this common factor in step 5. Thus establishing that $\boldsymbol{C}^*$ is $\boldsymbol{C}'$, by which $\boldsymbol{C}$ itself meets the definition of Cartesian tensor. □

Remark 3 In other words, tensoriality is *quotientive*, provided that the candidate object's cofactor is an arbitrary tensor.

Remark 4 E.g. in the u.d.d. case, the arenaic framing of the quotient rule is as follows. $\boldsymbol{T} \in \mathfrak{T}\text{ensor}\,(D, q_t, p_t)$ And $\boldsymbol{A} \in \mathfrak{T}\text{ensor}\,(D, q_a, p_a)$. Then $\boldsymbol{C}$ is established to belong to $\mathfrak{T}\text{ensor}\,(D, q_t - q_a, p_t - p_a)$.

Exercise 5 The Reader is welcome to check that the (l)-fold inner product versions of Theorem Q also holds.

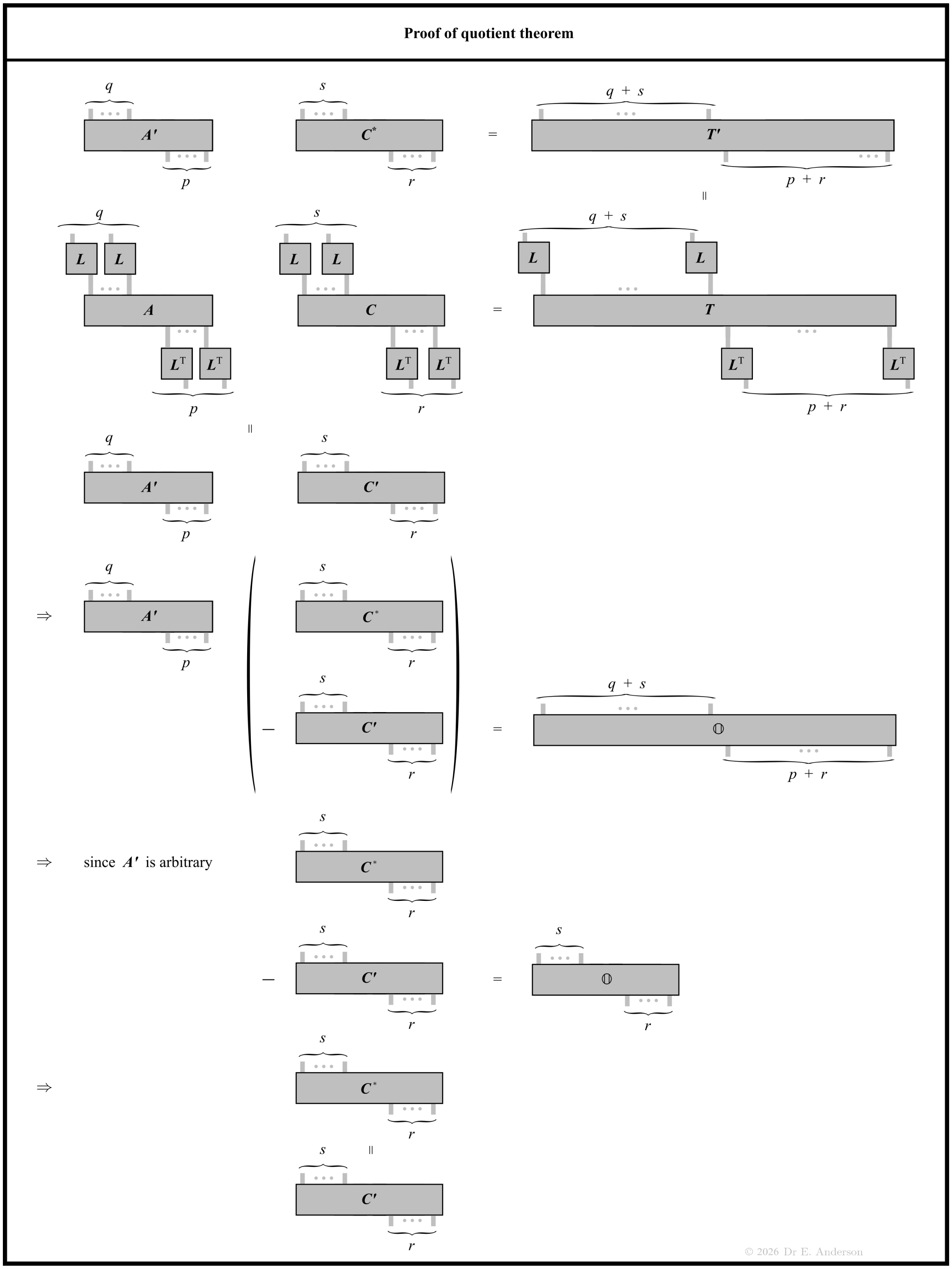

Proof of quotient theorem
since A′ is arbitrary
© 2026 Dr E. Anderson

Figure 26:

4.2 $SO(D)$ counterparts

4.2.1 Linearity

Proposition 1′ Let $\boldsymbol{P}$ and $\boldsymbol{O}$ be Cartesian pseudotensors of the same rank. Then so are the following.

S)

$$\lambda \boldsymbol{P} \ . \tag{42}$$

A)

$$\boldsymbol{P} \ + \ \boldsymbol{O} \ . \tag{43}$$

Corollary 1′

L) Let $\boldsymbol{P}_{(i)}$ $(i) \ = \ (1)$ to (k) be Cartesian pseudotensors of the same rank. Then so is

$$\sum_{(i) \, = \, (1)}^{(k)} \lambda_{(i)} \boldsymbol{P}_{(i)} \ . \tag{44}$$

D) This includes the difference of any 2 equal-rank pseudotensors,

$$\boldsymbol{S} \ - \ \boldsymbol{T} \ . \tag{45}$$

Remark 1 To mark $SO(D)$ dependence at the level of arenas, one can use such as

$$\mathfrak{T}\mathrm{ensor}(\ D, \, q, \, p; \, SO(D) \ ) \ .$$

4.2.2 Outer product

Proposition 2′

E) Let $\boldsymbol{T}$ be a Cartesian tensor and $\boldsymbol{P}$ be a Cartesian pseudotensor. Then their outer product $\boldsymbol{T}\,\boldsymbol{P}$ is a Cartesian pseudotensor.

O) Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be Cartesian pseudotensors. Then their outer product $\boldsymbol{P}\,\boldsymbol{Q}$ is a Cartesian tensor

Corollary 2′

O) Let $\boldsymbol{P}_{(j)}$ $(j) \ = \ (1)$ to (l) be Cartesian pseudotensors. Then the (l)-fold outer product

$$\prod_{(j) \, = \, (1)}^{(l)} \boldsymbol{P}_{(j)} \ = \ \bigotimes_{(j) \, = \, (1)}^{(l)} \boldsymbol{P}_{(j)} \tag{46}$$

is itself a Cartesian pseudotensor for (l) odd. Versus a Cartesian tensor for (l) even.

E) Let $\boldsymbol{T}_{(i)}$ $(i) \ = \ (1)$ to (k) be Cartesian tensors. And $\boldsymbol{P}_{(i)}$ $(i) \ = \ (1)$ to (l) be Cartesian pseudotensors.

Then the $(\,(k) \ + \ (l)\,)$-fold outer product

$$\prod_{(i) \, = \, (1)}^{(k)} \boldsymbol{T}_{(i)} \ \times \ \prod_{(j) \, = \, (1)}^{(l)} \boldsymbol{P}_{(j)} \ = \ \bigotimes_{(i) \, = \, (1)}^{(k)} \boldsymbol{T}_{(i)} \ \otimes \ \bigotimes_{(j) \, = \, (1)}^{(l)} \boldsymbol{P}_{(j)} \tag{47}$$

– with whichever ordering to its factors – is the following.

A Cartesian pseudotensor for (l) odd. And a Cartesian tensor for (l) even.

4.2.3 Self-contraction

Proposition 4′ The single self-contraction of a Cartesian pseudotensor is itself a Cartesian pseudotensor.

Corollary 4′ The l-fold self-contraction of a Cartesian pseudotensor is itself a Cartesian pseudotensor. Up to whichever value of l that its ranks support.

4.2.4 Inner products

Proposition 3′

O) Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be pseudotensors. Then any of their singly-contracted inner products $\boldsymbol{P} \cdot \boldsymbol{Q}$ are tensors.

E) Let $\boldsymbol{T}$ be a Cartesian tensor and $\boldsymbol{P}$ a Cartesian pseudotensor. Then any of their once-contracted inner products $\boldsymbol{T} \cdot \boldsymbol{P}$ is a Cartesian pseudotensor.

Corollary 3′

O) Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be Cartesian pseudotensors. Then their -contracted inner products $\boldsymbol{P} \textcircled{l} \boldsymbol{Q}$ are Cartesian tensors.

E) Let $\boldsymbol{T}$ be a Cartesian tensor and $\boldsymbol{P}$ be a Cartesian pseudotensor. Then their l-fold contracted inner products $\boldsymbol{T} \textcircled{l} \boldsymbol{P}$ are Cartesian pseudotensors.

In each case up to whichever value of l that their ranks support.

4.2.5 Quotient theorem

Corollary Q.1 (Cartesian pseudotensors' corollary of the quotient theorem)

i) Suppose that a Cartesian pseudotensor $\boldsymbol{P}$ is the outer product of an arbitrary Cartesian tensor $\boldsymbol{A}$ and a candidate object $\boldsymbol{C}$. Then $\boldsymbol{C}$ is itself a Cartesian pseudotensor.

ii) Suppose that a Cartesian pseudotensor $\boldsymbol{P}$ is the outer product of an arbitrary Cartesian pseudotensor $\boldsymbol{E}$ and a candidate object $\boldsymbol{C}$. Then $\boldsymbol{C}$ is itself a Cartesian tensor.

iii) Suppose that a Cartesian tensor $\boldsymbol{T}$ is the outer product of an arbitrary Cartesian pseudotensor $\boldsymbol{E}$ and a candidate object $\boldsymbol{C}$. Then $\boldsymbol{C}$ is itself a Cartesian pseudotensor.

Exercise 6 [Long] Prove all of this Subsec's results piecemeal. Present your answers in the vertical tensor networks formulation.

4.3 Parity unification

4.3.1 Parity conservation

Remark 1 Propositions 1 and 4 are multiplicatively unary. As such, Propositions 1 and 1′ and can be unified into a single parity-conserving Proposition $1^\sharp$. And Propositions 4 and 4′ into another: Proposition $4^\sharp$. Where 'parity-conserving' means that the input parity is the same as the output parity. I.e. $+$ to $+$ for tensors, or $-$ to $-$ for pseudotensors. Absorbing Corollaries as well, we arrive at the following sharper unified statements.

Proposition $1^\sharp$ Cartesian parity-s tensoriality is scaleable. And same-rank 'additive, subtractive and elsewise linearly-combineable.

Proposition 4$^\sharp$ Cartesian parity-s tensoriality is l-fold self-contractive. Up to whichever value of l that its ranks support.

4.3.2 Parity multiplication. i. Binary

Remark 1 Entertaining the possibility of pseudotensoriality in Cartesian tensors' binary products confers 2×2 input-object parities. I.e.

$$+ + \, , \quad + - \, , \quad - + \, , \quad - - \, .$$

Since these are deployed productively, they result in the following respective output-object partities.

$$+ \, , \quad - \, , \quad - \, , \quad + \, .$$

Unqualified outer and inner products however treat their 2 input factors interchangeably. Thus the 2 above middle cases collapse to a single middle case. Which we denote by E) standing for h*e*terogeneous inputs. The first case – for tensors – has been left unlabelled. While the last case – in which both inputs are pseudotensors – is denoted above by O). Standing for h*o*mogeneous inputs.

Proposition 2$^\sharp$ Let $\boldsymbol{T}$ be a Cartesian parity-t tensor and $\boldsymbol{U}$ be a Cartesian parity-u tensor. Then their outer product $\boldsymbol{T}\boldsymbol{U}$ is a Cartesian parity-$t\,u$ tensor.

Proposition 3$^\sharp$ Let $\boldsymbol{T}$ be a Cartesian parity-t tensor and $\boldsymbol{U}$ be a Cartesian parity-u tensor. Then each of the l-fold-contracted inner products $\boldsymbol{T} \; ⓛ \; \boldsymbol{U}$ supported is a Cartesian parity-$t\,u$ tensor.

4.3.3 ii. n-ary

Remark 1 Entertaining the possibility of pseudotensoriality in Cartesian tensors' n-ary products confers a large tower of input parities. However the single outputs only admit 2 parities. And these can readily be related to the input parity counts as follows. N_- input pseudotensors enforces output parity

$$(-)^{N_-} \, .$$

Irrespective of what N_+ of input tensors enter. This furthermore simplifies as follows.

$$(-)^{N_-} \quad = \quad \begin{cases} +1 & N_- \text{ even} \\ -1 & N_- \text{ odd} \, . \end{cases} \tag{48}$$

Corollary 2$^\sharp$ Let $\boldsymbol{T}_{(i)}$ $\;(i) = (1)$ to (f) be Cartesian parity-t_i tensors. Then the (f)-fold outer product

$$\prod_{(i)\,=\,(1)}^{(f)} \boldsymbol{T}_{(i)} \quad = \quad \bigotimes_{(i)\,=\,(1)}^{(k)} \boldsymbol{T}_{(i)} \tag{49}$$

– with whichever ordering to its factors – is the following. A Cartesian pseudotensor when N_- is odd. And a Cartesian tensor when N_+ is even.

Exercise 7 Prove this Subsection's results in a unified manner. Present your answers in the vertical tensor networks formulation.

End Remark 2 All in all, pseudotensoriality is thus linear and self-contractive.

Furthermore, if one views pseudotensoriality as $SO(D)$-tensoriality, then these properties are immediate from their holding for whichever kind of G-tensor. While pseudotensoriality is outer- and inner-productive, and quotientive modulo the usual arbitrariness of cofactor, in each case modulo a parity factor behaving as stated. For all that we have postponed parity-unification of the quotient theorem and its pseudotensor corollary until the end of the next Section.

5 Tensor symmetries

5.1 Introduction

5.1.1 Pairwise (anti)symmetry

Definition 1 Let $\boldsymbol{T}$ be a tensor. Suppose that exchanging a pair of $\boldsymbol{T}$'s indices, slots or legs preserves the value of $\boldsymbol{T}$. Then $\boldsymbol{T}$ is *symmetric* in our pair. If instead the sign is reversed, then $\boldsymbol{T}$ is *antisymmetric* in our pair.

Remark 1 A rank-2 object, is *symmetric* if

$$S_{a\,b} \;=\; S_{b\,a}\;.$$

While it is *antisymmetric* if

$$A_{a\,b} \;=\; -\,A_{b\,a}\;.$$

A simple index-free formulation is as follows.

$$\boldsymbol{S} \;=\; \boldsymbol{S}^{\mathrm{T}}\;,$$

$$\boldsymbol{A} \;=\; -\,\boldsymbol{A}^{\mathrm{T}}\;.$$

Remark 2 It is well-known that if a 2-tensor is (anti)symmetric in its indices in one frame, then it is likewise in all other frames. (Anti)symmetry is thus a well-defined tensorial property.

Modelling Assumption 1 For the rest of this Section, we assume that the field $\mathfrak{F}\mathrm{ield}$ we are working over is not of characteristic 2. Thus $2 \neq 0$ and so we can divide by 2. In the current Article's main context, $\mathfrak{F}\mathrm{ield} = \mathbb{R}$, so immediately $2 \neq 0$.

Lemma 1 A symmetric 2-tensor double-contracted into an antisymmetric 2-tensor returns zero.

Proof

$$S_{a\,b}\,A^{a\,b} \;=\; S_{a\,b}\left(\,-\,A^{b\,a}\right) \;=\; -\,S_{b\,a}\,A_{a\,b} \;=\; -\,S_{a\,b}\,A_{a\,b}$$
$$\Rightarrow\; 2\,S_{a\,b}\,A^{a\,b} \;=\; 0$$
$$\Rightarrow\; S_{a\,b}\,A^{a\,b} \;=\; 0\;.$$

Where the first step is by antisymmetry. The second makes 2 dummy-index changes. The third is by symmetry. The fourth groups like terms. And the fifth divides by 2, which we have guaranteed to be licit. □

Lemma 2 Every rank-2 tensor admits a decomposition into symmetric and antisymmetric parts.

Proof Our caveat ensures that division by 2 is licit. Then $0 = 1 - 1$ and rebracketing returns the following 2-term split.

$$\boldsymbol{T}_{a\,b} \;=\; \frac{1}{2}\left(T_{a\,b} + T_{b\,a}\right) + \frac{1}{2}\left(T_{a\,b} - T_{b\,a}\right)\;. \tag{50}$$

But the first term is symmetric and the second term is antisymmetric, so we have obtained such a sum. □

Structure 1 This split is underpinned by the following arena-level decomposition. Take our tensors act on the vector space $\mathfrak{V} = \mathbb{R}^D$. Then

$$\mathbb{R}^D \otimes \mathbb{R}^D \;=\; \mathfrak{S}\mathrm{ym}\left(\mathbb{R}^D\right) \oplus \mathfrak{A}\mathrm{ntisym}\left(\mathbb{R}^D\right) \;=\; \Sigma^2\left(\mathbb{R}^D\right) \oplus \Lambda^2\left(\mathbb{R}^D\right)\;.$$

Of the last two, rather more sources use the notation $\Lambda^2(\mathfrak{V})$ [11, 20, 33]. This is because the theory of forms enjoys many applications.

5.1.2 Further notations

Notational Remark 9 The symmetric and antisymmetric (pairwise) *Bach brackets* notations are as follows.

$$T_{(\,a\,b\,)} \;=\; \frac{1}{2}\,(\,T_{a\,b} \,+\, T_{b\,a}\,)\;.$$

$$T_{[\,a\,b\,]} \;=\; \frac{1}{2}\,(\,T_{a\,b} \,-\, T_{b\,a}\,)\;.$$

Notational Remark 10 Let us refer to further indices, slots or legs on an object that do not participate in its symmetries, as *spectator indices.*

$$T_{a\,\ldots\,b\,i\,j}$$

has a parallel symmetric–antisymmetric split for whichever of its index pairs, with the other indices cast as spectators. Suppose that these indices are not contiguous. Then it is customary to write the Bach brackets with internal bars bounding the spectators, for instance as follows.

$$T_{a\,\ldots\,(\,b\,|\,i\,|\,j\,)}$$

Bach brackets efficientize the notation.

Notational Remark 11 See Fig 27 for tensor network versions of each equation and notation covered so far in the current Section. With straight slash through some legs for symmetrization. And jagged slash for antisymmetrization. Spectator legs are not slashed. Suppose that some such are trapped between legs that are to feature along the same slash. Then this can be handled by the unslashed legs being drawn shorter than the slashed ones. So that the slashes miss them.

It is furthermore possible for an index to be a spectator as regards one slash while participating in another!

Notational Remark 12 We can furthermore cast each last expression in terms of permutations as follows.

$$T_{(a\,b)} \;=\; \frac{1}{|S_2|}\sum_{s\,\in\,S_2} T_{s(a)\,s(b)}\;. \tag{51}$$

And

$$T_{[a\,b]} \;=\; \frac{1}{|S_2|}\sum_{s\,\in\,S_2} \sigma(s)\,T_{s(a)\,s(b)}\;. \tag{52}$$

In each case with reference to the permutation group on 2 objects, S_2 . While the second case also brings in the sign $\sigma(s)$ of each element in our permutation group.

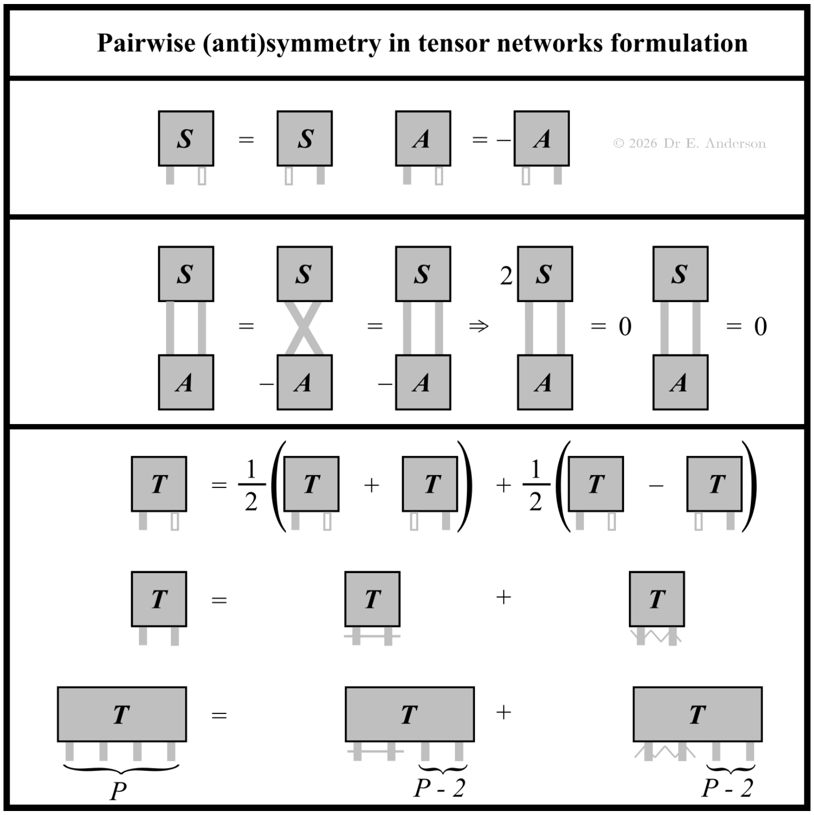


Figure 27:

5.1.3 The trace– tracefree and combined splits

Structure 1 The trace–tracefree split is as follows.

$$\boldsymbol{T} \;=\; \frac{T}{n}\mathbb{I} \;+\; \boldsymbol{T}^{\mathrm{T}} \;. \tag{53}$$

Thus inverting, an explicit form for the tracefree part is as follows.

$$\boldsymbol{T}^{\mathrm{T}} \;=\; \boldsymbol{T} \;-\; \frac{T}{n}\mathbb{I} \;. \tag{54}$$

Where $T \;=\; {}_{\mathrm{r}}\boldsymbol{T}$: the *trace* of $\boldsymbol{T}$. This is a map $\mathbb{R}^D \;\times\; \mathbb{R}^D \;\longrightarrow\; \mathbb{R}$.

Structure 2 The combined split – trace–tracefree split of the antisymmetric–symmetric split – is as follows.

$$\boldsymbol{T} \;=\; \boldsymbol{A} \;+\; \boldsymbol{S}^{\mathrm{T}} \;+\; \frac{T}{n}\mathbb{I} \;. \tag{55}$$

Notational Remark 13 See Fig 28 for tensor network versions of the current SSSec.

Exercise 5^- (square sums) Show that

$$ {}_{\mathrm{r}}\left(\boldsymbol{T}^2\right) \;=\; {}_{\mathrm{r}}\left(\boldsymbol{A}^2\right) \;+\; {}_{\mathrm{r}}\left(\boldsymbol{S}^2\right) \;. $$

$$ {}_{\mathrm{r}}\left(\boldsymbol{T}^2\right) \;=\; {}_{\mathrm{r}}\left(\boldsymbol{T}^{\mathrm{T}\,2}\right) \;+\; \frac{T^2}{n} \;. $$

$$ {}_{\mathrm{r}}\left(\boldsymbol{T}^2\right) \;=\; {}_{\mathrm{r}}\left(\boldsymbol{A}^2\right) \;+\; {}_{\mathrm{r}}\left(\boldsymbol{S}^{\mathrm{T}\,2}\right) \;+\; \frac{T^2}{n} \;. $$

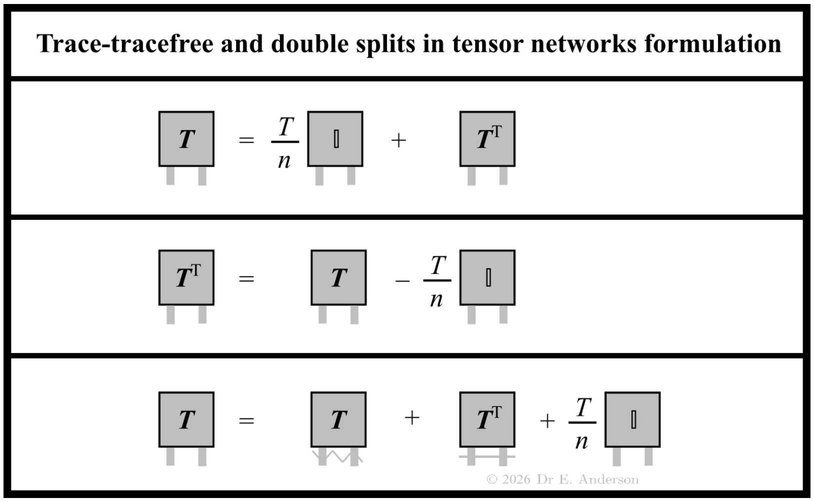


Figure 28:

5.2 A simpler extension

5.2.1 Minimum, total and partial (anti)symmetry

Remark 1 SSecs 5.1-2 considered the minimum nontrivial (anti)symmetrization: on 2 indices. One can however (anti)symmetrize larger sets of indices. If this is applied to all the indices present, it is total (anti)symmetrization In u.d.d. formulation , this is subject to the limitation that all the indices involved have to be on the same floor. Total (anti)symmetrization is the maximum possible amount and pairwise the minimum possible amount. Let us refer to all other instances as partial (anti)symmetrization.

Structure 1 The permutation group point of view provides a particularly clear picture of this generalization. In the total case, one passes from (51, 52) to the following.

$$T_{(\,a_1\,\dots\,a_P\,)} \;=\; \frac{1}{|S_P|}\sum_{s\,\in\,S_P} T_{s(\,a_1)\,\dots\,s(a_P)} \;=\; \frac{1}{P!}\sum_{s\,\in\,S_P} T_{s(\,a_1)\,\dots\,s(a_P)}\ . \tag{56}$$

$$T_{[\,a_1\,\dots\,a_P\,]} \;=\; \frac{1}{|S_P|}\sum_{s\,\in\,S_P} \sigma(s)T_{s(\,a_1)\,\dots\,s(a_P)} \;=\; \frac{1}{P!}\sum_{s\,\in\,S_P} \sigma(s)T_{s(\,a_1)\,\dots\,s(a_P)}\ . \tag{57}$$

While in the partial case with all M indices involved contiguous, say,

$$T_{(\,a_1\,\dots\,a_M\,)\,a_{M\,+\,1}\,\dots\,a_P\dots} \;=\; \frac{1}{|S_M|}\sum_{s\,\in\,S_M} T_{s(\,a_1)\,\dots\,s(a_P)\,a_{M\,+\,1}\,\dots\,a_P} \;=$$

$$\frac{1}{M!}\sum_{s\,\in\,S_M} T_{s(\,a_1)\,\dots\,s(a_P)\,\,a_{M\,+\,1}\,\dots\,a_P}\ . \tag{58}$$

$$T_{[\,a_1\,\dots\,a_M\,]\,a_{M\,+\,1}\,\dots\,a_P\dots} \;=\; \frac{1}{|S_M|}\sum_{s\,\in\,S_M} \sigma(s)T_{s(\,a_1)\,\dots\,s(a_P)\,a_{M\,+\,1}\,\dots\,a_P}$$

And

$$=\; \frac{1}{M!}\sum_{s\,\in\,S_M} \sigma(s)T_{s(\,a_1)\,\dots\,s(a_P)\,\,a_{M\,+\,1}\,\dots\,a_P}\ . \tag{59}$$

Example 3 The minimum example of total (anti)symmetry that is not just a pairwise (anti)symmetry is for $P \;=\; 3$. Here

$$6\,T_{(\,a\,b\,c\,)} \;=\; \sum_{s\,\in\,S_3} T_{s(a)\,s(b)\,s(c)} \;=\; T_{a\,b\,c} \;+\; T_{b\,c\,a} \;+\; T_{c\,a\,b} \;+\; T_{a\,c\,b} \;+\; T_{b\,a\,c} \;+\; T_{c\,b\,a}\ . \tag{60}$$

$$6\,T_{[\,a\,b\,c\,]} \;=\; \sum_{s\,\in\,S_3} \sigma(s) T_{s(a)\,s(b)\,s(c)} \;=\; T_{a\,b\,c} + T_{b\,c\,a} + T_{c\,a\,b} - T_{a\,c\,b} - T_{b\,a\,c} - T_{c\,b\,a}\;. \quad (61)$$

Example 4 The minimum example of a partial (anti)symmetry that is neither pairwise nor total is for $P = 4$. Namely,

$$T_{(\,a\,b\,c\,)\,d} \;=\; \frac{1}{6} \sum_{s\,\in\,S_3} T_{s(a)\,s(b)\,s(c)\,d}\;. \quad (62)$$

$$T_{[\,a\,b\,c\,]\,d} \;=\; \frac{1}{6} \sum_{s\,\in\,S_3} \sigma(s) T_{s(a)\,s(b)\,s(c)\,d}\;. \quad (63)$$

And all other orderings of the index that is being left out, totalling 4 in each case.

Remark 2 See Fig 29 for a tensor networks rendition of the current Article.

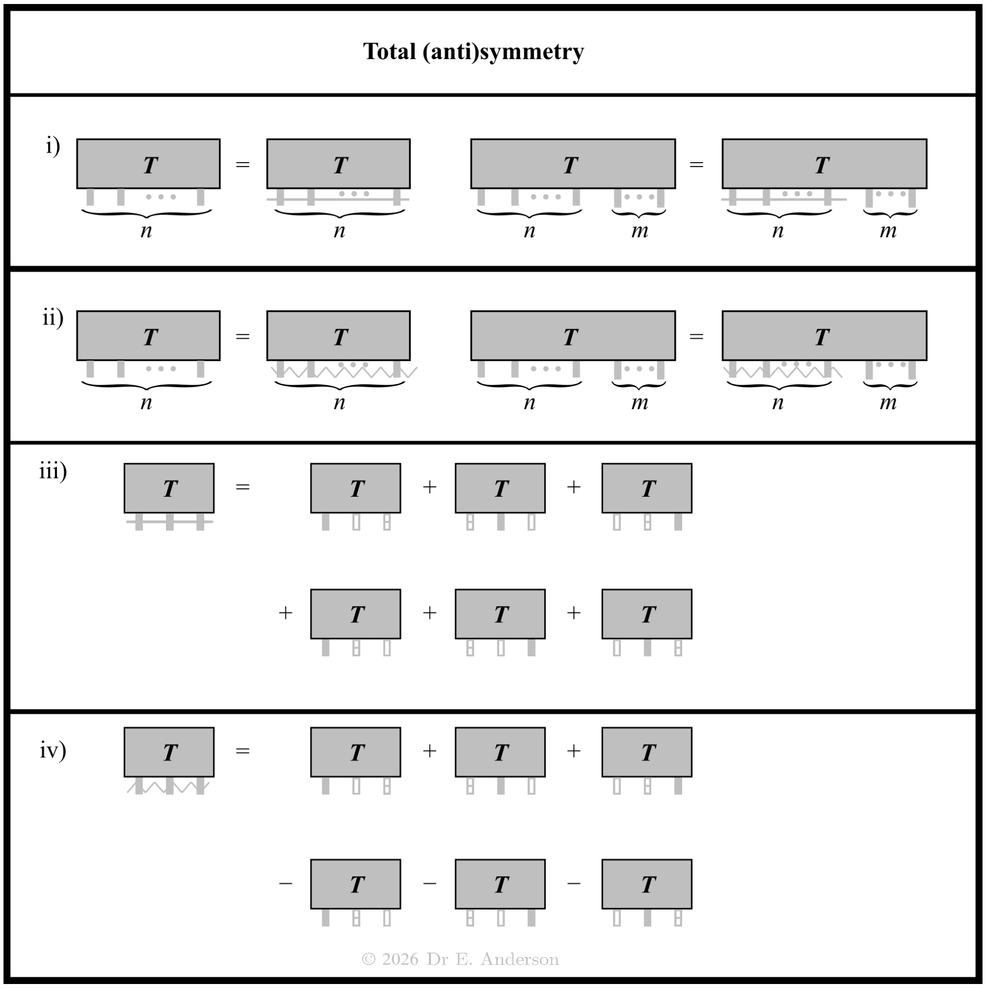


Figure 29:

5.2.2 Partition notation for symmetry type

Examples 0 and 1 The partition notation for the cases supporting no nontrivial symmetry between indices are as follows.

$$\emptyset$$

for scalars. And

$$\square$$

for vectors.

Example 2 The partition notation for pairwise (anti)symmetry is as follows.

$$\square\square \; , \; \begin{smallmatrix}\square\\\square\end{smallmatrix} \, .$$

This suggests the following arena-level notation

$$\mathfrak{V} \otimes \mathfrak{V} \;=\; \mathfrak{V}_{\square\square} \oplus \mathfrak{V}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \, . \tag{64}$$

Which then readily generalizes in parallel to the below developments.

Example 3 For 3-cycle symmetry,

$$\square\square\square \; , \; \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \, .$$

Example 4 While for 4-cycle symmetry,

$$\square\square\square\square \; , \; \begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix} \, .$$

Structure 2 The vertical ones are the conjugate partitions of the horizontal ones of the same length. Using Appendix B's notation, we can jointly denote (anti)symmetry using

$$[\square\square] \; , \; [\square\square\square] \; , \; [\square\square\square\square] \; . \tag{65}$$

Remark 1 The current Section applies regardless of the notion of tensor involved. So long as all of the indices involved in an element of symmetry are of the same conceptual type, leaving them open to permutation. As regards the current Article's scope, then, the current section applies just as well to Cartesian pseudotensors as to Cartesian tensors.

Exercise 8 a) Show that in dimension D ,

$$\#(\square\square\square) \;=\; \frac{D(D+1)(D+2)}{3!} \; , \; \#\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right) \;=\; \frac{D(D-1)(D-2)}{3!} \; ,$$

$$\#(\square\square\square\square) \;=\; \frac{D(D+1)(D+2)(D+3)}{3!} \; , \; \#\left(\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}\right) \;=\; \frac{D(D-1)(D-2)(D-3)}{3!} \; .$$

b) Prove the following.

Lemma 3 For rank $P := p + 1$

i)

$$\#\left(\underbrace{\square \cdots \square}_{P}\right) \;=\; \frac{D^{\overline{p}}}{P!} \;=\; \binom{D+p}{P} \;=\; \frac{1}{P!}\prod_{I=1}^{p} D + I \, . \tag{66}$$

ii)

$$\#\left(\left.\begin{matrix}\square\\\vdots\\\square\end{matrix}\right\} P\right) \;=\; \frac{D^{\underline{p}}}{P!} \;=\; \binom{D}{P} \;=\; \frac{1}{P!}\prod_{I=1}^{p} D - I \, . \tag{67}$$

Where $D^{\overline{p}}$ and $D^{\underline{p}}$ are the *rising* and *falling factorials* respectively [54].

5.3 A harder extension: including mixed symmetries

5.3.1 Existence, minimum occurrence and persistence

Remark 1 That

$$\#\,(\square\!\square) \;+\; \#\left(\begin{array}{c}\square\\[-4pt]\square\end{array}\right) \;=\; \#\,(\square^2)$$

underpins that

$$\square \;\otimes\; \square \;=\; \square\!\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array} \;=\; [\,\square\!\square\,] \;. \qquad (68)$$

These are of course reformulations of the basic split (50).

Remark 2 However, as regards

$$\bigotimes_{I\,=\,1}^{3} \square \,,$$

$$\#\,(\square\!\square\!\square) \;+\; \#\left(\begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\end{array}\right) \;<\; \#\,(\square^3) \;.$$

For

$$\frac{D\,(\,D\;+\;1\,)\,(\,D\;+\;2\,)}{3!} \;+\; \frac{D\,(\,D\;-\;1\,)\,(\,D\;-\;2\,)}{3!} \;=$$

$$\frac{D}{3}\left(\begin{array}{rl} & D^2\;+\;3\,D\;+\;2\\ + & D^2\;-\;3\,D\;+\;2\end{array}\right) \;=\; \frac{D\,\left(\,D^2\;+\;2\,\right)}{3} \;.$$

And

$$D^3 \;-\; \frac{D}{3}\,\left(\,D^2\;+\;2\,\right) \;=\; \frac{D}{3}\,\left(\,3\,D^2\;-\;D^2\;-\;2\,\right)$$

$$=\; \frac{2}{3}\,D\,\left(\,D^2\;-\;1\,\right) \;=\; \frac{2}{3}\,(\,D\;-\;1\,)\,D\,(\,D\;+\;1\,) \;<\; 0\;.$$

Unless $D \;=\; 0$ or 1 . Neither of which however support nontrivial theories of tensors.

This points to the existence of $\geq\;1$ mixed-symmetry species entering the following. The decomposition of our tensor triple product into a direct sum of irreducible pieces.

Remark 3 This existence of a nontrivial middle of mixed-symmetry species furthermore persists for all subsequent ranks. This is trivial from the partition representation. Alongside how fixed-N partition arenas for all subsequent N contain nontrivial middles [67].

All in all, only scalars, vectors and 2-tensors possess no mixed-symmetry parts entering the following.

$$(\text{ The } P\text{-tuple } \bigotimes \text{ of } \square\text{'s) decomposition into } \bigotimes \text{ (irreducible pieces) .}$$

5.3.2 A start on determining the nature of the minimum example

Example $P \;=\; 3$ One further piece is supported by rank-3 tensors. In partition notation,

$$2\,1 \;=\; \begin{array}{l}\square\!\square\\[-4pt]\square\end{array} \;=\; v \;.$$

This is the minimum example of a mixed symmetry.

Exercise 9 a) The following split represents a correct if not yet symmetry-specifying formulation for a rank-3 tensor in 2-D .

$$T_{a\,b\,c} \;=\; T_{(\,a\,b\,c\,)} \;+\; T_{(\,a\,b\,c\,)\;=\;0} \;. \qquad (69)$$

Investigate how many linearly-independent (LI) components each *word-type* – for us now $T_{a\,a\,a}$ and $T_{a\,a\,b}$ with no sums – contributes to each piece.

b) In $\geq$ 3-D, this split generalizes to

$$T_{abc} \;=\; T_{(abc)} + T_{[abc]} + T_{\substack{(\,a\;b\;c\,)\,=\,0\\ [\,a\;b\;c\,]\,=\,0}} . \tag{70}$$

Which word-types are now supported, and how many LI components does each contribute to each piece?

c)$^-$ Write out each of the above pure pieces as vectors of matrices sporting a minimum number of LI components.

Exercise 10 Throughout this Exercise, work with a rank-3 tensor in 2-D . Trawl though all possible components to show that 1 antisymmetry is not enough to capture the mixed symmetry. While considering pairwise antisymmetry in each position in turn is too much. And that even after considering linear dependency (LD), this suggestion yields too many degrees of freedom (d.o.f.) and thus must be discarded.

5.3.3 Applying a subcase of a subcase of how to tensor Young diagrams

Remark 1 While we have so far been taking on faith that there is only 1 kind of mixed-symmetry piece for rank 3 , a milder ambiguity impinges. Namely that 2 copies of $\ydiagram{2,1}$ feature in the following. The triple tensor product of $\ydiagram{1}$'s decomposition into a direct sum of irreducible pieces.

Remark 2 I.e.

$$\bigotimes_{I=1}^{3} \ydiagram{1} \;=\; \ydiagram{3} \oplus 2\,\ydiagram{2,1} \oplus \ydiagram{1,1,1} . \tag{71}$$

As follows from the simple rule for tensoring by atomos . Which Appendix B picks out as a subcase of Pieri's formula for tensoring by a 1-part partition. Which is in turn a subcase of the 'branching rule' conceptual type.

Remark 3 Applying Appendix B's furtherly shortened notation in terms of Young diagram conjugacy classes, (71) becomes the following.

$$\bigotimes_{I=1}^{3} \ydiagram{1} \;=\; [\ydiagram{3}] \oplus 2\,\ydiagram{2,1} . \tag{72}$$

Remark 4 But in 2-D , the following conjugation symmetry-breaking simplification occurs.

$$\bigotimes_{I=1}^{3} \ydiagram{1} \;=\; \ydiagram{3} \oplus 2\,\ydiagram{2,1} . \tag{73}$$

5.3.4 Determining the nature of the minimum example continued

Remark 1 Also Remark 2 of SSSec 5.3.1's count needs to be halved, yielding the following.

$$\#\left(\ydiagram{2,1}\right) \;=\; \frac{D\left(D^2 - 1\right)}{3} \;=\; \frac{(D-1)\,D\,(D+1)}{3} . \tag{74}$$

Exercise 11 a) Show that the overlapping-index combination
i) antisymmetry in last pair

$$T_{abc} \;=\; T_{a[bc]} ,$$

ii) symmetry in first pair

$$T_{abc} \;=\; T_{(ab)c}$$

is not a viable candidate for $\ydiagram{2,1}$.

b) But that the combination

i) antisymmetry in last pair

$$T_{a\,b\,c} \;=\; T_{a\,[\,b\,c\,]}\;, \tag{75}$$

ii) zero antisymmetric 3-cycle

$$T_{[\,a\,b\,c\,]} \;=\; \mathbb{O}_{a\,b\,c} \tag{76}$$

supports the correct number of components.

c) How about the symmetry–antisymmetry reversal of ii)? I.e.
i) symmetry in the last pair

$$T_{a\,b\,c} \;=\; T_{a\,(\,b\,c\,)}\;, \tag{77}$$

ii) zero symmetric 3-cycle

$$T_{(\,a\,b\,c\,)} \;=\; \mathbb{O}_{a\,b\,c}\;. \tag{78}$$

Remark 2 See Fig 30 for a tensor networks rendition of b)'s pair of symmetries.

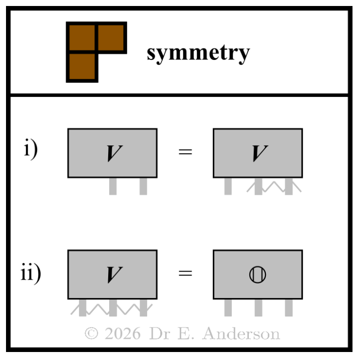


Figure 30:

5.3.5 Rank 4

Remark 1 Ensure that you have covered Exercise C.1 by this stage.

Example 4 Rank $P = 4$ is minimum to sport multiple types of mixed symmetry. Comprising 1 heteroconjugate pair,

$$\ydiagram{3,1}\;,\;\ydiagram{2,1,1}\;. \tag{79}$$

And a rather more interesting example,

$$\ydiagram{2,2}\;. \tag{80}$$

Among the famous Differential-Geometric tensors, the all-downstairs Riemann and Weyl curvature tensors enjoy this particular mixed symmetry.

Notational Remark 14 Following on from discussion in [39], the Kulkarni–Nomizu product encodes the same symmetry. These feature for instance in the 2 trace pieces by which the Weyl tensor differs from the Riemann tensor. The following related truer names and notation is suggested. *Square symmetry* and *square product*. after the corresponding Young diagram for the symmetry. To be denoted by this Young diagram itself. In a smaller size for the product for comensurability with other product- and more generally binary-relation-symbols.

The 2 correction terms in setting up the Weyl curvature tensor are then the following homogeneous and heterogeneous uses of the new symbol.

$$\boldsymbol{g} \mathbin{\rotatebox{45}{$\boxplus$}} \boldsymbol{g}\;,\quad \boldsymbol{g} \mathbin{\rotatebox{45}{$\boxplus$}} \boldsymbol{R}\;.$$

Each term furthermore comes with a dimension-dependent proportionality factor.] Observe that in rotating the square by $\frac{\pi}{4}$ radians, we not only get it to contain a boxed multiplication symbol. But also attain Appendix B's more advanced notation that is manifestly conjugation-invariant.

Remark 2 $\boxplus$ corresponds to the following symmetries. ‘

i) Pairwise antisymmetry.

$$T_{a\,b\,c\,d} \;=\; T_{a\,b\,[\,c\,d\,]} \;.$$

ii) Index-pair exchange symmetry, with 1 of the exchanged pairs carrying the previous antisymmetry.

$$T_{a\,b\,c\,d} \;=\; T_{c\,d\,a\,b} \;.$$

iii) Zero 3-cycle condition.

$$T_{a\,[\,b\,c\,d\,]} \;=\; 0_{a\,b\,c\,d} \;.$$

Which, in the context of the Riemann curvature tensor, is called the *first Bianchi identity* [22, 13]. See Fig 31 for a tensor networks rendition.

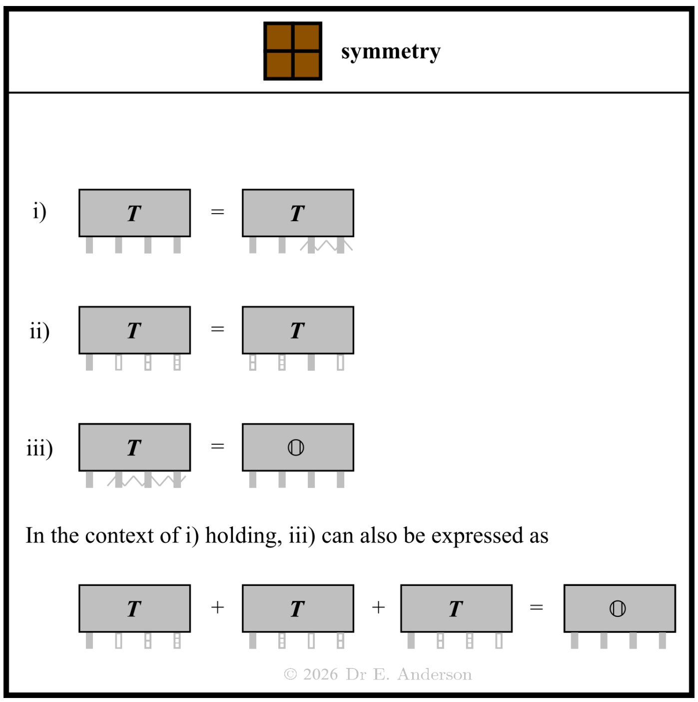


Figure 31:

5.3.6 Exercises

Exercise 12 Show that

$$\#\left(\boxplus\right) \;=\; \frac{D^2\,\left(D^2\;-\;1\right)}{12} \;=\; \frac{(\,D\;-\;1\,)\,D^2\,(\,D\;+\;1\,)}{12}\;. \qquad (81)$$

Exercise 13$^-$ Show that

$$\#\left(\boxplus\right) \;\overset{2\text{-}D}{=}\; 1\;, \qquad (82)$$

$$\#\left(\yng(3,1)\right) \;\overset{2\text{-}D}{=}\; 3\;. \qquad (83)$$

Exercise 14$^-$ Find

$$\#\left(\boxplus\right)$$

in 3- and 4-D . Show that

$$\#\left(\ydiagram{2,2}\right) \overset{3\text{-}D}{=} \#\left(\ydiagram{2}\right) . \tag{84}$$

But

$$\#\left(\ydiagram{2,2}\right) \overset{4\text{-}D}{>} \#\left(\ydiagram{2}\right) . \tag{85}$$

Pointer 1 This corresponds for instance to the 3-D Riemann curvature tensor containing no information in excess of its first contraction the Ricci curvature tensor. But not in 4-D , so there is a strong sense in which $D = 4$ is minimum for distinctive study of the Riemann curvature tensor. Furthermore, the Weyl curvature tensor encodes this further information. So this remarkable object is realized in 4-D , but is just zero in 3-D .

Exercise 15 Show that in arbitrary-D ,

$$\#\left[\ydiagram{3,1}\right] = \frac{D^2\left(D^2 - 1\right)}{6} = \frac{(D - 1)\,D^2\,(D + 1)}{6} . \tag{86}$$

Exercise 16 Find where among the 2-D $\ydiagram{2,2}$'s components the single supported d.o.f. resides.

5.3.7 More advanced Exercises

Exercise 17 [67] (Extent of mixed domination)

Let $\mathfrak{p} := \#(\text{pure d.o.f.})$ and $\mathfrak{M} := \#(\text{mixed d.o.f.})$.

Show that for rank $P = 3$,

$$\begin{array}{lllllll} \mathfrak{p} & = & \mathfrak{M} & \text{for} & D & = & 0 \text{ or } 2 \\ \mathfrak{p} & > & \mathfrak{M} & \text{for} & D & = & 1 \\ \mathfrak{p} & < & \mathfrak{M} & \text{for} & D & \geq & 3 \end{array} . \tag{87}$$

While for $P = 4$,

$$\begin{array}{lllllll} \mathfrak{p} & = & \mathfrak{M} & \text{for} & D & = & 0 \text{ or } 1 \\ \mathfrak{p} & < & \mathfrak{M} & \text{for} & D & \geq & 2 \end{array} . \tag{88}$$

Remark 1 Thus failing to spot that mixed symmetries exist rapidly and permanently degenerates into missing *most of* general tensors' d.o.f.

Exercise 18 For rank-4 tensors, work out how many components each word-type contributes to each of the following.

$$\ydiagram{4}\,,\quad \ydiagram{1,1,1,1}\,,\quad \ydiagram{2,2}\,,\quad \left[\ydiagram{3,1}\right] .$$

In each of 2-, 3- and 4-D .

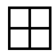
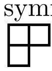
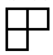
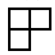
Pointer 2 On the one hand, the abovementioned Riemann to Weyl interconversion is a rank-4 example of the analogue of the trace-tracefree split. On the other hand, let us leave systematic consideration of rank-3 and -4 analogues of the trace-tracefree split to a later version of the current Review.

Exercise 19$^-$ Extend the elementary course's result that pairwise (anti)symmetry is tensorial to arbitrary symmetry. Also establish that the value of the trace, and being tracefree, are tensorial.

5.4 Symmetries of u.d.d. tensors

Remark 1 Suppose that we are given a tensor with q upstairs and p downstairs indices. Then a priori we can only consider permuting upstairs indices separately from permuting downstairs indices. Thus each u.d.d. tensor carries an upstairs-index permutation-symmetry class and a separate downstairs one. This can be encoded as the direct alias Cartesian product of each's partition's Young diagram.

Remark 2 If a metric is furthermore present, then one is further entitled to place all the indices on the same floor. Be it by the metric's lowering property or by its inverse's raising property. Then one can permute whichever indices, and one is back to encoding these using a single Young diagram. At least in some settings, however, one would regard purely downstairs and 'mixed-floor' versions of the tensor as different objects. Indeed as evidenced by their possessing symmetries that can be distinct from each other.

Example 1 A common example on manifolds is that Metric Geometry's all-downstairs Riemann curvature tensor carries more symmetries than the following. Its Affinely-natural counterpart with 1 raised index. By  to  . In this case, since there is only 1 upstairs index, there is no room for upstairs to contribute a nontrivial factor to the above Cartesian product . Via while one is realizing has  $\times$ $\square$, this is just equal to  .

5.5 Interplay with quotient rule

5.5.1 Contractions with symmetric pieces

Corollary Q.2 i) Let $\boldsymbol{S}$ be a rank-2 tensor that is symmetric but elsewise arbitrary. Suppose that this is double-contracted with a candidate object $\boldsymbol{C}$. And that the outcome is a tensor $\boldsymbol{T}$. Then $\boldsymbol{C}$ symmetrized over its contracted pair of indices, which we denote by $\boldsymbol{SC}$, is itself a tensor.

ii) Let $\boldsymbol{A}$ be a rank-2 tensor that is antisymmetric but elsewise arbitrary. Suppose that this is double-contracted with a candidate object $\boldsymbol{C}$. And that the outcome is a tensor $\boldsymbol{T}$. Then $\boldsymbol{C}$ antisymmetrized over its contracted pair of indices is itself a tensor.

<u>Proof</u> i) See Fig 32 for a tensor networks rendition. We cannot immediately cancel off $\boldsymbol{S}$, since it is not arbitrary. But $\boldsymbol{SC}$ double-contracted into an antisymmetric but elsewise arbitrary tensor $\boldsymbol{A}$ is zero by symmetry–antisymmetry. So the given $\boldsymbol{SC}$ double-contracted into $\boldsymbol{S}$ is equal to $\boldsymbol{SC}$ double-contracted into $\boldsymbol{S} + \boldsymbol{A} = \boldsymbol{Arb}$. Whose second factor is indeed now arbitrary, and thus can be cancelled off. So the standard quotient theorem applies to $\boldsymbol{SC}$, giving the claimed result. $\square$

Corollary Q.3 (with spectator indices) i) Let $\boldsymbol{S}$ be a rank-k tensor that is symmetric in 2 of its indices but elsewise arbitrary. Suppose that this is doubly-contracted with a candidate object $\boldsymbol{C}$. And that the outcome is a tensor $\boldsymbol{T}$. Then $\boldsymbol{C}$ symmetrized over its doubly-contracted indices is a tensor.

ii) The symmetry $\rightarrow$ antisymmetry counterpart of this also holds.

Exercise 20 Prove Corollaries Q.1.ii) and 2. Present your answers in tensor networks form.

Corollary Q.4 i) Let $\boldsymbol{S}$ be a rank-l tensor that is l-fold symmetric but elsewise arbitrary. Suppose that this is l-fold contracted with a candidate object $\boldsymbol{C}$. And that the outcome is a tensor $\boldsymbol{T}$. Then $\boldsymbol{C}$ symmetrized over its contracted l-fold of indices is a tensor.

ii) The symmetry $\rightarrow$ antisymmetry counterpart of this also holds.

iii) Both of these extend to include spectator indices as well.

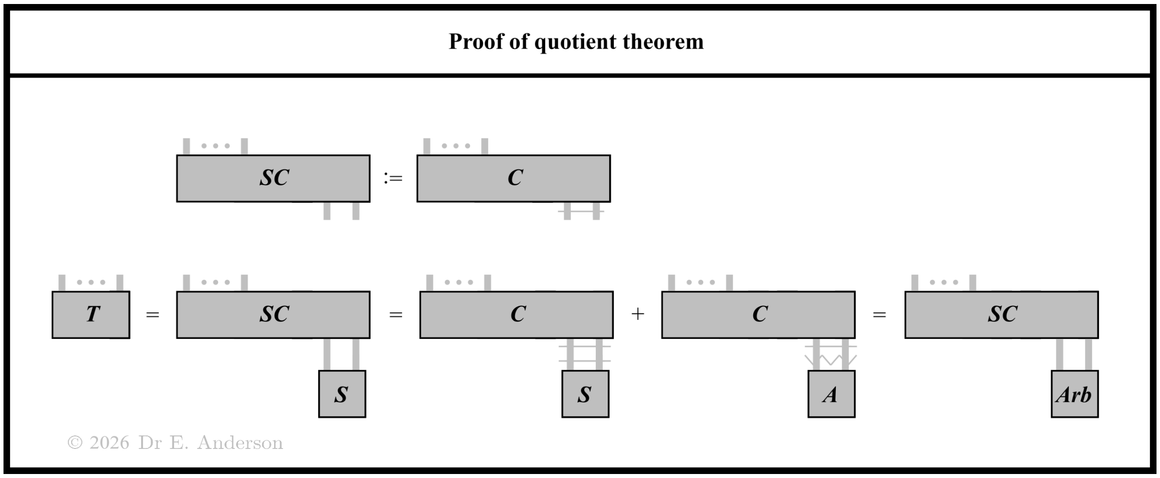


Figure 32:

Remark 1 From now on, we build in spectator indices without further comment.

Corollary Q.5 (general-symmetry extension of quotient theorem for tensors) Suppose that a tensor $\boldsymbol{M}$ carries the Young diagram $Y(\,\mathrm{p}(N)\,)$ of symmetries. And has an l-subset of these indices contracted into a candidate object $\boldsymbol{C}$. Let this l-subset furthermore realize the Young subdiagram $Y(\,\mathrm{p}(l)\,)$. Finally suppose that our product $\boldsymbol{M}\ \textcircled{l}\ \boldsymbol{C}$ is a tensor. Then the $Y(\,\mathrm{p}(l)\,)$-symmetrization of the contracted indices on $\boldsymbol{C}$ is itself a tensor.

Exercise 21$^+$ Prove Corollaries Q.3-4. Present your answers in tensor networks form.

5.5.2 Interplay with trace–tracefree split

Corollary Q.6 (trace-tracefree case) a) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the tracefree index pair of an elsewise arbitrary tensor $\boldsymbol{\mathcal{T}}$. So as to form a tensor $\boldsymbol{T}$. Then

$$\boldsymbol{C}^{\mathrm{T}} \;:=\; (\text{ tracefree part over an index pair of } \boldsymbol{C}\ )$$

is itself a tensor.

b) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the trace term's index pair of an elsewise arbitrary tensor $\mathbb{I}\,\boldsymbol{\mathcal{F}}$. Where $\boldsymbol{\mathcal{F}}$ carries purely spectator indices. So as to form a tensor $\boldsymbol{T}$. Then

$$\boldsymbol{C}^{\mathrm{t}} \;:=\; (\text{ trace part over an index pair of } \boldsymbol{C}\ )$$

is itself a tensor.

Corollary Q.7 (double-split case) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the symmetric-tracefree index pair of an elsewise arbitrary tensor $\boldsymbol{\mathcal{S}}^{\mathrm{T}}$. So as to form a tensor $\boldsymbol{T}$. Then

$$\boldsymbol{C}^{\mathrm{ST}} \;:=\; (\text{ tracefree part over an index pair of } \boldsymbol{C}\ )$$

is itself a tensor.

Remark 1 This result superposes 2 different types of split: symmetric-antisymmetric and trace-tracefree. It has no new partner results. For the antisymmetric partner works with just a symmetric concomitant piece. And so is covered by corollary Q.3. And the trace partner works with just an unsplit trace-free concomitant piece. And so is covered by corollary Q.6.

5.5.3 Interplay with pseudotensors

Corollary Q.7 (general-symmetry extension of quotient theorem for pseudotensors)

i) Suppose that a tensor $\boldsymbol{M}$ carries the Young diagram $Y(\,\mathrm{p}(N)\,)$ of symmetries. And has an l-subset of these indices l-fold contracted into a candidate object $\boldsymbol{C}$. Let this l-subset furthermore realize the Young subdiagram $Y(\,\mathrm{p}(l)\,)$. Finally suppose that our product $\boldsymbol{M} \textcircled{l} \boldsymbol{C}$ is a pseudotensor. Then the $Y(\,\mathrm{p}(l)\,)$-symmetrization of the contracted indices on $\boldsymbol{C}$ is itself a pseudotensor.

ii) Suppose that a pseudotensor $\boldsymbol{M}$ carries the Young diagram $Y(\,\mathrm{p}(N)\,)$ of symmetries. And has an l-subset of these indices l-fold contracted into a candidate object $\boldsymbol{C}$. Let this l-subset furthermore realize the Young subdiagram $Y(\,\mathrm{p}(l)\,)$. Finally suppose that our product $\boldsymbol{M} \textcircled{l} \boldsymbol{C}$ is a tensor. Then the $Y(\,\mathrm{p}(l)\,)$-symmetrization of the contracted indices on $\boldsymbol{C}$ is itself a pseudotensor.

iii) Suppose that a pseudotensor $\boldsymbol{M}$ carries the Young diagram $Y(\,\mathrm{p}(N)\,)$ of symmetries. And has an l-subset of these indices l-fold contracted into a candidate object $\boldsymbol{C}$. Let this l-subset furthermore realize the Young subdiagram $Y(\,\mathrm{p}(l)\,)$. Finally suppose that our product $\boldsymbol{M} \textcircled{l} \boldsymbol{C}$ is a pseudotensor. Then the $Y(\,\mathrm{p}(l)\,)$-symmetrization of the contracted indices on $\boldsymbol{C}$ is itself a tensor.

Corollary Q.8 a).i) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the tracefree index pair of an elsewise arbitrary tensor $\mathcal{T}$. So as to form a pseudotensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{T}}$ is itself a pseudotensor.

ii) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the tracefree index pair of an elsewise arbitrary pseudotensor $\mathcal{T}$. So as to form a tensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{T}}$ is itself a tensor.

iii) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the tracefree index pair of an elsewise arbitrary pseudotensor $\boldsymbol{\mathcal{T}}$. So as to form a pseudotensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{T}}$ is itself a tensor.

b).i) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the trace term's index pair on an elsewise arbitrary tensor $\mathbb{I}\,\boldsymbol{\mathcal{F}}$. Where $\boldsymbol{\mathcal{F}}$ carries purely spectator indices. So as to form a pseudotensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{t}}$ is itself a pseudotensor.

ii) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the trace term's index pair on an elsewise arbitrary pseudotensor $\mathbb{I}\,\boldsymbol{\mathcal{F}}$. So as to form a tensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{t}}$ is itself a pseudotensor.

iii) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the trace term's index pair on an elsewise arbitrary pseudotensor $\mathbb{I}\,\boldsymbol{\mathcal{F}}$. So as to form a pseudotensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{t}}$ is itself a tensor.

c) i) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the symmetric-tracefree index pair of an elsewise arbitrary tensor $\boldsymbol{\mathcal{S}}^{\mathrm{T}}$. So as to form a pseudotensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{ST}}$ is itself a pseudotensor.

ii) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the symmetric-tracefree index pair of an elsewise arbitrary pseudotensor $\boldsymbol{\mathcal{S}}^{\mathrm{T}}$. So as to form a tensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{ST}}$ is itself a pseudotensor.

iii) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the symmetric-tracefree index pair of an elsewise arbitrary pseudotensor $\boldsymbol{\mathcal{S}}^{\mathrm{T}}$. So as to form a pseudotensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{ST}}$ is itself a tensor.

Exercise 22 [Long] Prove Corollaries Q.7–8 piecemeal. Present your answers in tensor networks form.

5.5.4 Parity reformulation of four quartets of quotient theorems and corollaries

Remark 1 Entertaining the possibility of pseudotensoriality in the quotient theorem for Cartesian tensors now involves all 4 distinct parity cases. For, as compared with the previous product case, the quotient makes heterogeneous use of its inputs: outer product and arbitrary factor. In the order given in the SSSec on products, the first is tensorial, and the others are what we previously labelled as cases i) to iii).

Notational Remark 15 Indeed truer notation for these cases follows from their inputs: $+\,-$, $-\,+$ and $-\,-$ respectively.

Theorem Q$^\sharp$ (Parity unification of Cartesian (pseudo)tensor quotient theorem and corollaries) Suppose that a Cartesian parity-t tensor $\boldsymbol{T}$ is the outer product of the following. An arbitrary Cartesian parity-a tensor $\boldsymbol{A}$ And a candidate object $\boldsymbol{C}$. Then $\boldsymbol{C}$ is itself a Cartesian parity-$a\,t$ tensor.

Corollary Q.7$^\sharp$ (Parity unification of Cartesian (pseudo)tensor quotient theorem's general-symmetry extension)

Suppose that a Cartesian parity-m tensor $\boldsymbol{M}$ carries the Young diagram $Y(\,\mathrm{p}(N)\,)$ of symmetries. And has an l-subset of these indices l-fold contracted into a candidate object $\boldsymbol{C}$. Let this l-subset furthermore realize the Young subdiagram $Y(\,\mathrm{p}(l)\,)$. Finally suppose that our product $\boldsymbol{T} = \boldsymbol{M} \textcircled{l} \boldsymbol{C}$ is a Cartesian parity-t tensor. Then the $Y(\,\mathrm{p}(l)\,)$-symmetrization of the contracted indices on $\boldsymbol{C}$ is itself a Cartesian parity $m\,t$-tensor.

Corollary Q.8$^\sharp$ a) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the tracefree index pair of an elsewise arbitrary parity-a tensor $\boldsymbol{\mathcal{A}}$. So as to form a parity-t tensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{T}}$ is itself a parity-$a\,t$ tensor.

b) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the trace term's index pair of an elsewise arbitrary parity-b tensor $\mathbb{I}\,\boldsymbol{\mathcal{B}}$. Where $\boldsymbol{\mathcal{B}}$ carries purely spectator indices. So as to form a parity-t tensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{t}}$ is itself a parity-$b\,t$ tensor.

c) i) Suppose that a candidate object $\boldsymbol{C}$ is doubly-contracted into the symmetric-tracefree index pair of an elsewise arbitrary parity-s tensor $\boldsymbol{\mathcal{S}}^{\mathrm{T}}$. So as to form a parity-t tensor $\boldsymbol{T}$. Then $\boldsymbol{C}^{\mathrm{ST}}$ is itself a parity-$s\,t$ tensor.

Exercise 23 Prove this SSec's results in a unified manner. Present your answers in tensor networks form.

Exercise 24 Work out the arenas involved in each result.

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