Tensor Networks throughout the Basic Theory of STEM. III. Linear Algebra

E. Anderson* **and K. Everard**

Abstract

We continue our program of using tensor networks throughout the basic theory of STEM, now in basic Linear Algebra. We make the upstairs-downstairs distinction between vectors and their covector duals, in anticipation of Differential Geometry and Relativity applications. This choice renders available Penrose's default that the tensor network's legs are to be vertical. We furthermore introduce colour-coding to distingush primary-space component indices and basis labels: grey and beige respectively. This is part of our more specific use of tensor networks' rainbow-candy vertical variant.

We consider 1) homogeneous-linear coordinate transformations, and 2) changes of basis. The interplay between these requires a beige version of 1). 3) Inner products and norms. 4) Orthonormalization and 5) changes of orthonormal bases, specializing 1) and 2) and incurring the Drawing and Visualization ambiguity in the cover figure. 6) Adjoint operators and 7) dual bases. 8) Bilinear and multilinear forms, alongside their specialization to alternating forms, a standard application of which is to determinants.

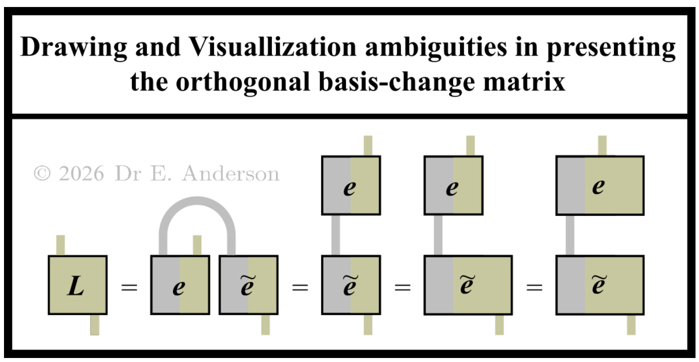


Figure 1:

This Article is ⟨3⟩: accessible to third-year undergraduates.

Cite as E. Anderson and K. Everard, "Tensor Networks throughout the Basic Theory of STEM. III. Linear Algebra", Online Encyclopaedia of Tensors and other Multi-Linear Arrays, institute-theory-stem.org/oetoma-tensor-networks-in-vector-algebra/ (2026).

* Corresponding author: Dr.E.Anderson.Maths.Physics *at* protonmail.com . Institute for the Theory of STEM

1 Introduction to our notation using components and bases

Notational Remark 1 The current Encyclopaedia [17] investigates how the Combinatorial and Visual tensor networks notation [7, 45, 15, 16, 22, 21] fares throughout the basic theory of STEM. An occasional alias for this notation in the Lie Theory and Physics literatures is *birdtracks*. This refers to the lines – a coordinate-free replacement for indices – that emanate from the boxes denoting the tensors [2, 3, 12, 23, 17] and other multi-linear arrays [42, 30, 33, 22, 17].

Structure 1 Let $\boldsymbol{v}$ be a vector in $\mathbb{R}^n$. If this is given in components, v^i, then these are with respect to some choice of basis vectors.

A Physicist might call this some choice of reference frame. The standard Cartesian basis [13] is denoted by

$$\{\,\boldsymbol{e}_i\,\}_{i\,=\,1}^{n} \ .$$

This corresponds to a rectangular frame of reference. Which forms a little square in $\mathbb{R}^2$, a little cube in $\mathbb{R}^3$ and so on. If we are using such a basis, then

$$\boldsymbol{v} \ = \ \sum_{i\,=\,1}^{n} \boldsymbol{e}_i\, v^i \ .$$

Suppose that for instance we take $\boldsymbol{e}_1$ to be the unit vector along the x-axis, $\hat{\boldsymbol{x}}$. Then the corresponding component v^1 is the component of our vector along the x-axis. Which can be viewed, more Geometrically, as the outcome of projecting our vector onto the x-axis.

Notational Remark 2 Let us next entertain various more efficient notations for this. $\boldsymbol{e}_i$ is really an object with 2 indices. Comprising a vector index hidden in its boldness, and a basis member label, i. Since we assign upstairs indices to vectors, this is, in fully-indexed form,

$$e^k{}_i \ .$$

On the one hand, k and i both run over 1 to n. On the other hand, there is a sense in which they are not the same kind of indices. One way out would be to use different fonts. Or to have a, b, c denote a different kind of label from i, j, k. Or to use (i) for basis member labels.

Notational Remark 3 Our first way out is however to use

$$e^k{}_i \ .$$

I.e. grey for spatial indices to beige for basis member labels.

It is then a reasonable question whether the e should itself be grey, beige, both or neither...

The democratic answer is both. This is better than neither, because it says which 2 kinds of index it carries. This will be rather useful in more advanced subjects, in which numerous species of index are present. Indeed, our ab initio dull choice of grey for space and beige for 'internal labels' reflects that these labels are ubiquitous infrastructure labels. For we are reserving more iconic colours for such as the many different kinds of indices that feature in Lie Theory; see the Conclusion for further details.

There is a temporary inconvenience in that it is not easy to colour our object-symbol e half in one colour and half in another. Firstly, this could well require a figure-file 'glyph' to produce as a LaTeX macro. But this does not scale well in the sense that one would need several minutes per case per letter per font per combination of colours to produce these. Secondly, it would not be easy to read, and even less easy to print or to read off a print-out.

Notational Remark 4 We soak much of this up with our second way out. Which is to place one's index-free equations[1] in the figure environment, and box up each object-symbol.

The boxes are then readily large enough to nitidly multi-colour. Now one only needs 1 box shape per multilinear-array rank. And only 1 box decomposition per n-colour object, for us today a vertical halving for our half-grey and half-beige object. All of this is object-letter and font-independent, requiring just 2 or 3 clicks to deploy each extra such. While each colour combination for a box needs recolouring operations, we target this with an n-colour palette. With n small enough for the colours to be widely recognizably nameable, and also distinguishable even in prevalent cases of colourblindness.

♠ ◇ ♣

Notational Remark 5 A priori, there is an infinity of shapes that the legs could take. We prefer vertical legs wherever possible. In upstairs-downstairs distinct presentations, this trivializes leg shape when applicable, and goes back to Penrose [7].

It does not take particularly technically-complex examples to force other than vertical line-interval legs however. The current Article shows this to apply even for basic Linear Algebra.

Remark 1 The current Encyclopaedia's Drawing and Visualization system is moreover well capable of firstly finding incorrect, inconsistent or conflated indices in the literature. Secondly, of working with 1 order of magnitude more index species than is usually attempted in any previous literature. Thirdly, it provides clearer pedagogy. Both by emphasizing and subsequently tracking precisely what each index is, and by visually encoding relation between index types.

Finally this pedagogical motivation strongly suggests fielding our system for basic enough courses to have millions of students per year. Such as Linear Algebra [22], Flat Geometry [21] or Combinatorics [17] And beginner-level courses in each of Geometry of Surfaces [46, 42], Differential Geometry [42] and Lie Theory [47].

As opposed to thousands or less per year that consider the more advanced and specialized topics that we point to in the Conclusion.

[1] These are useful for introducing and conceptualizing about multi-linear objects. Albeit they are often held to be less useful for calculating. For all that Cvitanovic [45] has shown us that calculations themselves can be efficiently done within a tensor networks formulation.

2 Homogeneous-linear coordinate transformations and changes of basis

Remark 1 Fig 2.a) casts a vector $\boldsymbol{v}$ in grey, with Subfig b) providing a linear transformation $\boldsymbol{M}$ acting upon this.

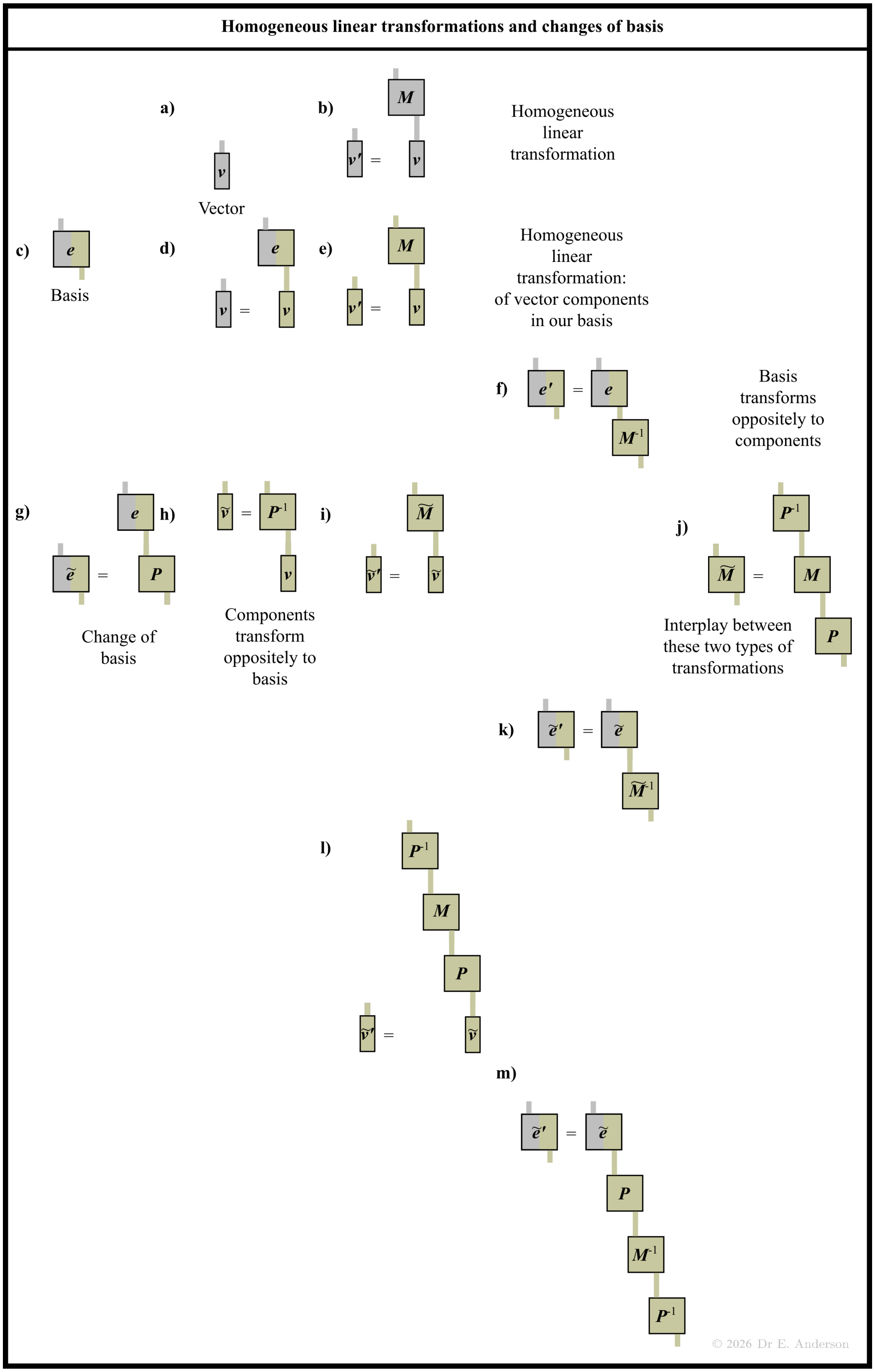
Remark 2 Subfig c) introduces a linear basis (LB) [1, 10, 48, 13, 22] of vectors, with vectorial content in grey now accompanied by basis-label content in beige. Subfig d) expresses our vector in terms of our basis, recasting it in beige. We subsequently require the beige counterpart of our linear transformation: Subfig e).

Remark 3 Bases transform oppositely, as per Subfig f). Let us distinguish a change of basis matrix $\boldsymbol{P}$ to the tilded basis from the plain basis (Subfig g). Oppositeness now returns Subfig h)'s transformation.

Remark 4 Posit the tilded-basis version of the transformation matrix in Subfig i) Then straightforward consistency fixes this to take the form in Subfig j). Subfig k) is the tilded-basis version of Subfig f). Subfig l) substitutes Subfig j)'s relation in Subfig i). Finally Subfig m) substitutes Subfig j)'s relation in Subfig k).

Pointer 1 Extending to inhomogeneous-linear transformations would in fact be entertaining not Linear Algebra but Affine Algebra.[2] We leave this for another occasion.

[2]See Sample Chapter 10 of [22] for an outline in index notation, or [32, 28] for more extensive accounts; a tensor networks version shall eventually appear here [17]...



Homogeneous linear transformations and changes of basis
a)
b)
Homogeneous
linear
transformation
Vector
c)
Basis
d)
e)
Homogeneous
linear
transformation:
of vector components
in our basis
f)
Basis
transforms
oppositely to
components
g)
h)
i)
j)
Change of
basis
Components
transform
oppositely to
basis
Interplay between
these two types of
transformations
k)
l)
m)
© 2026 Dr E. Anderson

Figure 2:

3 Inner-product spaces

3.1 Inner products and norms

Remark 1 These represent equipping our vector space with further layers of structure [1, 14, 48, 11]. Building up inner product spaces and normed spaces respectively. This enables the further basic and widely applicable notions of normalization and orthogonality, and their composite: orthonormality. This also has valuable applications in Functional Analysis [35, 24, 43].

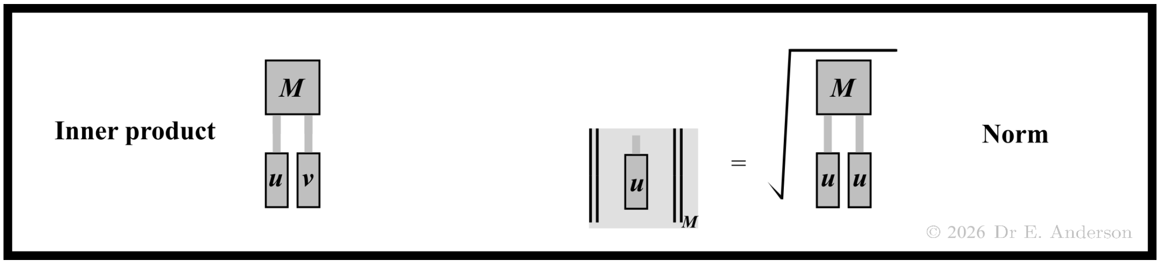


Figure 3:

Notational Remark 6 The LHS denotes a scalar function of a vector. By placing a vectorial box – with one upstairs leg – within a larger paler scalar box: with no legs. In treatments using just the one norm, the $\boldsymbol{M}$ subscript would often be dropped. And likewise when $\boldsymbol{M} = \mathbb{I}$: the Euclidean norm, which is a basic and very widely occurring.

3.2 Orthonormalization

Remark 1 Suppose that we are given any LB of vectors $\boldsymbol{v}_i$ for a finite-d vector space $\mathfrak{V}$. Then it is well-known that [13, 14, 48] an orthonormal (*ONLB*) can be constructed.[3] By picking a first input vector and normalizing it to $\hat{\boldsymbol{v}}$. Next projecting this out from a second input vector, and then normalizing the ensuing vector. And so on, via the following generalization. Upon picking the jth input vector $\boldsymbol{v}_j$, we project out all previously-picked input vectors: $\boldsymbol{v}_1$ to $\boldsymbol{v}_{j-1}$. Since we have only a finite number of input vectors, this process terminates when we have picked all of the input vectors.

We exhibit this procedure in tensor network notation in Fig 4. Including showing that all stages therein correspond to projecting and then normalizing. Just with the first projector being trivial, since there is not yet any previously-picked input vector to project out...

[3]This is often referred to as *Gram–Schmidt orthonormalization*.

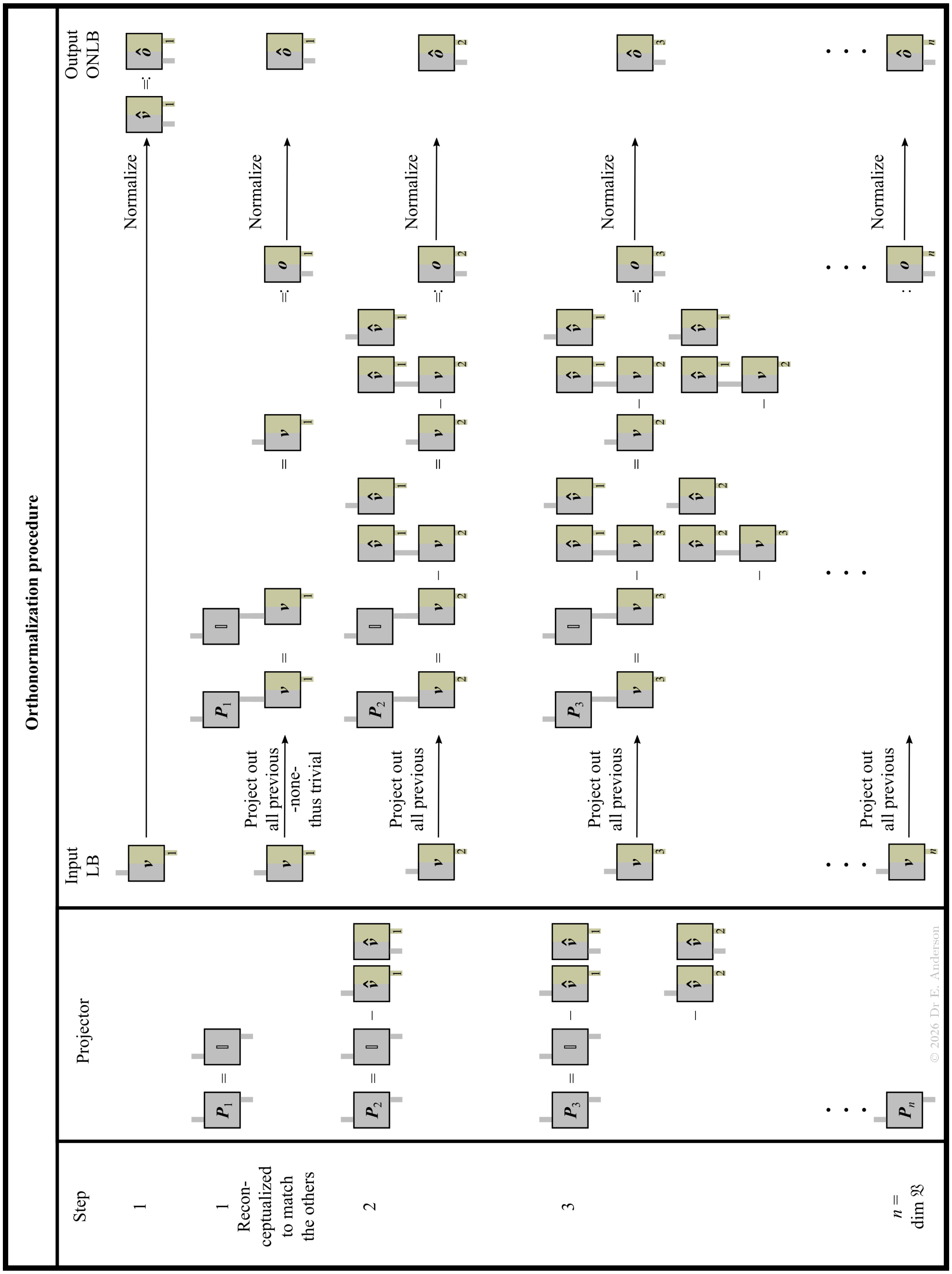

Orthonormalization procedure
Step
Projector
Input LB
Output ONLB
1
1 Recon-ceptualized to match the others
2
3
n = dim 𝔙
Normalize
Project out all previous -none- thus trivial
Project out all previous
© 2026 Dr E. Anderson

Figure 4:

3.3 Change of ONLB

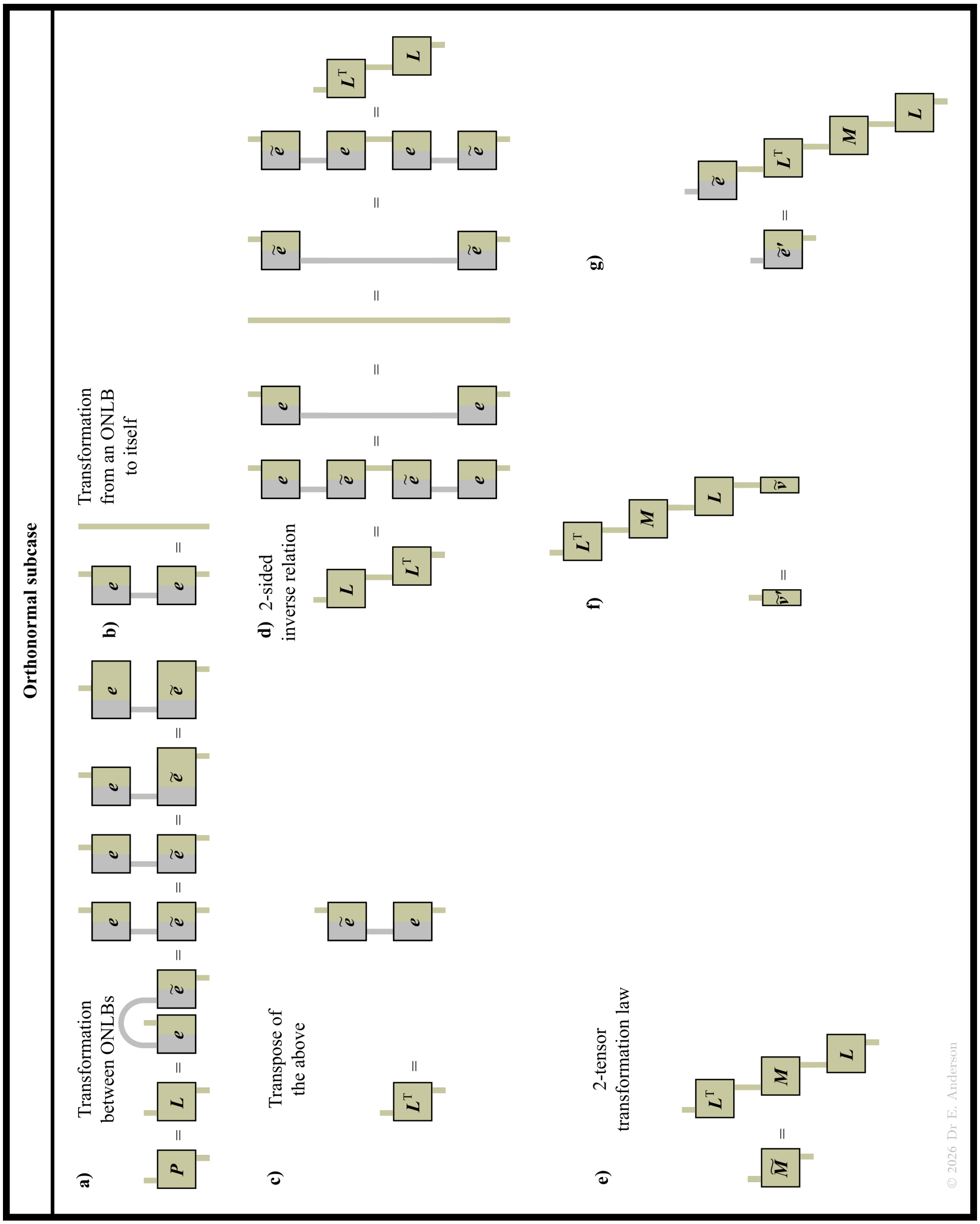


Figure 5:

Remark 1 In this context, the LB transformation matrix $\boldsymbol{P}$ specializes to the $\boldsymbol{L}$ detailed in Fig 5.a). Where the first form uses the Euclidean metric, and the last three, the Kronecker delta. With the last two reinstating the order of the indices that the second form dropped. The first at the cost of using boxes of distinct lengths to encode arrays of the same rank. While the second homogenizes this, at the expense of using more space, and departing from the convention of depicting 2-arrays as squares.

Remark 2 Further specializing to transforming from an ONLB to itself returns the identity. In Kronecker delta form: rank $(1, 1)$. This is depicted in Subfig b), and the form taken by the transpose in Subfig c). Subfig d) spells out how this transpose serves as a 2-sided inverse for $\boldsymbol{L}$.

Remark 3 The transformation matrix $\boldsymbol{M}$ now obeys the *Cartesian* 2-*tensor transformation law* [2, 5, 12], as per Subfig e).

Remark 4 Finally Subfigs f) and g) are the corresponding specializations of Fig 2.l) and m) respectively.

3.4 Dual bases

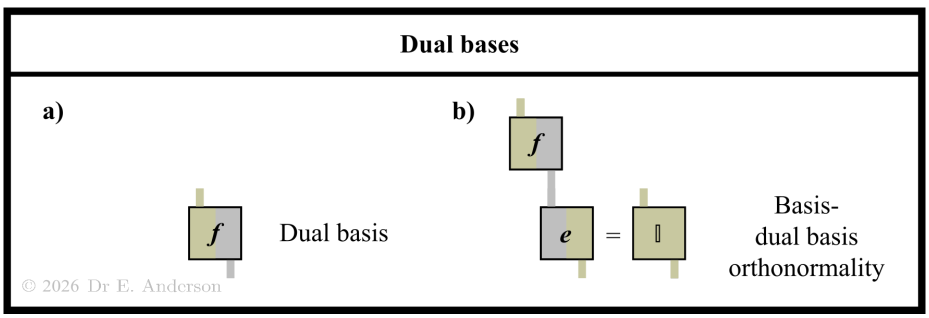


Figure 6:

Remark 1 Suppose that we are given an LB for vectors. Then a corresponding one for covectors

$$\{ f_i \}_{i=0}^{n}$$

is provided by the relation in Fig 6 [1, 13, 14, 48].

Exercise 1 Formulate the interplay between change of cobasis and homogeneous-linear transformations in terms of tensor networks.

3.5 (Self-)adjoint linear operators

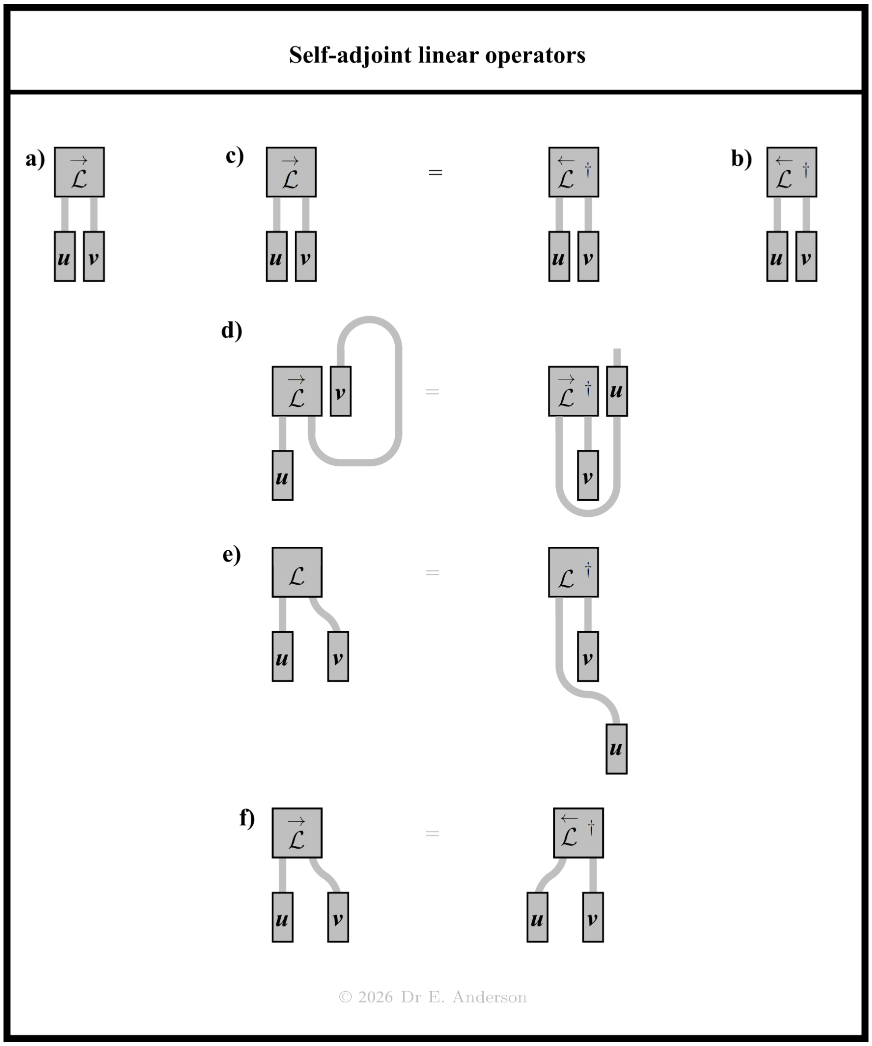


Figure 7:

Remark 1 Let $\mathcal{L}$ denote a *linear operator* and $\mathcal{L}^{\dagger}$ denote its *adjoint*. The arrow in Fig 7.a) signifies that $\mathcal{L}$ acts on $\boldsymbol{v}$: forwards. While that in Subfig b) signifies that $\mathcal{L}^{\dagger}$ acts on $\boldsymbol{u}$: backwards. For a *self-adjoint operator*, the outcome of these two actions coincides (Subfig c).

In the real finite case,

$$\mathcal{L} \;=\; \boldsymbol{M} \quad \text{and} \quad \mathcal{L}^{\dagger} \;=\; \boldsymbol{M}^{\mathrm{T}} \; .$$

And forwards or backwards action is immaterial.

Pointer 1 Suppose that one has a *linear differential operator* [4, 22]. And one adheres to the convention that derivatives act to the right. In this setting, $\boldsymbol{u}$, $\boldsymbol{v}$ might be infinite vectors, functions, or both at once. Then a departure from pure verticality is forced in formulating self-adjointness. With Subfig d) providing a particular minimalistic presentation.

Subfig f) follows suit as regards how one of us depicted div in [19]. Proceeding reflection-symmetrically at the cost of retaining forwards and backwards arrows. While Subfig e) dispenses with arrows at the cost of losing reflection symmetry.

4 Multilinear forms

4.1 From bilinear to multilinear

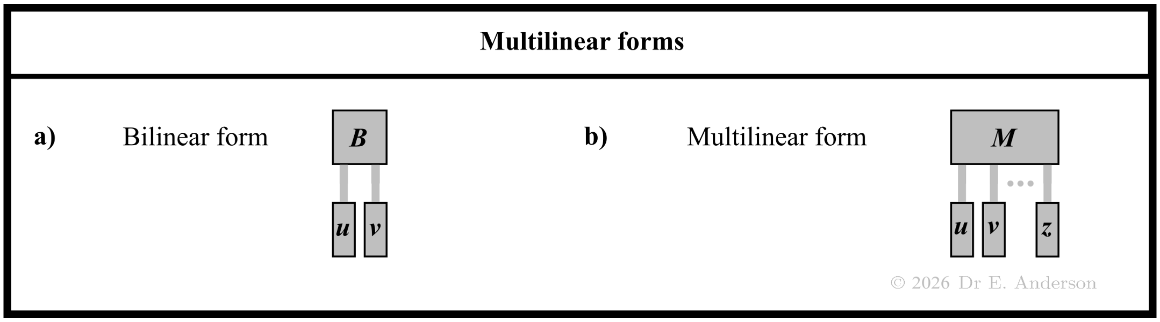


Figure 8:

Structure 1 *Bilinear forms* [1, 13, 48] are a somewhat distinct alternative in abstraction to inner products (Fig 8.a). Now readily generalizing from 2- to n-slot machines for producing scalars out of collections of 2 to n vectors. I.e. the *multilinear forms* [1, 9] that we depict in tensor network notation in Subfig b).

4.2 The alternator and its application to determinants

Structure 2 *Alternating forms* [1] (Fig 9.a) are the totally-antisymmetric subcase of multilinear forms. Rendering them forms in a a second sense [6, 30], concerning the restriction of downstairs tensors to the totally-antisymmetric subcase. Here the multilinear machine is the downstairs version of the *alternator*, alias *alternating tensor Levi–Civita tensor* and ϵ-*tensor*. That we already presented the 3- d version of in [18]

Structure 3 In terms of the alternator, the *determinant* of a matrix takes the form in Fig 9.b).[4] Where $\boldsymbol{a}_1$ to $\boldsymbol{a}_n$ are the columns of the underlying matrix. E.g. [13] makes use of this formulation in 3-d .

Proposition 1 Let $\boldsymbol{A}$ be a square matrix of size n .
a.i) det is linear in each input.

ii)
$$\det A \;=\; \det A^{\mathrm{T}} \; .$$

b)
$$\det \boldsymbol{A} \;=\; 0$$

if any of the below conditions hold.

i) 2 columns are the same.

ii) A column is a linear combination of other columns.

iii) The columns are linearly dependent.

iv) The row counterparts of whichever of i) to iii).

[4]Determinants also admit [48, 10, 13, 22] matrix, longhand-Algebraic and permutation-group formulations.

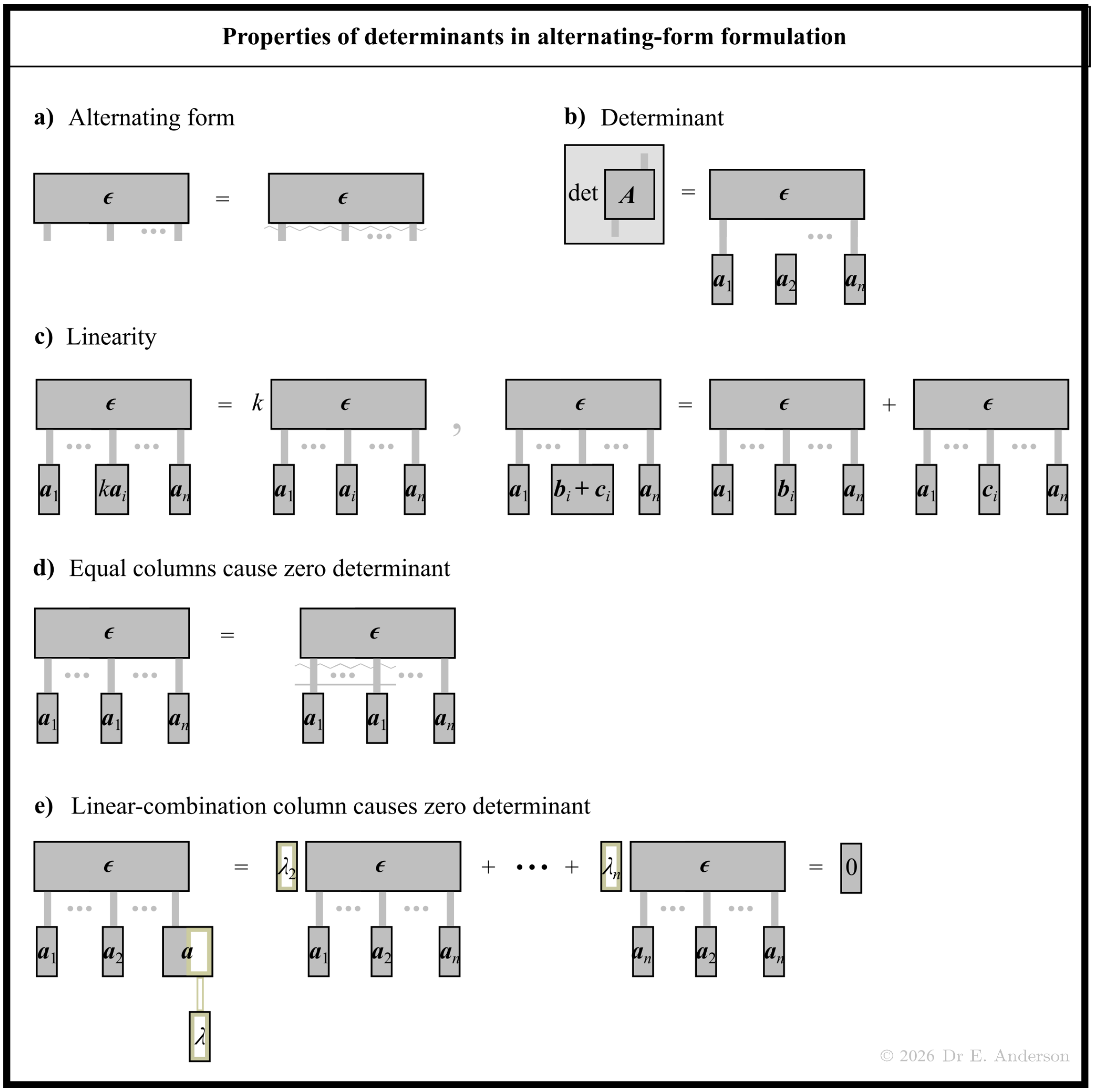


Figure 9:

<u>Proof</u> a.i) is manifest from its multilinear formulation. ii) These 2 expressions' tensor networks are the same shape. b.i) This follows by symmetry–antisymmetry. ii) This ensues from a.i) and b.i). iv) Row results follow from column counterparts (or vice versa) by a.ii). □

Notational Remark 7 See Subfigs c)–e) for tensor networks rendition of the proofs of a.i) and b.i)-ii). We here use a candy-pattern on a beige background to encode a linearly-independent set index. This indicates some relation to the corresponding plain beige: a linearly-independent set that furthermore manages to span. The Conclusion further justifies the longer-term need to use candy-patterning as well as colouring. Finally, it is for such use of candy-patterning that we draw our tensor networks with rather thick legs.

Exercise 2$^-$ Prove v), providing also a tensor networks diagram for it.

Proposition 2 det is

i) binarily productive:

$$\det(\boldsymbol{A}\,\boldsymbol{B}) \;=\; \det \boldsymbol{A} \det \boldsymbol{B}\ .$$

ii) Finitely productive:

$$\det\left(\prod_{i\,=\,1}^{k} \boldsymbol{A}_i\right) \;=\; \prod_{i\,=\,1}^{k} \det \boldsymbol{A}_i\ .$$

<u>Proof</u> in 3-d subcase. i) Firstly make 2 uses of the alternator formulation of determinant. Secondly evoke the full Kronecker delta theorem in 3-d [18]. Next use multilinearity, the unit property, and the definition of matrix multiplication, Finish off with the alternator formulation of determinant once more. See Fig 10 for how these moves are strung together. $\square$

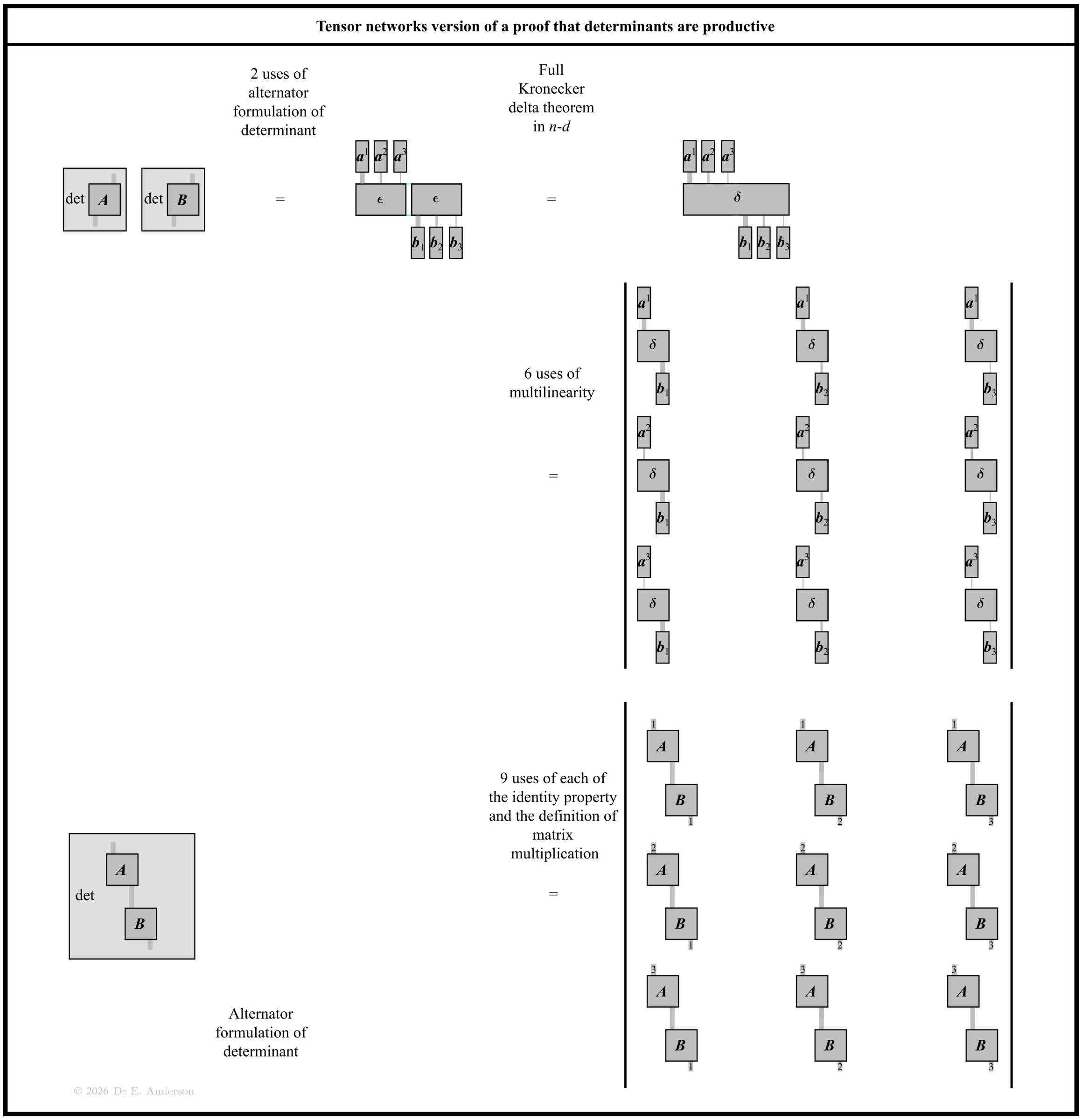


Figure 10:

Exercise 3$^-$ Prove ii), providing also a tensor networks diagram for it.

4.3 Interlude: the general-n alternator and Kronecker delta theorem

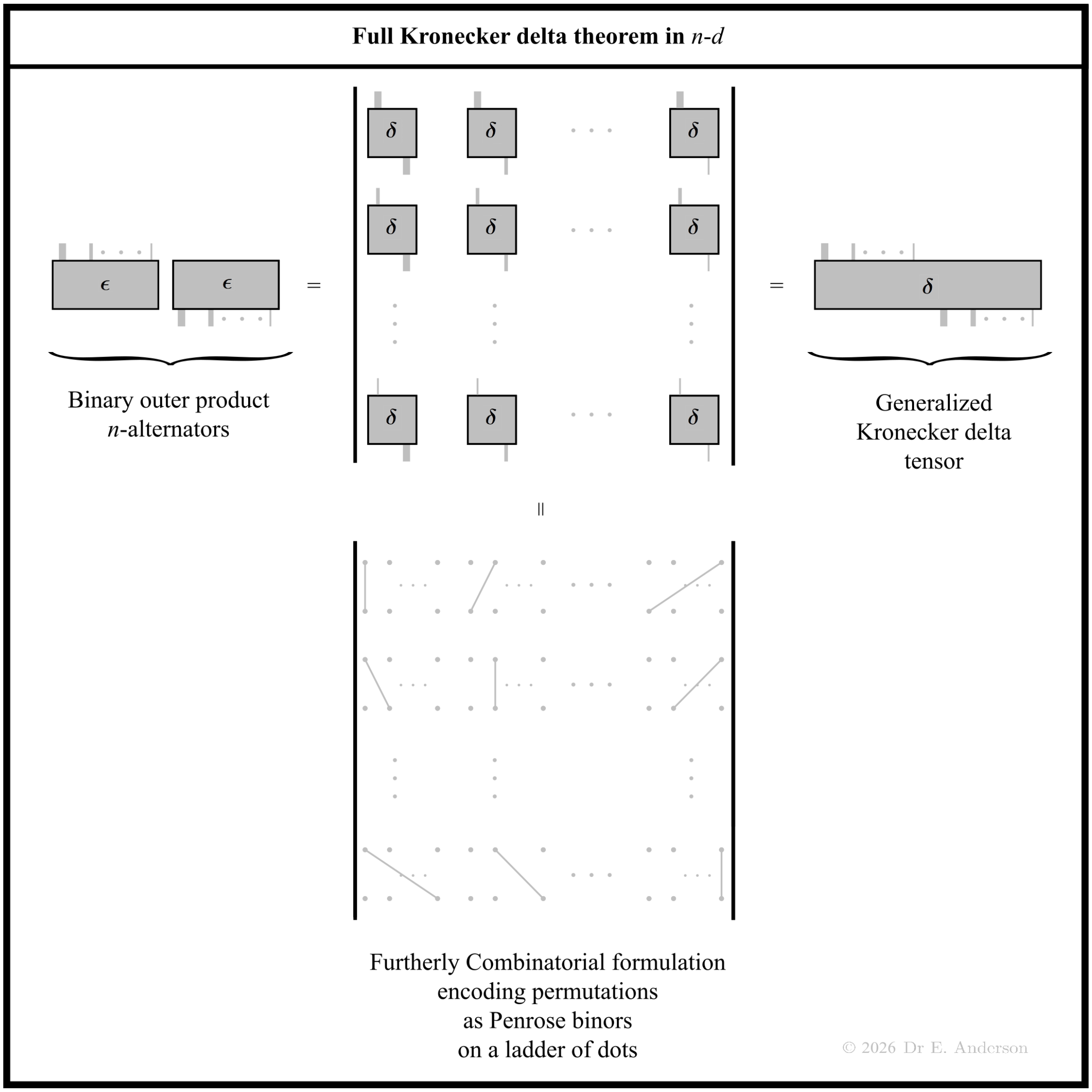


Figure 11:

Remark 1 On the one hand, Vector Algebra [18] is often considered just in 3- or 2-d. For Physics and Geometry use, as rooted in Euclidean space and the Euclidean plane.

On the other hand, basic Matrix Algebra often considers other, or arbitrary, dimension. So we require the general-n alternator to handle the determinants arising in Matrix Algebra.

And so here is the definition of the general-n flat-space alternator [8].

Definition 1 In components,

$$\epsilon_{a_1\, a_2\, \dots\, a_n} \quad := \quad \begin{cases} +1 & a_1\, a_2 \dots a_n \quad \text{an even permutation of} \quad 1\, 2 \dots n \\ -1 & a_1\, a_2 \dots a_n \quad \text{an odd permutation of} \quad 1\, 2 \dots n \\ \ 0 & \qquad\qquad\qquad \text{otherwise} \end{cases} \quad .$$

Structure 4 The full Kronecker delta theorem is now as per Fig 11: a tensor networks rendition of e.g. [8]. Prompting the definition of what Lovelock and Rund [8] refer to as the *generalized Kronecker delta tensor* at the end of this Figure.

Exercise 4 Work out the suite of contractions supported by the full Kronecker delta theorem in arbitrary-d . Present your answer in tensor networks form.

4.4 And back to applying alternators to determinats

Exercise 5 Provide a tensor networks formulation for Proposition 1 and 2's properties of determinants a) for 2-d matrices and b) for n-d matrices. In each case, include an extension of the above productivity proof.

5 Conclusion

5.1 Summary

Remark 1 We have recast Linear Algebra in vertical tensor network form. This serves to present the tensor network notational system in a simple setting. That is considerably more familiar to students than any of the collection of more advanced settings in which tensor networks are a more established notation.

Remark 2 As highlights, we documented presentational ambiguities for orthonormal linear bases and in an aside on (self)-adjoint linear differential operators. We also expanded [18]'s treatment of alternators and the Kronecker delta theorem to matrices of arbitrary size for application to determinants. And then casting basic properties of determinants in the tensor networks language, including proving these properties internally to this formulation.

Smaller Pointers A-B We however leave Linear Algebra's eigentheory, and matrix decompositions, for further occasions.

5.2 Larger Pointers

Specialized Pointers a-f Here are some more advanced settings in which tensor networks have been adopted [34, 37, 44, 50]. Spinor Calculus, Knot, Link and Braid Theory, anyons or Statistical Mechanics' Yang–Baxter equation.

Larger Pointers 1-8 While for instance the following subject matters would benefit from more use of tensor networks. 1) Differential Geometry [49, 52, 25] and 2) General Relativity [33, 30, 31]. The current Encyclopaedia [17] shall set these up over the next decade.

3) Lie Theory [47, 29, 27], for which Cvitanovič [45] pioneered tensor networks under the name of birdtracks. One of us has built a $p = 26$-colour palette to more comprehensively cover Lie Theory [51, 17]. Parts of Lie Theory are in turn usefully applicable to 1), and 2). To 4) The Principles of Dynamics [51] via such as Poisson algebras [39] and Dirac's treatment of constrained dyanmical systems [26]. And to 5) Flat Geometry, 6) the N-body problem 7) Differential Equations [38] and 8) Invariant Theory [40, 41].

Notational Remark 8 So far, Lie Theory and applications require around $p = 60$ species of index. This is in excess of the Introduction's considerations of palettes, while the choice of $p = 26$ is acceptable.

Here are some reasons for this proliferation of conceptual classes of index. Reference to multiple Lie subalgebras within a single system of equations may be required. While Dirac's constrained dynamical systems benefiting from immediately visually discerning between primary first-class, secondary first-class...

This gives the longer-term context due to which we adopt candy-patterning. In the current Article, with just 3 types of index in play, this could have been avoided by sole use of colour-coding. But the above Lie-Theoretic context and its applications over-stretch the practical applicability of a colour-coding palette. This furthermore permits say all the Lie algebra generator sets to be immediately recognizable by their blue backgrounds. Or the candidate constraint sets by their red backgrounds, which are promoted to fireopal backgrounds once confirmed to be consistent.

So all in all, we put forward a *rainbow-candy vertical variant of tensor networks.*

Acknowledgements We thank S. Sánchez and A. Ford for co-developing 'figure factories' for fast accurate technical drawing. And the Applied Combinatorics and Topology Discussion Group.

References

[1] P. Halmos, *Finite-Dimensional Vector Spaces*, (Springer-Verlag, N.Y. 1942).

[2] H. Jeffreys and Bertha Swirles Jeffreys, *Mathematical Physics* (C.U.P., Cambridge, 1956).

[3] M.R. Spiegel, *Vector Analysis (And An Introduction to Tensor Analysis)* (McGraw-Hill, N.Y. 1959).

[4] C. Lanczos, *Linear Differential Operators* (Van Nostrand, London 1961; Dover, Mineola NY 1997).

[5] R.C. Wrede, *Introduction to Vector and Tensor Analysis* (Self-published 1963; Dover, Mineola N.Y. 1972).

[6] H. Flanders, *Differential Forms. With Applications to the Physical Sciences* (General Publishing Company, Toronto, Canada 1963; Dover, Mineola N.Y. 1989).

[7] R. Penrose "Angular Momentum: an approach to Combinatorial Spacetime," in *Quantum Theory and Beyond* ed. T. Bastin (C.U.P., Cambridge 1971); reprinted *Collected Works Vol 2. Works from 1968-1972* (O.U.P., N.Y. 2011).

[8] D. Lovelock and H. Rund, *Tensors, Differential Forms and Variational Principles* (Wiley, New York 1975; Dover, Mineola NY 1989).

[9] S. Lang, *Algebra* (Springer, 2002).

[10] A.F. Beardon *Algebra and Geometry* (C.U.P., Cambridge 2005).

[11] H. Dym, *Linear Algebra in Action* (A.M.S, Providence RI 2006).

[12] D.C. Kay, *Tensor Calculus* (McGraw–Hill, N.Y. 2011).

[13] T.W. Körner, *Vectors, Pure and Applied: A General Introduction to Linear Algebra* (C.U.P., Cambridge 2012).

[14] S. Axler, *Linear Algebra Done Right* 3rd ed. (Springer, Cham, Switzerland 2015).

[15] M. Abadi et al, "TensorFlow: a System for Large-Scale Machine Learning", arxiv:1605:08965.

[16] C. Roberts et al, "TensorNetwork: A Library for Physics and Machine Learning", arxiv:1905:01330.

[17] *Online Encyclopaedia of Tensors and other Multi-Linear Arrays*, institute-theory-stem.org/online-encyclopaedia-of-tensors-and-other-multi-linear-arrays/ .

[18] E. Anderson, "Tensor Networks throughout the Basic Theory of STEM. I. Vector Algebra", Online Encyclopaedia of Tensors and other Multi-Linear Arrays, institute-theory-stem.org/oetoma-tensor-networks-in-vector-algebra/ (2026).

[19] "II. Vector Calculus", institute-theory-stem.org/oetoma-tensor-networks-in-vector-calculus/

[20] E. Anderson and K. Everard, "Eigentheory of $\mathbb{R}$-Symmetric Matrices from a Graph and Order Theory point of view, for [17].

[21] E. Anderson, *The Structure of Flat Geometry*, forthcoming (2026); see its Homepage conceptsof-shape.space/geometry/ for news and related articles.

[22] *Linear Mathematics*, forthcoming (2028); see its Homepage conceptsofshape.space/linear-mathematics/ for news and further samples.

Pointer references

[23] L.P. Eisenhart, *Riemannian Geometry* (P.U.P., Princeton 1926; 1949).

[24] N. Dunford and J.T. Schwarz, *Linear Operators. Part I. General Theory*, (1958; Wiley, 1988).

[25] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* Vol. 1 and 2 (1963; 1969; Wiley, New York 1996).

[26] P.A.M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York 1964).

[27] J.-P. Serre, *Lie Algebras and Lie Groups* (Benjamin, New York 1965).

[28] E. Snapper and R.J. Troyer, *Metric Affine Geometry*, (Springer, 1971; Dover, Mineola, N.Y. 1989).

[29] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications.* (Wiley, New York 1974; Dover, Mineola N.Y. 2002).

[30] C.W. Misner, K. Thorne and J.A Wheeler, *Gravitation* (Freedman, San Francisco 1973).

[31] S.W. Hawking and G.F.R. Ellis, *The Large-Scale Structure of Space-Time* (C.U.P, Cambridge 1973).

[32] G.E. Martin, *Transformation Geometry: An Introduction to Symmetry*, (Springer, 1982; 1996).

[33] R.M. Wald, *General Relativity* (U. Chicago P., Chicago 1984).

[34] R. Penrose and W. Rindler, *Spinors and Space-time: Volume 1, Two-spinor Calculus and Relativistic Fields* (C.U.P., Cambridge 1984).

[35] N. Young, *An introduction to Hilbert Space* (C.U.P., Cambridge 1988).

[36] T.W. Körner, *Fourier Analysis* (C.U.P., Cambridge 1988).

[37] L.H. Kauffman, *Knots and Physics* 2nd ed. (World Scientific, Singapore 1993).

[38] P.J. Olver, *Applications of Lie Groups to Differential Equations*, (Springer, 1993).

[39] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds* (Springer, Basel 1994).

[40] P.J. Olver, *Equivalence, Invariants, and Symmetry*, (C.U.P., Cambridge 1995).

[41] *Classical Invariant Theory*, (C.U.P., Cambridge 1999).

[42] B. O'Neill, *Elementary Differential Geometry* 2nd ed. (Elsevier, Amsterdam 2002; 1st ed. dates back to 1966).

[43] J.B. Conway, *A Course in Functional Analysis* (Springer, 2007).

[44] C. Nayak, S.H. Simon, A. Stern, M. Freedman and S. Das Sarma, "Non-Abelian Anyons and Topological Quantum Computation", Rev. Mod. Phys. **80** 1083 (2008), arXiv:0707.1889.

[45] P. Cvitanović, *Group Theory, Birdtracks, Lie's, and Exceptional Groups* (P.U.P., Princeton NJ 2008).

[46] A.N. Pressley, *Elementary Differential Geometry* (Springer, London 2010).

[47] J. Stillwell, *Naive Lie Theory* (Springer, N.Y. 2010).

[48] S. Lipschutz and M. Lipson, *Linear Algebra* 6th ed. (McGraw-Hill, New York 2018).

[49] J.M. Lee, *Introduction to Riemannian Manifolds* (Springer, New York 2018).

[50] *Encyclopedia of Knot Theory* ed. C. Adams, Erica Flapan, Allison Henrich, L.H. Kauffman, L.D. Ludwig and S. Nelson (CRC Press, Boca Raton FL 2021).

[51] E. Anderson, "Lie Theory Suffices to Resolve the Problem of Time", in *Time and Science* **3** ed. P. Harris and R. Lestienne (World Scientific, Singapore 2023).

[52] J.M. Lee, *Introduction to Complex Manifolds* (A.M.S., Providence RI 2024).