The Gauss–Bonnet Theorem

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Chapter 1

Gauss–Bonnet in Spherical Geometry

1.1 Spherical Geometry preliminaries

1.1.1 Great circles

Definition 1 Let Π^{O} be a plane that passes through the centre O of the unit sphere $\mathbb{S}^2$. Then a *great circle* [53, 56, 54, 61] (Fig 1.1.a) is the curve traced out by

$$\mathbb{S}^2 \quad \cap \quad \Pi^{\mathrm{O}} \ . \tag{1.1}$$

Naming Remark 1 This formulation dates back to Ancient Greece [47, 48, 49], under the name *greatest circle*. Which incorporates that no circle larger than this fits on a sphere.

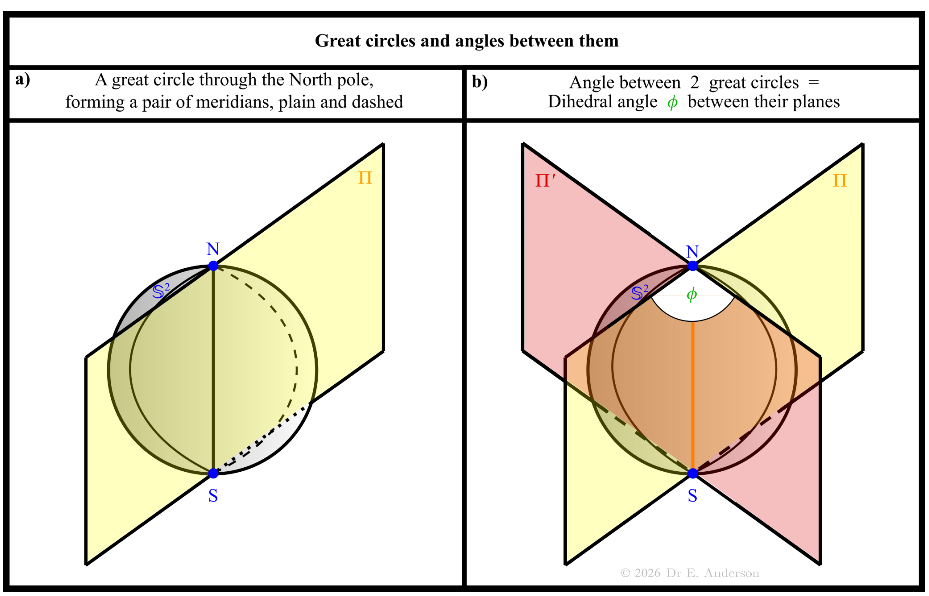

Figure 1.1:

1.1.2 Dihedral angle

Structure 1 Next take a pair of planes Π_i , $i = 1, 2$ in $\mathbb{R}^3$. Then straightforwardly the *dihedral angle* between these is given by the following.

$$\cos \phi = - \underline{\boldsymbol{n}}_1 \cdot \underline{\boldsymbol{n}}_2 . \tag{1.2}$$

For outward-pointing normals $\boldsymbol{n}_i$ to the planes Π_i . Where 'outward-pointing' is relative to the sector of angle ϕ under consideration.

Suppose furthermore that

$$\Pi_i = \Pi_i^{\mathrm{O}} .$$

I.e. that both of our planes pass through the sphere's centre O . Then the preceding paragraph can furthermore be interpreted as giving the angle between the 2 corresponding great circles,

$$\mathbb{S}^2 \cap \Pi_i^{\mathrm{O}} . \tag{1.3}$$

See Subfig b).

1.1.3 Lunes

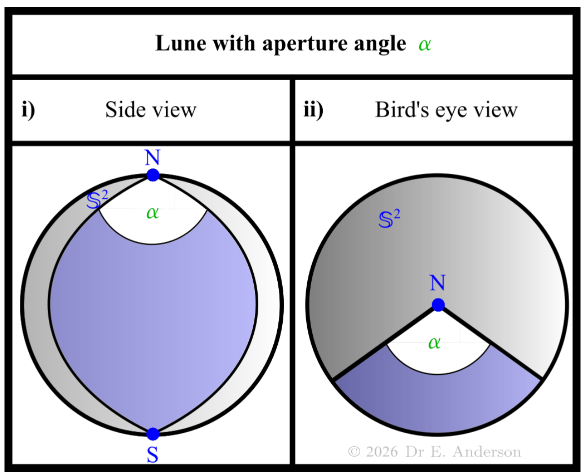

Figure 1.2:

Definition 1 A *lune* [53] alias *digon* [58] is the region between 2 distinct arcs of great circle.

Remark 1 These arcs necessarily meet at 2 antipodal points. W.l.o.g. place these at the the North and South poles, N and S . So the bounding half-great circles are *meridians*.[1]

Notational Remark 1 Lunes are characterized by a single parameter, which we can take to be [53, 54, 58, 57, 62, 40]: *aperture angle* α . This is between each lune's bounding meridians: the great-circle incarnation of dihedral angle. And gives rise to the notation $\mathbb{L}_\alpha{}^2$. See Fig 1.2 for sideways and bird's eye views of a lune. The 2 in this notation is a dimensional tracker that remembers that this is a piece of $\mathbb{S}^2$ rather than the p-sphere $\mathbb{S}^p$. After all, notation for half-planes and hemispheres does often carry such a reminder...

Exercise 1$^-$ Strightforwardly work out the following. Including use of Lemma 1 to establish Lemma 2 and vice versa.

[1]This is a term borrowed from the $\mathbb{S}^2$ model of the surface of the Earth in Geography.

Lemma 1 (unit-sphere area)

$$A\left(\mathbb{S}^2\right) \;=\; 4\,\pi\;. \tag{1.4}$$

Lemma 2 (lune area).

$$A\left(\mathbb{L}^2{}_\alpha\right) \;=\; 2\,\alpha\;. \tag{1.5}$$

Remark 1 For the above to make sense as regards the Geometry subcase of Physical dimensions, we need to clarify that we are talking about the following dimensionless area quantity.

$$\frac{Area}{1^2} \;=\; \frac{Area}{r_{\mathbb{S}}{}^2}$$

for a unit sphere.

1.2 Area of a great-circle triangle on the unit sphere

1.2.1 Internal-angle formulation

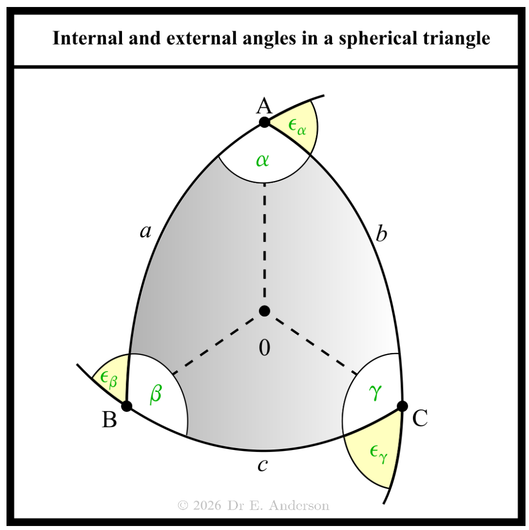

Figure 1.3:

Theorem S-3-I) For a great-circle triangle on the unit sphere, the area

$$A \;=\; I \;-\; I(F) \;=\; I \;-\; \pi \;=\; X\;. \tag{1.6}$$

Notational Remark 1 The first form here introduces the *internal angles* in a spherical triangle. Using Euler's 3-cycle notation α, β, γ for the angles in whatever Geometry's notion of triangle (Fig 1.3). From which we distill the following presentationally-useful adapted variable:

$$(\text{ interior angle sum })\;,\quad I \;:=\; \sum_{\text{3-cycles}} \alpha\;.$$

The first form is furthermore a relative-difference formulation: relative to any nondegenerate reference triangle $\triangle_{\mathrm{F}}$ in flat space F. Here any will do, since a basic result of Euclidean Geometry is that all of these share a fixed angle sum of ' 2 right angles' i.e. π radians in modern terms. In the third form, we introduce a spherical triangle's *spherical excess* (*angle sum*) 3 : relative to the case of a flat triangle... Which terminology is due to Legendre [51], and yet which notion goes back to Hariot [50].

Proof 1) Draw the 3 great circles whose arcs form our primary triangle $\triangle$. Which we take to be nondegenerate but elsewise arbitrary (Fig 1.4). These intersect pairwise at 3 further points not in $\triangle$. These further points are straightforwardly antipodal to $\triangle$'s 3 vertices.

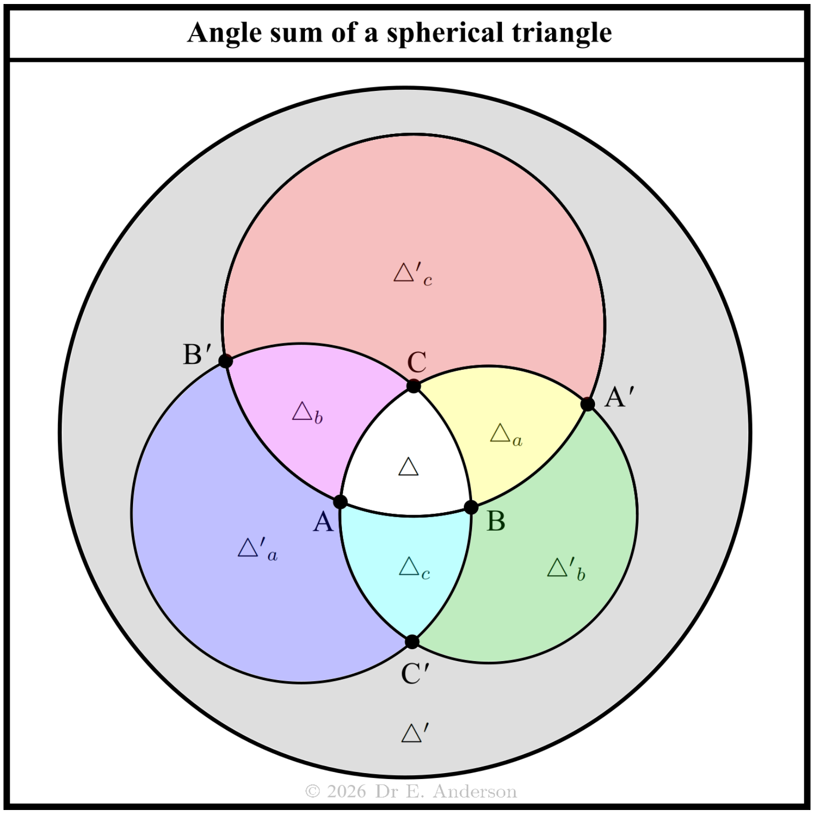

Figure 1.4:

In this way, we have constructed the antipodal triangle $\triangle'$ of our primary triangle. And 6 further triangles $\triangle_a$ and $\triangle'_a$. Where the index on the first of these 3-cycles of triangles indeed encodes which side they share with $\triangle$. And the second 3-cycle of triangles is antipodal to the first. So that their index indicates which side they share with $\triangle'$.

This said, consult Fig 1.4 once more to get a feel for our 8 spherical triangles. Observe in particular that this construct is extending our original triangle to an in-general irregular octahedron on our sphere... Let us celebrate by extending the index a to an index o that runs over our octahedron's antipodal face pairs. This takes one further value: blank!

2) The above construct can be viewed as applying the isometry on $\mathbb{S}^2$ that sends each point to its antipode. Which is represented by the matrix $-\mathbb{I}$ that sends $\triangle$ to $\triangle'$. And thus also the o-indexed version of this sending.

All 2-d isometries are area-preserving. Thus

$$A_{\mathrm{t}} = A'_{\mathrm{t}} \, . \tag{1.7}$$

Where t stands for (discrete) *total*. I.e. summed over o .

But also our 8 spherical triangles constitute a partition of $\mathbb{S}^2$. So

$$A_{\mathrm{t}} \; + \; A'_{\mathrm{t}} \;\; = \;\; 4\,\pi \; . \tag{1.8}$$

So we have a pair of simultaneous linear equations (1.7, 1.8). These are solved by

$$A_{\mathrm{t}} \; = \; 2\,\pi \; = \; A'_{\mathrm{t}} \; . \tag{1.9}$$

3) Additionally, each

$$\triangle \;\; \cup \;\; \triangle_a$$

is a lune with aperture angle α_a . Thus applying the lune area lemma,

$$A \; + \; A_a \; = \; 2\,\alpha_a \; .$$

So summing over 3-cycles,

$$3\,A \; + \; S \; = \; I \; . \tag{1.10}$$

Where

$$S \; := \sum_{\text{3-cycles}} A_a \; .$$

And where the first term just sums a constant over the 3-cycle.

But also splitting the first equality of (1.9)'s o-index sum into 3-cycle and blank parts,

$$A \; + \; S \;\; = \;\; 2\,\pi \; . \tag{1.11}$$

So we have arrived at a further pair of simultaneous linear equations.

Between which we can readily eliminate S to solve for A :

$$2\,I \;\; = \;\; 3\,A \; + \; 2\,\pi \; - \; A \;\; = \;\; 2\,A \; + \; 2\,\pi \;\; \Rightarrow$$

$$A \; = \; I \; - \; \pi \; . \square \tag{1.12}$$

Remark 1 This proof is in effect a Geometrical trick to pass from areas of coordinate-adapted regions – lunes – to areas of an unadapted type of region: spherical triangles.

Remark 2 For the sake of completeness, the other half of our second pair of simultaneous equations' solution is

$$S \; = \; 3\,\pi \; - \; I \; .$$

1.2.2 External-angle formulation

Theorem S-3 For a great-circle triangle on a unit sphere,

$$A \;\; = \;\; 2\,\pi \; - \; E \; . \tag{1.13}$$

Where now

$$(\text{ exterior angle sum }) \; , \quad E \; := \sum_{\text{3-cycles}} \epsilon_\alpha \; .$$

Proof We can infinitesimally define internal and external angles. So Flat Geometry's supplementarity of angles along a line that is cut by another line applies. So

$$\alpha \; + \; \epsilon_\alpha \; = \; \pi \; . \tag{1.14}$$

Thus summing over 3-cycles,

$$I \; + \; E \;\; = \;\; 3\,\pi \; .$$

So

$$A \;\; = \;\; I \; - \; \pi \;\; = \;\; 3\,\pi \; - \; E \; - \; \pi \;\; = \;\; 2\,\pi \; - \; E \; . \; \square$$

1.2.3 Extension to a great-circle polygon

Theorem S-N-I For a great-circle polygon[2] P_N on a unit sphere,

$$A_N \; = \; I_N \; - \; T\,\pi \; . \tag{1.15}$$

Notational Remark 1 We use N subscripts for P_N versions of our notions. While using the shorthand notation for elementary triangles that

$$Z \; := \; Z_3 \; .$$

Notational Remark 2 Also our N-a-gon can be triangulated into great-circle triangles, to which theorem S.3-I) can be applied. By a trivial induction, our triangulation contains

$$T \; := \; N \; - \; 2$$

triangles. Providing an elegant Combinatorial interpretation for the further shifted adapted variable that features in this Theorem. For which T is a truer-notation.

Theorem S-N

$$A_N \; = \; 2\,\pi \; - \; E_N \; . \tag{1.16}$$

<u>Proof</u> Pick some triangulation. Then

$$Area_N \; = \; \sum_{t\,=\,1}^{T} (\, I \; - \; \pi \,) \; = \; I_N \; - \; T\,\pi \; . \tag{1.17}$$

Where the second step is by our triangulation not introducing any new portions of internal angle. So the sum of our T triangles' internal angles coincides with that of the original N-a-gon's. Alongside the t-sum of the constant π .

Next sum

$$\alpha \; + \; \epsilon_\alpha \; = \; \pi$$

over all N angles.

$$I_N \; + \; E_N \; = \; N\,\pi \; .$$

Apply this to theorem S-N-I to exchange internal for external angles. Finally use the cancellation that

$$N \; - \; T \; = \; N \; - \; (\, N \; - \; 2 \,) \; = \; 2$$

to obtain theorem S-N. □

Remark 1 Observe that the external-angle version more cleanly generalizes its triangle counterpart. Here the coefficient remains 2 rather than requiring an adapted variable that varies with N .

[2] All polygons in the current Review have a finite number of vertices.

1.3 Corollaries

Corollary S-3-I For a spherical geodesic triangle,

$$I \geq \pi \; .$$

With equality only if it is degenerate, in the sense of a zero-area spherical triangle.

Remark 1 Contrast with flat space triangles, for which

$$I = \pi \; .$$

This is in fact an early indication that spheres possess everywhere-positive curvature.

Remark 2 For completeness, we include the two following less-used results.

Corollary S-3 For a spherical geodesic triangle,

$$E \leq 2\pi \; .$$

Corollary S-N For a spherical geodesic polygon,

$$E_N \leq n\pi \; .$$

Notational Remark 3 Here we evoke the common [64] 'relative' shifted adapted variable (indexing LI relative separations)

$$n := N - 1 \; .$$

Corollary S-N-I For a spherical geodesic polygon,

$$I_N \geq T\pi \; .$$

1.4 Some further motivation

1.4.1 Above we sneaked in three little versions of Gauss–Bonnet...

Remark 1 Theorems S-3-I, S-3 and S-N are already little versions of the Gauss–Bonnet theorem.

Notational Remark 2 The truer-nomenclature in use here is S for sphere, 3 for triangle and N for N-a-gon. With external-angle formulations as a default, so finally I stands for the internal-angle exception.

1.4.2 From great circles to geodesics

Structure 1 The Gauss–Bonnet theorem [42, 39, 45, 31, 24, 134, 34, 41] is more generally for a not-necessarily-geodesic polygonal region of a surface. Various components are needed to form a bridge from the above spherical results to this. A first component for this bridge is that a more modern point of view a sphere's great circles are indeed its geodesics, as established by the Calculus of Variations.

Exercise 2 Find the geodesics of $\mathbb{S}^2$ using the Calculus of Variations. Show that the formula thus obtained indeed coincides with the great-circle description.

1.4.3 Reappraising the lune area lemma

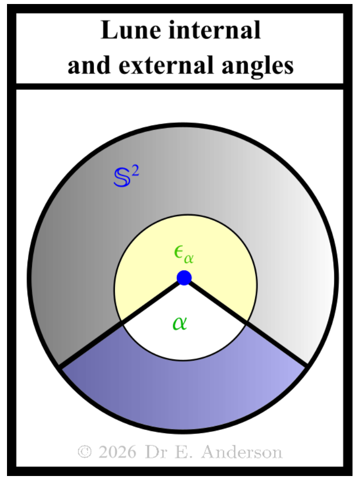

Figure 1.5:

Remark 1 Actually, we sneaked in a fourth little version of Gauss–Bonnet. For the lune area lemma can be re-envisaged as **theorem S.2.I**. It furthermore enjoys the following further reformulation.

Theorem S.2 For a unit-circle lune, with reference to Fig 1.5,

$$A\left(\mathbb{L}_\alpha{}^2\right) \;\;=\;\; 2\,(\,\pi \;-\; \epsilon_\alpha\,)\;\;.$$

Exercise 3$^-$ Establish this, and also write both theorem S.2.I and S.2 in 2-cycle form.

1.4.4 Reappraising the sphere area lemma

Structure 2 There is however also a Gauss–Bonnet theorem for the whole of a compact-without-boundary surface. And the whole sphere is itself an example of a compact-without-boundary surface. To celebrate, we accord the notation **theorem S.CWB** to lemma 1 (sphere area).

Remark 1 For all that we have not yet explained why an area formula for chunks of sphere has anything to do with the Gauss–Bonnet theorem... Conventional formulations of which inter-relate the following.

1) The *total Gaussian curvature* of our chunk: a Geometrical, and more generally Analytical expression.

2) The Euler characteristic of our chunk: a Combinatorial and more generally Topological expression.

Remark 1 This reveals 1) and 2) as two further components for the above-mentioned bridge, which we unpack in Chapters 2 and 3 respectively.

Pointer 1 This separately reveals that the Gauss–Bonnet theorem is of *index theorem type* [123]. I.e. a theorem unexpectedly and fruitfully managing to equate a priori unrelated Analytic and Topological objects.

Chapter 2

Extrinsic curvature

2.1 Of curves in $\mathbb{R}^2$

2.1.1 Curves

Remark 1 For now, we are working in $\mathbb{R}^n$, so Fig 2.1's conceptualization suffices.

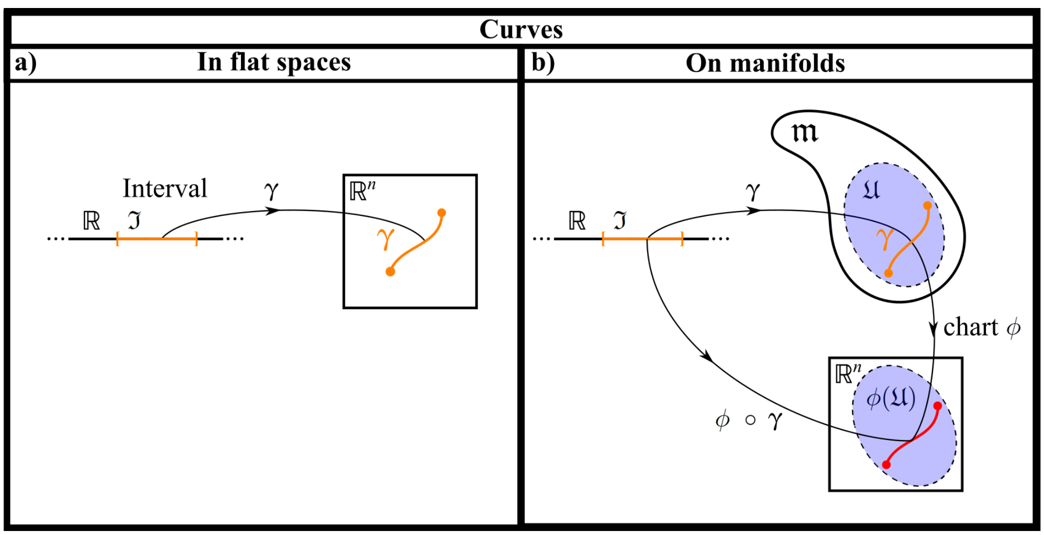

Figure 2.1:

Remark 1 Curvature is a departure from flatness. For curves, the flat objects to compare against are straight lines.

2.1.2 A first tangent formulation

Approach 1 Our first specific conceptualization sources the following seven-step procedure.[1]

1) We entertain the following more specific concept. The *deviation* of our path-length parametrized curve from the corresponding tangent line at some particular fixed value of our parameter, s_0. Observe that this is a pointwise notion: well capable of having different values at distinct points along our curve.

2) By [42] Taylor's theorem,

[1]This can be viewed as an expanded version of the account in [42].

$$\boldsymbol{\gamma}(\,s_0\,+\,\Delta s\,)\ \ =\ \ \boldsymbol{\gamma}(\,s_0\,)\ +\ \dot{\boldsymbol{\gamma}}\,\Delta s\ +\ \frac{1}{2!}\,\ddot{\boldsymbol{\gamma}}(\Delta s)^2\ . \tag{2.1}$$

Where our derivative terms are evaluated at s_0 . And where our displaced parameter value is small enough that we can neglect $\geq$ 3rd-order terms.

3) We next form the *discrepancy*

$$\Delta\boldsymbol{\sigma}\ :=\ \boldsymbol{\sigma}(\,s\,+\,\Delta s\,)\ -\ \boldsymbol{\sigma}(\,s\,)\ . \tag{2.2}$$

By virtue of being a difference, this is a relative quantity.

4) Next dot it with the unit normal vector $\boldsymbol{N}$ at s_0 .

[This indeed makes sense at the displaced value as well. This is due to our working in $\mathbb{R}^2$, in which vectors can simply be displaced between base points.]

Thus forming the '*normal discrepancy*',

$$\underline{\boldsymbol{N}}\cdot\underline{\gamma}''\ .$$

5) Since the normal is perpendicular to the tangent plane, the first-order terms cancel. Thus only second-order terms are left,

$$\underline{\boldsymbol{N}}\cdot\Delta\underline{\gamma}\ \ =\ \ \frac{1}{2}\,\underline{\boldsymbol{N}}\cdot\underline{\gamma}''\,(\Delta s)^2\ . \tag{2.3}$$

6) Claim: Furthermore,

$$\underline{\boldsymbol{N}}\cdot\Delta\underline{\gamma}\ \ =\ \ \frac{1}{2}\,||\underline{\gamma}''||\,(\Delta s)^2\ . \tag{2.4}$$

For, firstly since $\boldsymbol{\gamma}$ is unit-speed,

$$0\ =\ 1'\ =\ \left(\,||\gamma||^2\,\right)''\ \ =\ \ 2\,\underline{\gamma}'\cdot\underline{\gamma}''$$

$$\Rightarrow\ \boldsymbol{\gamma}''\ \perp\ \boldsymbol{\gamma}'\ \Rightarrow\ \boldsymbol{\gamma}''\ \parallel\ \boldsymbol{N}\ \ \Rightarrow$$

$$\boldsymbol{\gamma}''\ \ =\ \ c\,\boldsymbol{N}\ . \tag{2.5}$$

For some, by convention positive, constant c . Thus

$$\underline{\boldsymbol{N}}\cdot\underline{\gamma}''\ \ =\ \ c\,||\boldsymbol{N}||^2\ \ =\ \ c\ \times\ 1\ =\ c\ .$$

Where the second step uses that $\boldsymbol{N}$ is a unit normal. But also (2.5) $\Rightarrow$

$$||\boldsymbol{\gamma}''||\ \ =\ \ ||\,c\,\boldsymbol{N}\,||\ \ =\ \ |c|\,||\boldsymbol{N}||\ \ =\ \ c\ \times\ 1\ =\ c\ . \tag{2.6}$$

Where the second step is by the scaling property of norms. And the third is by positivity and the unit-normal condition.

Equating (2.5 , 2.6)

$$\underline{\boldsymbol{N}}\cdot\underline{\gamma}''\ \ =\ \ ||\gamma''||\ .$$

Finally substitute this in (2.3) to obtain (2.4) .

7) We then take this norm to provide a positive-signed notion of curvature, according to

$$\kappa\ :=\ ||\gamma''||\ .$$

2.1.3 Extrinsic versus intrinsic curvature

Remark 1 In the current setting, in which our primary space is a curve – 1-d – the above is the only nontrivial type of curvature supported. Upon transcending to higher-dimensional primary spaces, the above notion of curvature's most immediate descendants are extrinsic notions of curvature. Now in a context in which a priori distinct notions of intrinsic curvature are also supported. The extrinsic point of view here means relative to some ambient space, at present $\mathbb{R}^2$. While the intrinsic point of view is internal to the primary space itself.

2.1.4 A second tangential approach

Remark 1 Since we assume that the Reader has done at least 2 years of university-level Mathematics, it is probable that you have previously derived most of the following explicit formulae. If any of them are unfamiliar, derive them as **Exercise 1**.

Approach 2 Firstly entertain Cartesian coordinates. Then

$$\kappa \;=\; \left| \frac{\mathrm{d}\psi}{\mathrm{d}s} \right| .$$

Where ψ is the angle between the tangent line at S and the x-axis.

Furthermore, using x as parameter, and $' := \frac{\mathrm{d}}{\mathrm{d}x}$,

$$\kappa \;=\; \frac{|\, y'' \,|}{(\, 1 \;+\; y'^{\,2} \,)^{\frac{3}{2}}} . \tag{2.7}$$

Finally switching to an arbitrary parameter ν with $\dot{} := \frac{\mathrm{d}}{\mathrm{d}\nu}$,

$$\kappa \;=\; \frac{|\, \dot{x}\,\ddot{y} \;-\; \dot{y}\,\ddot{x} \,|}{(\, \dot{x}^2 \;+\; \dot{y}^2 \,)^{\frac{3}{2}}} .$$

Finally, transforming to plane polar coordinates,

$$\kappa \;=\; \frac{\left|\, r^2 \;+\; 2\, r_\theta{}^2 \;-\; r\, r_{\theta\theta} \,\right|}{(\, r^2 \;+\; r_\theta{}^2 \,)^{\frac{3}{2}}} .$$

Observe that all of these expressions involve second derivatives. Furthermore, at a stationary point, (2.7) collapses to a double derivative. Somewhat more generally, a second derivative serves as an approximation to this notion of curvature. While in the previous Subsec's formulation, the magnitude of a second derivative is an exact expression for this notion of curvature...

2.1.5 Tangent and normal as a start on setting up an orthonormal frame

Approach 3 $\dot{\gamma}(\mathrm{S})$ can furthermore be interpreted as the *unit tangent vector* at S ,

$$\boldsymbol{T}(\mathrm{S}) .$$

Consequently

$$\dot{\boldsymbol{T}}(\mathrm{S}) \;=\; \ddot{\gamma}(\mathrm{S}) .$$

$$\kappa(\mathrm{S}) \;=\; ||\dot{\boldsymbol{T}}|| .$$

And (2.5) becomes

$$\boldsymbol{T}' \;=\; \kappa\, \boldsymbol{N} . \tag{2.8}$$

2.1.6 The magnitude-of-acceleration interpretation

Remark 1 We now consider contexts in which $\boldsymbol{\gamma}$ tracks position or displacement. While the parameter s is interpreted as a timefunction t .

Remark 2 Then the tangent vector is a *velocity vector*,

$$\boldsymbol{v} \ := \ \dot{\boldsymbol{x}} \ .$$

Where now $\dot{} \ := \ \frac{\mathrm{d}}{\mathrm{d}t}$.

Its magnitude is then a *speed scalar*,

$$v \ := \ ||\boldsymbol{v}|| \ = \ ||\dot{\boldsymbol{x}}|| \ .$$

Remark 3 So

$$1 \ = \ ||\boldsymbol{T}|| \ = \ ||\dot{\boldsymbol{x}}|| \ = \ ||\boldsymbol{v}| \ =: \ v$$

receives the extra interpretation of a *unit-speed* condition.

$$\boldsymbol{a} \ = \ \dot{\boldsymbol{v}} \ = \ \ddot{\boldsymbol{x}}$$

is an acceleration. Finally, curvature is here magnitude of acceleration.

$$\kappa \ = \ ||\ddot{\boldsymbol{x}}|| \ = \ ||\boldsymbol{a}|| \ =: \ a \ .$$

This is of course a specialization of the above observations concerning second derivatives.

2.1.7 Radius- and centre-of-curvature

Structure 2 The *radius of curvature* (*roC*) is the reciprocal of the curvature,

$$roC(\mathrm{P}) \ = \ \rho(\mathrm{P}) \ = \ \kappa^{-1}(\mathrm{P}) \ . \tag{2.9}$$

This can be interpreted as the radius of a tangent circle at P (Fig 2.2.a). If our incipient curve is itself a circle (Subfig b), then it is its own tangent circle,

$$\mathrm{roC} \ = \ r_{\mathbb{S}} \ .$$

Structure 3 This circle's centre is called the *centre of curvature* (*coC*). It is at

$$\boldsymbol{\gamma}(s) \ + \ \rho(s)\,\boldsymbol{N}(s) \ = \ \boldsymbol{\gamma}(s) \ + \ \rho^2(s)\,\boldsymbol{T}'(s)\gamma(s) \ + \ \rho^2(s)\,\boldsymbol{\gamma}''(s) \tag{2.10}$$

Remark 1 If we let our circle be arbitrarily large, a line ensues, which sports zero curvature: no deviation form a line (Subfig c).

Remark 2 As one moves along a curve, in general the roC changes. So the coC traces out a nontrivial locus (Subfig d) of some interest.

Remark 3 The circle of curvature is more than just a tangent, in that it matches the curve's first and second derivatives. This is the kissing property, by which the circle of curvature is a kissing circle alias osculating circle.

Exercise 2$^-$ If our curve is a circle, in which ways are the current and previous subsection's structures somewhat less extrinsic?

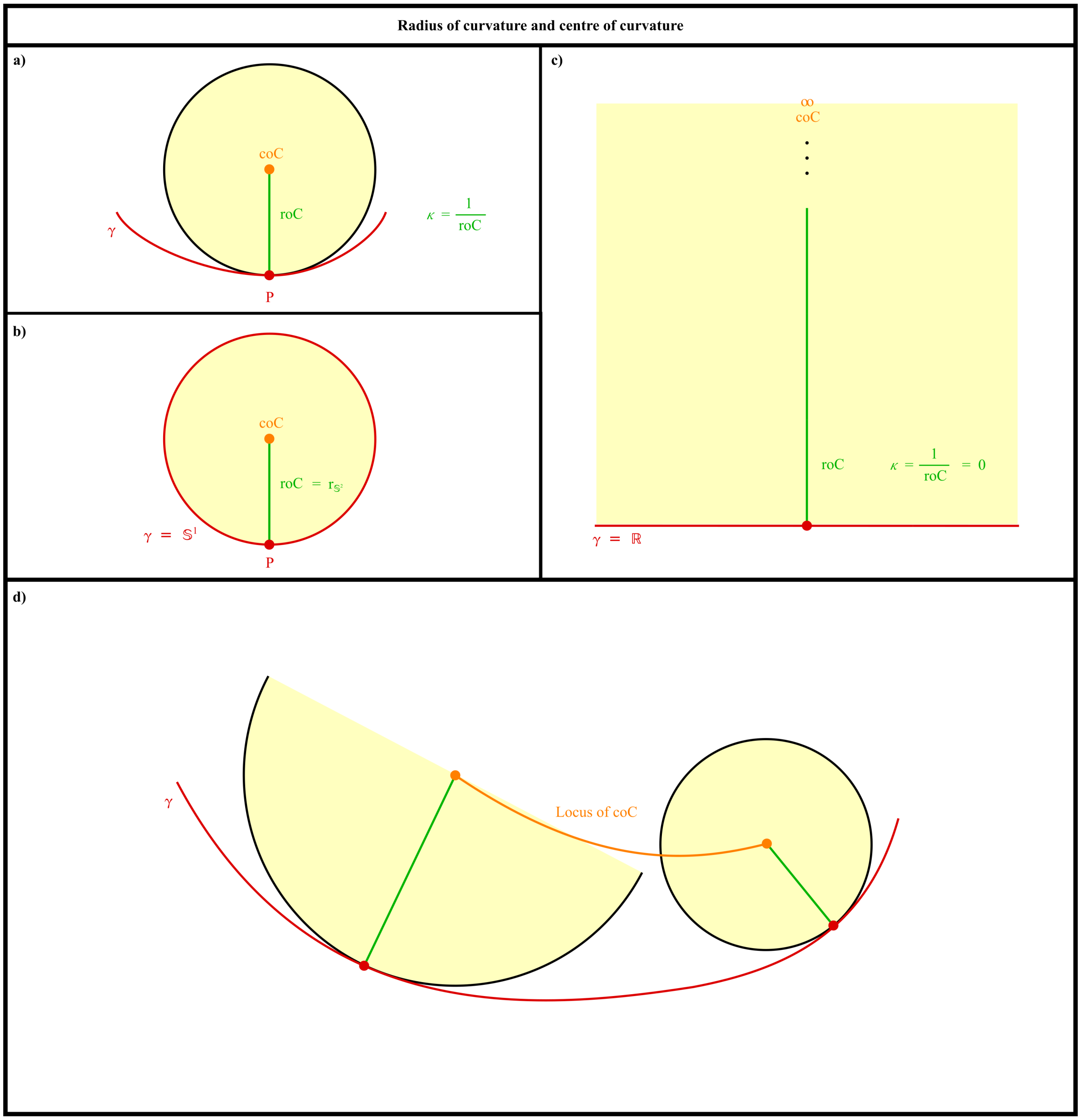

Figure 2.2:

2.2 Of curves in $\mathbb{R}^3$

2.2.1 The binormal

Remark 1 In 3-d , the sequence of kissing planes picked out by $\boldsymbol{T}$ and $\boldsymbol{N}$ in general wobbles from side to side as one progresses along our curve. We encode this via the unit normal to the kissing plane. Since we already have a normal to the curve, we need a new name and notation for this.

Structure 1 The *binormal* is

$$\underline{\boldsymbol{B}}(s) := \underline{\boldsymbol{T}}(s) \times \underline{\boldsymbol{N}}(s) . \tag{2.11}$$

Where via the cross product, we have smuggled in a right-hand convention for our frame.

2.2.2 The TBN, alias Frenet, frame

Structure 2 All three of $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$ are unit vectors and mutually pependicular. Thus these form an orthonormal linear basis (ONLB) at each point along our curve. Thus also cycling

$$\underline{\boldsymbol{T}}(s) \ := \ \underline{\boldsymbol{N}}(s) \ \times \ \underline{\boldsymbol{B}}(s) \ . \tag{2.12}$$

$$\underline{\boldsymbol{N}}(s) \ := \ \underline{\boldsymbol{B}}(s) \ \times \ \underline{\boldsymbol{T}}(s) \ . \tag{2.13}$$

This is called a *TBN frame* alias *Frenet frame* (see [26, 45] for introductions [42, 39] for further details).

2.2.3 Twisting, or pitching is measured by (extrinsic) torsion

Lemma 1

$$\boldsymbol{B}'(s) \ \propto \ \boldsymbol{N}(s) \ . \tag{2.14}$$

Exercise 3 Prove this.

Structure 1 Modulo a further sign convention, this constant of proportionality is the torsion τ, as follows.

$$\boldsymbol{B}'(s) \ = \ -\,\tau(s)\,\boldsymbol{N}(s) \ . \tag{2.15}$$

This torsion follows the above curvature's suit in being extrinsic. It provides a measure of departure from planarity by 'pitching', 'twisting' or 'turning' [85] out of the plane.

2.2.4 The Serret–Frenet equations

Lemma 2

$$\boldsymbol{N}'(s) \ = \ \tau\,\boldsymbol{B}(s) \ - \ \kappa\,\boldsymbol{T}(s) \ . \tag{2.16}$$

<u>Proof</u>
Differentiate (2.13):

$$\underline{\boldsymbol{N}}'(s) \ = \ (\,\underline{\boldsymbol{B}}(s) \ \times \ \underline{\boldsymbol{T}}(s)\,)' \ = \ \underline{\boldsymbol{B}}'(s) \ \times \ \underline{\boldsymbol{T}}(s) \ + \ \underline{\boldsymbol{B}}(s) \ \times \ \underline{\boldsymbol{T}}'(s)$$

$$= \ -\,\tau(s)\,\underline{\boldsymbol{N}}(s) \ \times \ \underline{\boldsymbol{T}}(s) \ + \ \underline{\boldsymbol{B}}(s) \ \times \ \kappa(s)\,\underline{\boldsymbol{T}}(s) \ = \ \tau(s)\,\underline{\boldsymbol{T}}(s) \ \times \ \underline{\boldsymbol{N}}(s) \ - \ \kappa(s)\,\underline{\boldsymbol{T}}(s) \ \times \ \underline{\boldsymbol{N}}(s)$$

$$= \ \tau(s)\,\underline{\boldsymbol{B}}(s) \ - \ \kappa(s)\,\underline{\boldsymbol{T}}(s) \ .$$

Where the second step is by the product rule. The third step uses the TNB approach's definitions of κ and τ. The fourth step makes 2 uses of antisymmetry. And the fifth step uses (2.11, 2.12) . □

Structure 1 We now collect (2.8, 2.16, 2.15) into the following *Serret–Frenet equations.*

$$\boldsymbol{T}' \ = \ \kappa\,\boldsymbol{N} \ , \tag{2.17}$$

$$\boldsymbol{N}' \ = \ -\,\kappa\,\boldsymbol{T} \ + \ \tau\,\boldsymbol{B} \ , \tag{2.18}$$

$$\boldsymbol{B}' \ = \ -\,\tau\,\boldsymbol{N} \ . \tag{2.19}$$

Remark 1 This is a first-order homogeneous-linear system.

And well-determined: 3 equations in 3 unknowns.

2.2.5 The matrix and dual-vector formulations

Structure 2 Since the above is homogeneous-linear, it can be repackaged into the following matrix form.

$$\overline{\overline{\boldsymbol{E}}}' = \underline{\overline{\boldsymbol{A}}} \cdot \overline{\overline{\boldsymbol{E}}} \, . \tag{2.20}$$

Where grey encodes spatiality to black encoding basis memebers. And

$$\boldsymbol{A} := \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix} . \tag{2.21}$$

Structure 3

$$\underline{\overline{\boldsymbol{A}}} = \underline{\underline{\overline{\boldsymbol{\epsilon}}}} \cdot \overline{\boldsymbol{\omega}} \tag{2.22}$$

further repackages 3-d 's curvature-and-torsion information into the following dual vector.

$$\boldsymbol{\omega} := \begin{pmatrix} \tau \\ 0 \\ \kappa \end{pmatrix} . \tag{2.23}$$

2.2.6 Double-constancy returns the helices

Exercise 4$^-$ Show that a curve in $\mathbb{R}^3$ for which κ, τ are both constant must take the following helix form.

$$\gamma(s) = \begin{pmatrix} a \cos \psi \\ \pm\, a \sin \psi \\ b\, \psi \end{pmatrix} . \tag{2.24}$$

Interpret this sign ambiguity. Also find where the circle and straight line are hiding special subcases.

2.2.7 Scalar triple product formulae for 3-d curvature and torsion

Exercise 5 Show that

$$\kappa = \frac{|\,\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''\,|}{||\boldsymbol{\gamma}'||^3} .$$

And

$$\tau = \frac{[\,\boldsymbol{\gamma}', \boldsymbol{\gamma}'', \boldsymbol{\gamma}'''\,]}{||\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''||^2} .$$

2.3 Of curves in $\mathbb{R}^{\leq 4}$

2.3.1 Jordan frames

Remark 1 Using $\{\, \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3 \,\}$ in place of $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$, we can readily extend to any finite size of ONLB.

Remark 2 With the 3-d specific cross product replaced by p-forms supported in arbitrary-d .

2.3.2 The Jordan equations

==

Structure 1 These start with the arbitrary-length version of (2.20) [39]. $\boldsymbol{A}$, which in the present context is called the *aspect matrix*, remains antisymmetric, and a band matrix. In the sense that nonzero entries solely occur in the first upper and lower parallels of the principal diagonal. Which are opposite-sign versions of each other. So for a curve in 4-d , curvature and torsion have one further comrade. Going up by 1 concomitant with each increase in d .

Exercise 6$^+$ Show that 4-d 's sole further comrade involves at most fourth-order derivatives. Also parallel the above 3-d working to derive the ' 4-d Jordan equation' analogue of the Serret–Frenet equation. Finally give the dual $\boldsymbol{A}$ explicitly in components.

Pointer 1 This point of view [74] extends to the formulation of *Cartan structure equations* [78, 39, 83, 87]. Which adds a further layer of differential equations for an object carrying one further derivative: a further formulation of intrinsic curvature. With for instance substantial Differential Geometry, GR, Lie Theory and Invariant Theory applications.

2.4 Of a surface by 'tangential routes'

2.4.1 Surfaces in $\mathbb{R}^3$ and beyond

Structure 2 A first notion of 2-surface (often just called surface), generalizing the plane within $\mathbb{R}^3$ is depicted in Fig 2.1.b). Where $\mathfrak{V}$ and $\phi(\mathfrak{V})$ are open sets. And we are placing a local 2-d coordinate grid on our local patch. The coordinate patch is local, and does not necessarily extend, even if the 2-surface does.

Structure 1 Let $\mathfrak{U} \subset \mathbb{R}^2$ be an open neighbourhood. Shrinking $\mathfrak{U}$ if necessary, we entertain a smooth parametrization

$$\phi \ : \ \mathfrak{U} \longrightarrow \boldsymbol{\mathfrak{S}} \ .$$

With parameters u, v serving as coordinates for the image of our map, $\phi(\mathfrak{U})$. Which serves as a *patch of surface* that is smoothly parametrized as given.

Remark 2 Covering the general surface requires multiple coordinate patches.

2.4.2 Extrinsic curvature from slicing a surface with a plane

Structure 3 Given a surface $\boldsymbol{\mathfrak{S}}$, we can pick a point s and then place a plane through it. Within this plane, the surface forms a curve. And so we can uplift Subsec 2.1.2's procedure.

We can furthermore apply this pointwise over the whole surface. And planewise through each point.

2.4.3 From deviation from the tangent plane

We here parallel SSec 2.1.2's approach to the curvature of a curve.

1) Once again we consider the deviation of our object from the corresponding tangent flat at a fixed parameter value. Though our object is now a surface. The fixed parameter value is now a pair of numbers (u_0, v_0) picking out the point p . And the tangent flat is a tangent plane, $\boldsymbol{\mathfrak{T}}_{\mathrm{p}}(\boldsymbol{\mathfrak{S}})$.

2) Once again evoke [42] Taylor's theorem. Albeit now in the 2- rather than just 1-variable case. Neglect $\geq$ 3rd-order terms once more.

This returns

$$\boldsymbol{\sigma}(u_0 + \Delta u \, , \ v_0 + \Delta v) \ = \ \boldsymbol{\sigma}(u_0, v_0) + \boldsymbol{\sigma}_u \Delta u + \boldsymbol{\sigma}_v \Delta_v + \frac{1}{2!}\left(\boldsymbol{\sigma}_{u\,u}(\Delta u)^2 + 2\,\boldsymbol{\sigma}_{u\,v} \Delta u \,\Delta_v + \boldsymbol{\sigma}_{v\,v}(\Delta v)^2 \right) . \tag{2.25}$$

Where all of the derivative terms are evaluated at (u_0, v_0) .

3) Form the discrepancy

$$\Delta\boldsymbol{\sigma} \ := \ \boldsymbol{\sigma}(\, u \ + \ \Delta u \ , \ \ v \ + \ \Delta v \,) \ - \ \boldsymbol{\sigma}(\, u, \, v \,) \ . \tag{2.26}$$

4) We next dot with the normal vector $\boldsymbol{n}$ at $(\, u, \, v \,)$.

5) Since the normal is perpendicular to the tangent plane, the first-order terms cancel. Thus only second-order terms are left.

6) Set

$$L \ := \ \boldsymbol{\sigma}_{u\,u} \cdot \boldsymbol{n} \ , \tag{2.27}$$

$$M \ := \ \boldsymbol{\sigma}_{u\,v} \cdot \boldsymbol{n} \ , \tag{2.28}$$

$$N \ := \ \boldsymbol{\sigma}_{v\,v} \cdot \boldsymbol{n} \ . \tag{2.29}$$

Then

$$\Delta\boldsymbol{\sigma} \ = \ \frac{1}{2} \ [\ L\,(\Delta u)^2 \ + \ 2\,M\,\Delta u\,\Delta v \ + \ N\,(\Delta v)^2 \ ] \ .$$

So in the infinitesimal limit,

$$II \ := \ L\,\mathrm{d}u^2 \ + \ 2\,M\,\mathrm{d}u\,\mathrm{d}v \ + \ N\,\mathrm{d}v^2 \ . \tag{2.30}$$

7) This is both the direct analogue of κ in Sec 2.1 And a structural parallel of the 2-d subcase of the *Geometric metric* alias *first fundamental form*.

Notational Remark 1 Indeed, II denotes *second fundamental form*. While this is an older notation and name for our curvature object, the name corresponds to the aspect in which we have arrived at the object... The notation is outdated by mismatch; $\mathrm{d}^2 II$ cures this. Let us pair it with $\mathrm{d}^2 I$ for the Geometrical metric line element $\mathrm{d}^2 s$. We shall be studying metrics in more detail in the next Chapter.

2.4.4 From eigentheory to principal curvatures

Remark 1 W.l.o.g. $\boldsymbol{k}$ is symmetric. For it arises contracted with 2 copies of $\mathrm{d}\boldsymbol{u}$. So if it had an antisymmetric piece, then this would in any case be annihilated by symmetry–antisymmetry.

Remark 2 Thus $\boldsymbol{k}$ has 2 real eigenvalues, corresponding to orthogonal eigenvectors. Let us denote these eigenvalues by

$$\kappa_1 \ \geq \ \kappa_2 \ .$$

These are termed the *principal curvatures*, corresponding to the principal directions along each's eigenvector.[2]

Remark 3 We can also arrive at these by applying the first Subsec 2.1.2's procedure at an arbitrary point for a LB of planes.

2.4.5 The corresponding invariants

Remark 1 2-d matrices have just 2 invariants: trace and determinant. For $\boldsymbol{k}$, these are

$$tr\,\boldsymbol{k} \ \ = \ \ \kappa_1 \ + \ \kappa_2 \ ,$$

$$det\,\boldsymbol{k} \ \ = \ \ \kappa_1\,\kappa_2 \ .$$

This follows from firstly invariants' independence from choice of LB of the underlying matrix. Secondly, from the convenient choice of the eigenbasis, in which the determinant takes a simplified form.

[2] Indeed, 'principal' is an alias for 'eigen'. Also at least these κ's and their descendants are denoted by Greek letters in the manner of eigenvalues.

2.4.6 Mean and Gaussian curvature

Remark 1 Geometers' traditional choice for a LB of invariants of $\boldsymbol{K}$ is in fact the following.[3]

$$(\text{Mean curvature}) \, , \quad H \; := \; \frac{\kappa_1 \, + \, \kappa_2}{2} \quad = \quad \frac{1}{2} \, {}_{\mathrm{r}} \boldsymbol{K} \; .$$

$$(\text{Gauss curvature}) \, , \quad K \; := \; \kappa_1 \, \kappa_2 \quad = \quad \det \boldsymbol{K} \; .$$

Notational Remark 2 Some truer notation for these is as follows.

$$(\text{Mean curvature}) \, , \quad h \; = \; \langle k \rangle \; = \; \frac{\kappa_1 \, + \, \kappa_2}{2} \quad = \quad \frac{1}{2} \, {}_{\mathrm{r}} \boldsymbol{K} \; . \tag{2.31}$$

And

$$(\text{Gauss curvature}) \, , \quad K_{\mathrm{G}} \quad = \quad \kappa_1 \, \kappa_2 \quad = \quad \det \boldsymbol{K} \; . \tag{2.32}$$

The first of these follows from reserving of capital letters for squared quantities: *weak Conway variables* [63, 64]. And then a standard notation for the mean. The second has a G subscript standing for 'Gauss' or 'Geometrically-significant', given Sec **??**'s development. This keeps it distinct from the extrinsic curvature tensor's family of quantities (unsubscripted k whose square is K).

Exercise 7$^-$ Determinant is not the only way of thinking about the Gaussian curvature. Show that for a 2-d surface,

$$tr \, \boldsymbol{K}^2 \; - \; (tr \, \boldsymbol{K} \,)^2 \; .$$

recovers $\det \boldsymbol{k}$ up to a constant of proportionality to be determined.

2.5 Of a surface by 'normal routes'

Remark 1 Many more advanced sources, in both the Geometry of Surfaces and Differential Geometry, consider extrinsic curvature from not a tangential but a normal point of view. As the rate of change of normals along curves or across surfaces...

To be fair, even the above tangential accounts make use of normals.

Motivation 1 Normals are perpendicular to tangents. So turning the normal is equivalent to turning the tangent space that it is perpendicular to. This is a duality in the sense of a dimension-1 object and a codimension-1 object. Or, at the level of forms, between a 1-form and a hypersurface form.

Motivation 2 A deeper motivation for conceptualizing in terms of normals comes from the following two Subsecs' maps.

2.5.1 The Gauss map

Structure 1 Working in the above coordinate patch, we now pick a normal vector at each point $\mathrm{S} \, \in \, \phi(\mathfrak{U})$ as follows.

$$N(\mathrm{s}) \; := \; \widehat{\phi_u \, \times \, \phi_v} \quad = \quad \frac{\phi_u \, \times \, \phi_v}{| \, \phi_u \, \times \, \phi_v \, |} \; . \tag{2.33}$$

This furnishes a smooth map

$$N \; : \; \phi(\mathfrak{U}) \; \longrightarrow \; \mathbb{R}^3 \; .$$

Structure 2 A *unit-normal vector field* on a surface is a map

$$N \; : \; \mathfrak{S} \; \longrightarrow \; \mathbb{S}^2 \; . \tag{2.34}$$

[3] This is a LB of invariants themselves. So this and both of the previous paragraphs all involve LBs on different spaces!

Which we take to be smooth. And perpendicular to the tangent plane at S . I.e. such that for each point $s \in \mathfrak{S}$,

$$N(\mathrm{s}) \perp \mathfrak{T}_{\mathrm{s}}\mathfrak{S} \ .$$

Not all $\mathfrak{S}$ globally admit a such. It is precisely the *orientable* $\mathfrak{S}$ that do. For setting up the above involves *picking an orientation*. Indeed, Gauss [2] introduced (2.34) so as to handle orientations: an immediately global notion which subsequently turned out to be Topologically contentful. (2.34) is consequently known as the *Gauss map*. We will explain orientability in further detail in Chapter 4.3.

2.5.2 The Weingarten map

Structure 1 The following further derivative map based on the above surface to sphere map is the *Weingarten map*.

$$\mathrm{d}n(\mathrm{s}) \ : \ \mathfrak{T}_{\mathrm{s}} \longrightarrow \mathfrak{T}_{n(\mathrm{s})}\,\mathbb{S}^2 \ .$$

This quantifies how the normal $\boldsymbol{n}$ varies over our surface $\mathfrak{S}$.
Remark 1 The Weingarten map provides a first coordinate-free formulation of the exttrinsic curvature.

2.5.3 The shape operator

Remark 2 Furthermore,

$$\mathfrak{T}_{\mathrm{s}}\mathfrak{S} \ \ \| \ \ \mathfrak{T}_{\boldsymbol{n}(\mathrm{s})}\mathbb{S}^2 \ .$$

Giving the simplified 'endomorphism' perspective

$$\mathrm{dn}_{\mathrm{s}} \ : \ \mathfrak{T}_{\mathrm{s}}\mathfrak{S} \longrightarrow \mathfrak{T}_{\mathrm{s}}\mathfrak{S} \ . \tag{2.35}$$

Naming Remark 1 Let us call this second coordinate-free formulation of extrinsic curvature the *shape operator*.

By its many and named aspects, the extrinsic curvature of a 2-d surface is a citizen of Kallista.

Notational Remark 3 S_{s} and L_{s} are common notations for the shape operator at point $\mathrm{s} \in \mathfrak{S}$.

Chapter 3

Some more global notions of curvature

Remark 1 In this brief Chapter, we consider a region $\mathfrak{R}$ in a surface $\mathfrak{S}$. This is of 'intermediate globality' between the whole surface and pointwise versions. Which is motivated by three of these objects featuring in our main Gauss–Bonnet theorem chapter.

3.1 The total Gaussian and mean curvatures

Definition 1 The *total Gaussian curvature* is the following integral thereover.

$$K_{\mathrm{TG}} := \iint_{\mathfrak{R}} K_{\mathrm{G}} \, \mathrm{d}S \; . \tag{3.1}$$

Structure 1 To complete the set of invariants at this intermediate level of globality, the *total mean curvature* is as follows.

$$h_{\mathrm{T}} \; := \; \iint_{\mathfrak{R}} h \, \mathrm{d}S \;\; = \;\; \frac{1}{2} \iint_{\mathfrak{R}} k \, \mathrm{d}S \;\; =: \;\; \frac{k_{\mathrm{T}}}{2} \; .$$

Where the last two terms switch to the trace of the extrinsic curvature and its total version.

3.2 Area to Gauss–Bonnet relation for $\mathbb{S}^2$

Remark 1 The dimensionless area (3.2) takes the following further forms.

$$\frac{Area}{1^2} \;\; = \;\; \frac{Area}{{r_{\mathbb{S}}}^2} \;\; = \;\; \frac{Area}{RoC} \;\; = \;\; K_{\mathrm{G}} \times Area \;\; = \;\; K_{\mathrm{TG}} \; . \tag{3.2}$$

Which is then

$$\iint_{\mathfrak{R}} K_{\mathrm{G}}, \mathrm{d}S$$

Because the sphere is of constant curvature, so we can pull K_{G} outside of the region's integral. Leaving just an area integral.

3.3 Intrinsic reformulation of the total Gaussian curvature

Definition 1 The *total Ricci scalar curvature* is

$$R_{\mathrm{T}} := \iint_{\mathfrak{R}} R \, \mathrm{d}S \tag{3.3}$$

Remark 1 By Gauss' outstanding theorem, this is related to the total Gaussian curvature by

$$R_{\mathrm{T}} \;=\; 2\, K_{\mathrm{T}} \ . \tag{3.4}$$

Inverting to,

$$G_{\mathrm{T}} \;=\; \frac{1}{2}\, K_{\mathrm{T}} \ . \tag{3.5}$$

3.4 The total geodesic and normal curvatures

Definition 1 The *total geodesic curvature* is

$$\kappa_{\mathrm{Tg}} \;:=\; \oint_{\partial\mathfrak{R}} \kappa_{\mathrm{g}} \, \mathrm{d}s \tag{3.6}$$

Remark 1 While the above is the last object that we need for the Gauss–Bonnet theorem, it makes good conceptual sense to complete the normal–geodesic split at this intermediate level of globality, as follows.

Definition 2 The *total normal curvature* is

$$\kappa_{\mathrm{Tn}} \;:=\; \oint_{\partial\mathfrak{R}} \kappa_{\mathrm{n}} \, \mathrm{d}s \ . \tag{3.7}$$

Pointer 7 See e.g. [89, 84, 79, 91, 90] for more about these last 3 Chapters' notions of curvature and yet further notions of curvature.

Chapter 4

The Euler characteristic

4.1 An invariant of polyhedra on the sphere

Proposition 1 Suppose that a sphere is cut up into F polygonal faces. Using a total of V vertices and E edges. Then

$$F - V + E = 2 . \tag{4.1}$$

Proof Denote our polygons by P_f , $f = 1$ to F . Each has $V_f = N_f$ vertices and consequently $E_f = N_f$ edges. And interior angle sum

$$I_f := I_{N_f} .$$

So by Theorem S-N-I,

$$A(P_f) = I_f - (N_f - 2)\pi .$$

Sum over f in our third equality, using F as subscript for sums over f .

$$4\pi = A\left(\mathbb{S}^2\right) = A_F = I_F - \pi N_F + 2\pi F$$

$$= (2\pi)V - \pi(2E) + 2\pi F = 2\pi(V - E + F) .$$

Where working backwards the second step uses the additive property of area under partition. And the first is by the sphere area lemma.

While working forwards, in the fourth step expand to obtain 3 summations. The first patches together the polygons to form the angles around V points, which add up to 2π about each. The second's 2 absorbs how our cut-up into polygons double-counts edges. And the third pulls out a constant and then just sums up 1's. The fifth step takes out a common factor of 2π . Finally cancel this out to obtain the result. □

Remark 1 This value of 2 is independent of the cut-up into polygons is used on our sphere. Each of which returns a spherical polyhedron. Which can furthermore be flattened into a polyhedron in $\mathbb{R}^3$.

4.2 Alternative Graph Theory proof

4.2.1 Statement

Sharper Proposition 2 [32] Let H be a polyhedron in $\mathbb{R}^3$ such that the following hold.

i) Any 2 of its vertices can be connected by a chain of edges.

ii) Any cycle in H consisting of straight-line segments (not necessarily edges) separating H into 2 pieces. Then

$$V \;-\; E \;+\; F \;=\; 2\;.$$

Remark 1 These caveats prepare us for Sec 4.3's Topological generalization. As regards outlining a second style of proof for $\mathbb{S}^2$, some basic Graph Theory preliminaries [93, 94] are as follows.

4.2.2 Supporting basic Graph Theory

Definition 1 A *graph* G [93, 94] consists of a set $\mathfrak{V}$ of *vertices* and a set $\mathfrak{E}$ of *edges*, each interlinking 2 vertices. A graph is *simple* if no edge runs from a vertex to itself (no loops). And each vertex pair supports ≤ 1 edge (no multi-edges). The *order* of a graph G is its number of vertices, $V := |\mathfrak{V}|$. Its *size* is its number of edges $E := |\mathfrak{E}|$. The *degree* alias *valency* of a vertex v is the number of edges $d(v)$ emanating from it.

Remark 1 For graphs, connected is equivalent to path-connected. A graph which is not connected is disconnected.

Definition 3 A *cycle* in a graph is a set of its edges. With each starting from the vertex at which the previous ended. While the last edge ends on the first vertex. A *tree* [93, 94] T is a cycle-free connected graph.

Lemma 1

$$V(\mathrm{T}) \;-\; E(\mathrm{T}) \;=\; 1\;.$$

<u>Proof</u> Base step. For the tree of order 1 – the point –

$$V \;-\; E \;=\; 1 \;-\; 0 \;=\; 0\;.$$

Induction hypothesis. Suppose that this holds for a tree of order k :

$$V(k) \;-\; E(k) \;=\; 1\;.$$

Then any tree of order $k + 1$ can be constructed as follows. By attaching a new point via a new edge somewhere on some tree of order k.

Induction step. But then

$$V(\,k+1\,) - E(\,k+1\,) \;=\; V(k) + 1 - (\,E(k) \;+\; 1\,) \;=\; V(k) - E(k) + 1 - 1 \;=\; 1 + 0 \;=\; 1\;.$$

Where the first step is by the end of the previous paragraph. The second reorders, and the third applies the induction hypothesis. So the claim is true $\forall\, k \,\in\, \mathbb{N}$ by induction.

Definition 4 Given a connected graph G, a *spanning tree* for it is any tree that contains all of G's vertices.

Lemma 2 Any connected graph admits ≥ 1 spanning tree S.

<u>Proof</u> We can systematically form a spanning tree by the following procedure. Keep on removing edges from G until no cycles are left. All the while prohibiting our sequence of graphs with less and less edges from becoming disconnected... □

4.2.3 The main proof

Proof i) evokes a subcase of Definition 1. Thus by Definition 4 we can pick some specific S.

Next form a dual graph $\widetilde{\mathrm{S}}$ of S in the following vertex-to-face sense. Assign a vertex $\widetilde{\boldsymbol{v}}$ to each face. And an edge $\widetilde{\boldsymbol{e}}$ between a pair of such vertices whenever their corresponding faces satisfy the following 2 conditions.

a) They are adjacent.

b) They intersect on an edge not in S.

Claim 1: $\widetilde{\mathrm{S}}$ is itself connected.

For if it were not, then it would be separated by a cycle in S. But S is a tree, so there is no such cycle.

Claim 2: $\widetilde{\mathrm{S}}$ is itself a tree.

For suppose that $\widetilde{\mathrm{S}}$ contained a cycle C. Then this would separate H by ii). This obstructs joining up T with edges. Contradiction! Because T is connected. Thus there is no cycle C. So $\widetilde{\mathrm{S}}$ meets the definition of a tree.

But then Lemma 1 applies to both S and $\widetilde{\mathrm{S}}$, giving step 2 of the following. So

$$2 \;=\; 1 \,+\, 1 \;=\; [\,V(\mathrm{S}) \,-\, E(\mathrm{S})\,] \,+\, \left[\,V\left(\widetilde{\mathrm{S}}\right) \,-\, E\left(\widetilde{\mathrm{S}}\right)\right]$$

$$=\; V(\mathrm{S}) \,-\, \left[\,E(\mathrm{S}) \,+\, E\left(\widetilde{\mathrm{S}}\right)\right] \,+\, V\left(\widetilde{\mathrm{S}}\right) \;=\; V \,-\, E \,+\, F\,.$$

Where step 3 rebrackets. While Step 4 uses both the role that H's faces play in $\widetilde{\mathrm{S}}$. And that G's edge-set has been partitioned into the following pieces. The edge-set of our choice of spanning tree S. And all other edges constituting the edge-set of a further face-dually defined tree $\widetilde{\mathrm{S}}$. □

Remark 1 How we established Claim 1 is dodgy. However, some fairly basic machinery of triangulations can abridge this gap [32]. Style of proof 1 in some ways goes back to Descartes [1]; specific use of spherical angles started with Legendre [52]. While style of proof 2 began with von Staudt [92].

4.3 Suitably nice polyhedral decompositions

4.3.1 Topology as an abstraction of continuity

Structure 1 2 Topological spaces are *homeomorphic* if there is a continuous map between them that also has a continuous inverse. This is Topology's most direct notion of equivalence, marking the subject as an abstraction of continuity.

4.3.2 Modelling Topological surfaces by suitably nice N-angulations

Structure 2 From now on, we entertain suitably nice polyhedral decompositions of connected Topological (2-)surfaces.

A) *Triangulations* have all-triangular faces.

B) *Quadrangulations* or *rectangular decompositions* have all-quadrilateral faces.

C) N-*angulations* have all-N-a-gon faces for a fixed N .

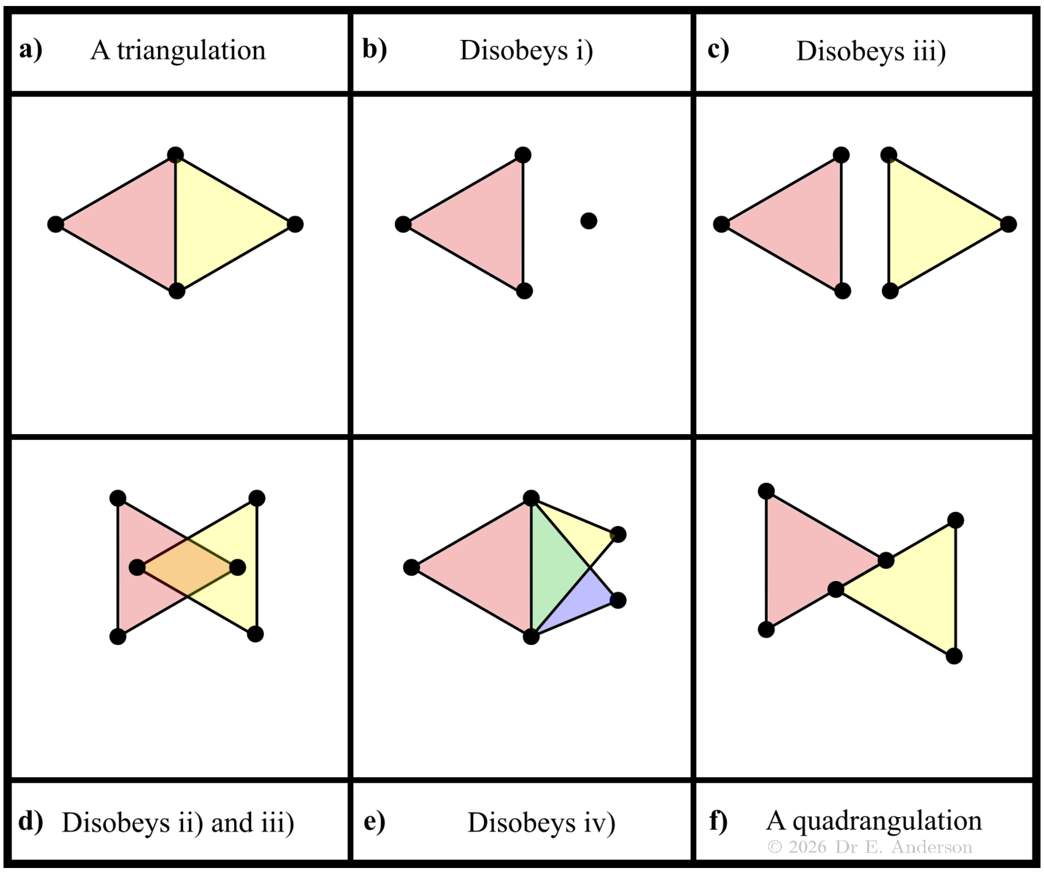

Figure 4.1:

Structure 3 We take 'suitably-nice' to mean the following[1]

i) Each point lies on ≥ 1 polygon.

[1]Though for some purposes, other conditions might be chosen. We may eventually include an Appendix to cover this source of variety, alongside A), B), C) extended's source of variety.

ii) Each pair of polygons' interiors are disjoint.

iii) Each pair of polygons meets on a common edge (endpoint vertices included) or on a common vertex.

iv) Each edge is an edge of precisely 2 polygons (outer face included).

Remark 1 See Fig 4.1 for an example of a triangulation, a quadrangulation, and various examples of not fitting nicely together.

4.3.3 Graph Theory's dual modelling arenas

Remark 1 Each of A), B), C) is an example of *face regularity*. I.e. the face-dual of Graph Theory's notion of *regularity for vertices*. This picks out graphs in which all the vertices have the same degree.

Remark 2 Under this correspondence, triangulations correspond to *cubic graphs*. I.e. the class [99] that is the focus of much of higher Graph Theory. Quadrangulations correspond to quartic graphs. And N-angulations to N-regular graphs.

Remark 3 Cubic graphs are furthermore used more generally as a model for much Topological content in all of Graph Theory.

Remark 4 Much Topological content? I.e. the portion that [98] is left after keeping on deleting all degree-1 vertices until none are left ('*serial defoliation*'). Alongside excising all degree-2 vertices without spoiling the graph class's modelling assumptions ('*homeomorphing*'). In particular, simple graphs do not permit collapse of triangles down to multi-edges or loops [97].

Remark 5 The modelling itself then involves resolving each vertex of degree ≥ 4 as follows. Replace it by enough cubic vertices to accommodate all of the edges emanating from the original degree ≥ 4 vertex. For instance a degree-4 vertex, its edges and the vertices at each's other end form the 4-star subgraph Which is resolved by the quadruped subgraph: 2 internal cubic vertices (Fig 4.2).

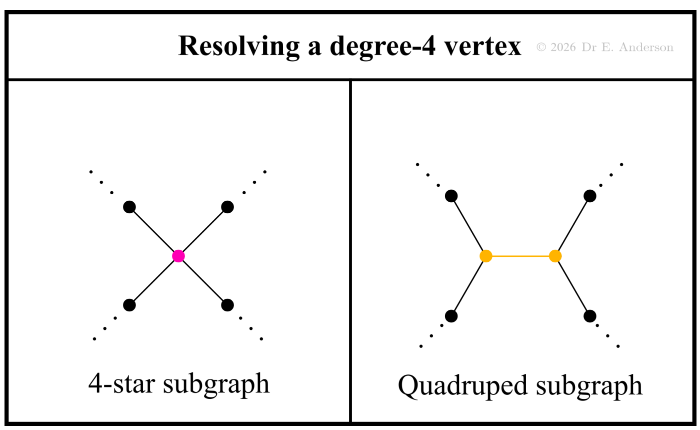

Figure 4.2:

4.3.4 Resolving N-angulations

Remark 1 At the dual level, an N-angulation can be turned into a finer triangulation model by triangulating each of its faces. There are various ways of doing this.

I) '*Little-triangulate*' it. Cut it up by a choice of non-intersecting diagonals. Which number $T := N - 2$; we already encountered this approach in Chapter 1...

II) *Stellate* it, alias *stellarly subdivide* it. I.e. introduce an internal vertex. And then edges from it to each of the N vertices.

III) *Barycentrically subdivide* it. I.e. firstly introduce a central vertex. Next introduce a vertex at the midpoint of each edge. Finally add edges from it to each of our $2N$ vertices.

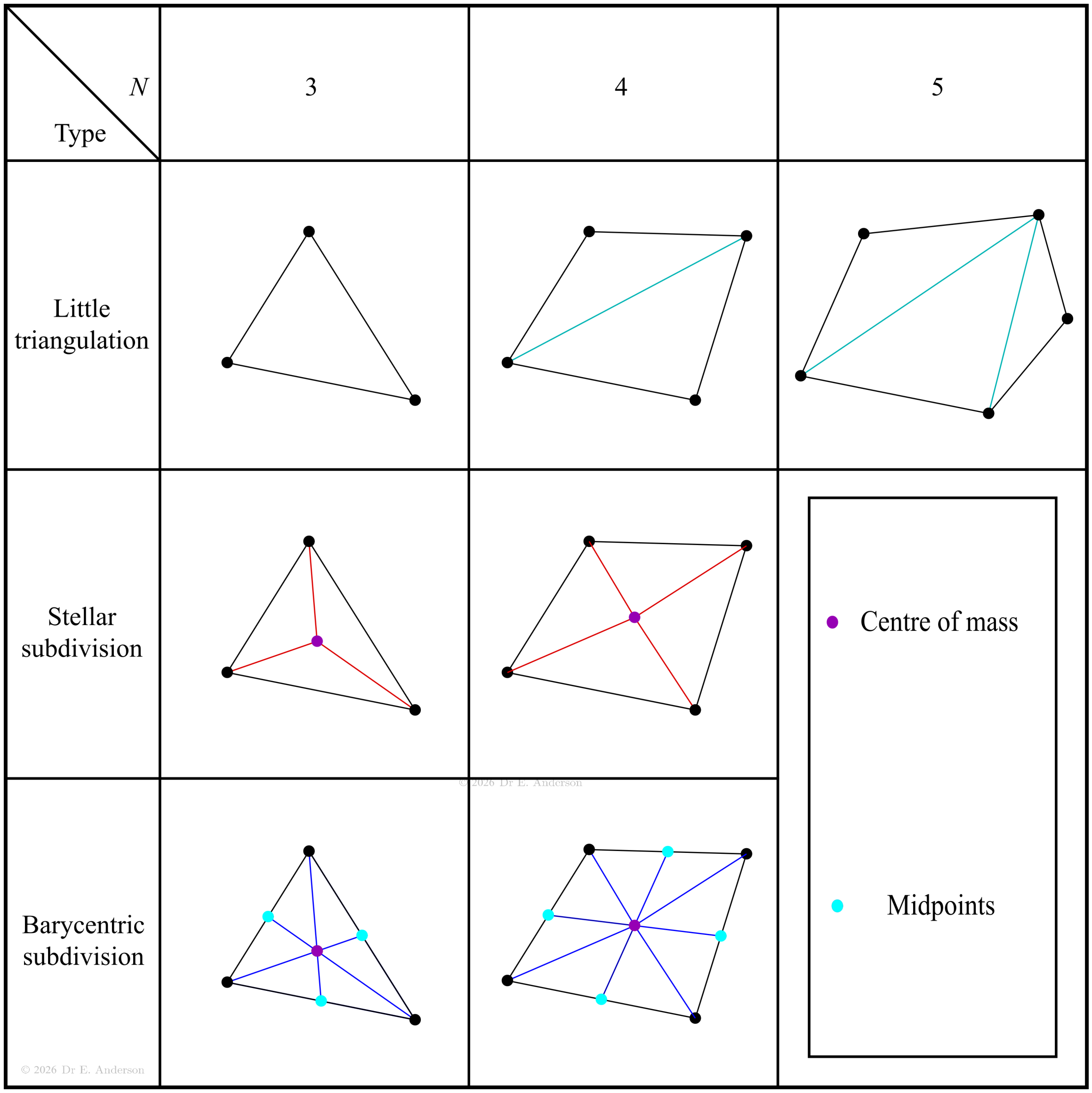

Figure 4.3:

Remark 2 On the one hand, I) has the advantage of minimizing the number of introduced elements.

Remark 3 On the other hand, II) and III) have canonical advantages, in the sense of no ambiguity in what objects are picked.

Remark 4 Though for some purposes there is an issue with where the extra vertex is to be placed.

Suppose that the polygon is star-shaped. Then placing the extra vertex at a star-point ensures straight edges. There is still a Drawing and Visualization [95, 100] placing ambiguity in which star point is used.

Next suppose that the polygon is furthermore convex. Then the centre of mass of the vertices – taken to each have unit mass – is a star-point. And constitutes a uniquely-distinguished such. Indeed 'barycentre' and 'centre of mass' are synonyms. If this choice is available and made, ii) and iii) can be viewed as centre-of-mass methods. From a Geometrical point of view, this belongs to the Affine level of structure [60, 59, 64]...

Remark 5 These matters start to illustrate that notions of convexity play a role in polyhedral models. [Star-shaped and star-point are themselves in the convexity family of notions...] They go on to play a role in specifically the suite of styles of proof of the Gauss–Bonnet theorem. As such, we may eventually include an Appendix on these as well. Finally observe that for some of our purposes, the above use of 'straight' needs to be supplanted by 'geodesic'. And yet much of the theory of convexity turns out to persist under this substitution!

Exercise 1 [96] For which N does triangulating the N-a-gon become ambiguous? View the 2 ensuing cases as some top and bottom notions. What is the first N to have a further middle case? What poset height functions can you espy for the triangulations? What is the minimum N to exhibit a nontrivial poset, in the sense of not just a chain? How many distinct triangulations are supported by arbitrary N ?

4.3.5 (N, M)-angulations, and so on

Definition 1 An *angulation* is our name for a polyhedral decomposition that fits nicely together. We also use (N, M)-*angulation* for the following. A face (N, M)-biregular [97] polyhedral subdivision that is suitably nice.

Example 1 The case of this referred to in the current Review is the $(3, 4)$-angulation. I.e. an angulation by just a nontrivial mixture of triangles and quadrilaterals. Where 'nontrivial' means that ≥ 1 exemplar of each is present...

4.4 The Euler characteristic

4.4.1 Back to $V - E + F$

Structure 4 More generally,

$$V - E + F \tag{4.2}$$

turns out to be independent of [23, 25] how one sufficiently nicely polyhaedrally decomposes a surface. It captures Topological information. As opposed to finer features of a surface. Such as assigning a metric, or an embedding into $\mathbb{R}^3$ which brings in an extrinsic curvature. So (4.2) is a function of our surface $\mathfrak{S}$'s topology, and nothing but its topology, $\mathfrak{S}_\tau$. We thus write

$$\chi\,(\mathfrak{S}_\tau) = V - E + F\,. \tag{4.3}$$

Where χ is termed the *Euler characteristic* [32, 119, 40, 23].

Remark 1 This Topological information has some capacity to discern between surfaces, starting with the following result.

CWB 2-manifold classification theorem These are all the sphere modulo insertion of g handles or k twists.

Structure 5 Here in both cases, the insertion starts by deleting 2 discs.

A) For a handle, a cylinder is then glued in to abridge between the deleted discs.

Which has the effect of Topologically joining on a torus $\mathbb{T}^2$.

B) While for a twist, a Möbius strip is glued in likewise.

Which has the effect of Topologically joining on a real-Projective plane $\mathbb{RP}^2$.

Structure 6 *Orientability* plays a major role below. This is a consistent assignation of certain objects over the entirety of a space.

In the context of a Topological surface $\mathfrak{S}$ as modelled by a sufficiently nice N-angulation, here is a suitable such consistency. On the boundary of each face, one can place an anticlockwise circuit of arrows. Were the surface non-orientable, tying to extend this pattern across it eventually runs into the following obstruction. One returns to a face that has already been assigned its arrows. Only to find that now continuing the pattern would involve placing a clockwise circuit of arrows on its boundary...

Remark 2 Cylinders and tori are orientable, while Möbius strips and real-projective spaces are not. So handle insertion retains orientability, while twist insertion does not.

Lemma 1 For surfaces $\mathfrak{X}$, $\mathfrak{Y}$ and Topological join $\#$, the following holds.

$$\chi(\,\mathfrak{X} \,\#\, \mathfrak{Y}\,) = \chi(\mathfrak{X}) + \chi(\mathfrak{Y}) - 2\,. \tag{4.4}$$

<u>Proof</u> Model $\mathfrak{X}$, $\mathfrak{Y}$ by angulations $K_\mathfrak{X}$, $K_\mathfrak{Y}$. Such that each contains ≥ 1 triangle. Thus we can pick a triangle in each.

Remove these. [They model the above-mentioned discs, which they are homeomorphic to...] And attach the ensuing deletions to each other by means of a triangular prism-surface tube. [Which models the above-mentioned cylindrical tube, which it is homeomorphic to...

This operation imparts

$$\Delta V = 0\,, \quad \Delta E = 3\,, \quad \Delta F = -2 + 3 = 1\,.$$

Thus
$$\chi(\mathfrak{X} \# \mathfrak{Y}) - (\chi(\mathfrak{X}) + \chi(\mathfrak{Y})) =: \Delta\chi = \Delta V - \Delta E + \Delta F = 0 - 3 + 1 = -2 . \Box$$

4.4.2 The orientable subcase

Definition 1 The *genus* of an orientable surface $\mathfrak{S}$ is the number of handles that it contains. The case with g handles is the *g-holed torus* $\mathbb{T}^2{}_g$

Theorem 1
$$\chi\left(\mathbb{T}_g{}^2\right) = 2\left(1 - g\left(\mathbb{T}_g{}^2\right)\right) .$$

Proof Part 1) Compute χ for $K_{\mathbb{T}^2}$: some polyhedral model of $\mathbb{T}^2$.

Part 2) Use the joining Lemma to adjoin further tori piecemeal.

Part 3) Establish model-independence of the calculation, by which
$$\chi\left(K_{\mathbb{T}_g{}^2}\right) = \chi\left(\mathbb{T}_g{}^2\right) .$$

For Part 1, cut out a triangular prism from another. While matching between the 2 prisms' vertices with further edges. Thus
$$\chi(K_{\mathbb{T}^2}) = V - E + F = 3 \times 4 - 3 \times 8 + 3 \times 4 = 0 .$$

For Part 2), proceed inductively.

Base step.
$$\chi\left(\mathbb{S}^2\right) = 2 .$$

Induction hypothesis. Suppose that the theorem holds for some $g = n$.

Induction step. Then for $g = n + 1$,
$$\chi\left(K_{\mathbb{T}^2{}_{n+1}}\right) = \chi(K_{\mathbb{T}^2{}_n} \# K_{\mathbb{T}^2}) = \chi(K_{\mathbb{T}^2{}_n}) + \chi(K_{\mathbb{T}^2})$$
$$= 2(1 - n) + -2 = 2[1 - (n + 1)] .$$

Where the third step uses both the induction hypothesis and the first part.

Part 3 is harder, so we only point to some styles of proof.

Style of proof 1) Ahlfors and Sario [23] do this in the context of Riemann surfaces.

Style of proof 2) Massey [25] does this for rectagular decompositions. This can be rendered compatible with the above presence of triangles by the following observation. Stellating a square incurs
$$\Delta V = 1 , \quad \Delta E = 4 , \quad \Delta F = -1 + 4 = 3 .$$

Thus
$$\Delta\chi = 1 - 4 + 3 = 0 .$$

Style of proof 3 One can also arrive at this by specializing more general Topological workings. For instance by leaning on Chapter 6's observation.

[Which passes the buck to establishing that a straightforward model of *homology* is purely-Topological in content.]

Or by using that χ is a subcase of *Lefschetz number* [32].

Remark 1 For CWB orientable $\mathfrak{S}$, we have inversion
$$g(\mathfrak{S}) = 1 - \frac{\chi(\mathfrak{S})}{2} .$$

Examples Starting with the more intuitive quantity in each case, the sphere has

$$g\left(\mathbb{S}^2\right) \;=\; 0\;, \quad \text{and so} \quad \chi\left(\mathbb{S}^2\right) \;=\; 2\;.$$

While the torus has

$$g\left(\mathbb{T}^2\right) \;=\; 1\;, \quad \text{and so} \quad \chi\left(\mathbb{T}^2\right) \;=\; 0\;.$$

All other oriented CWB surfaces have negative χ .

4.4.3 The non-orientable subcase

Definition 1 The *demigenus* k of a non-orientable surface is the number of $\mathbb{RP}^2$'s attached to it.

Theorem 2

$$\chi\left(\mathbb{RP}_k{}^2\right) \;=\; 1 \;-\; k\left(\mathbb{RP}_k{}^2\right)\;.$$

Exercise 2 Find a simple polyhedral model for $\mathbb{RP}^2$, and thus prove the non-orientable counterparts of Parts 1 and 2.

Remark 1 At least in the context of Riemann surfaces, one can systematically generalize to the non-orientable case by considering double covers.

Remark 2 For nonorientable CWB surfaces $\mathfrak{N}$, inversion returns

$$k(\mathfrak{N}) \;=\; 2 \;-\; \chi(\mathfrak{N})\;.$$

Examples The real-Projective plane has

$$k\left(\mathbb{RP}^2\right) \;=\; 1\;, \quad \text{and so} \quad \chi\left(\mathbb{RP}^2\right) \;=\; 1\;.$$

The Klein bottle

$$\mathbb{K}^2 \;\overset{\text{homeo}}{\sim}\; \mathbb{RP}^2 \,\#\, \mathbb{RP}^2\;.$$

has

$$k\left(\mathbb{K}^2\right) \;=\; 2\;, \quad \text{and so} \quad \chi\left(\mathbb{K}^2\right) \;=\; 0\;.$$

All the others have negative χ .

4.4.4 Arena of CWB 2-manifolds

Notational Remark 1 We use here the notation $\mathbb{RP}^2{}_k$ as a parallel of $\mathbb{T}^2{}_g$.

Remark 1 So on the one hand, demigenus uniquely characterizes $\mathbb{RP}^2{}_{2\,g\,+\,1}$.

Remark 2 On the other hand, $\mathbb{RP}^2{}_{2\,g}$ and $\mathbb{T}^2{}_g$ need a second parameter to split them. Namely the *sign-valued orientation parity* o . With

$$o\left(\,\mathbb{RP}^2{}_{2\,g}\right) \;=\; -\,1\;.$$

Versus

$$o\left(\,\mathbb{T}^2{}_g\right) \;=\; +\,1\;.$$

Exercise 3 Show that adding a twist and a handle to a sphere is homeomorphic to adding 3 twists. Thus completing the preliminaries to form the arena of CWB 2-surfaces in Fig 4.4.

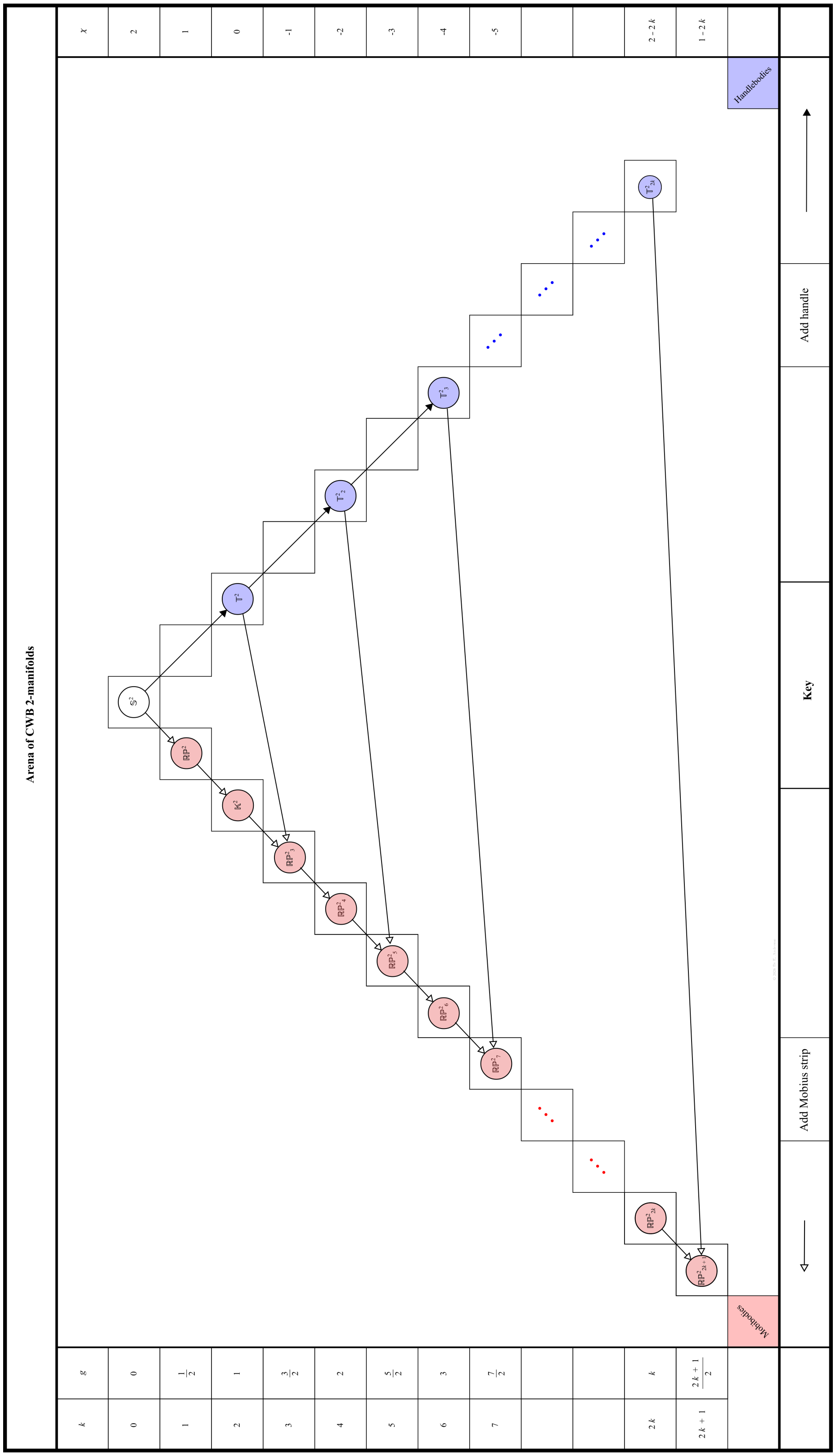
Arena of CWB 2-manifolds
Handlebodies
Mobiusbodies
Key
Add handle
Add Mobius strip

Figure 4.4:

4.4.5 Starting with a disc instead of a sphere

Remark 1 We also need to consider the disc $\mathbb{D}^2$. This is not included above since it possesses a boundary. It is oriented. Looking at a polyhedral model for the disc amounts to throwing away 1 face from a polyhedral model of a sphere. Thus

$$\chi\left(\mathbb{D}^2\right) = \chi\left(\mathbb{S}^2\right) - 1 = 2 - 1 = 1 .$$

Handles or twists can be added to a disc instead of to a sphere. For handles, this returns

$$\chi = 2(1 - g) - 1 = 1 - 2g .$$

Inverting to

$$g = \frac{1 - \chi}{2} .$$

While for twists,

$$\chi = 2 - k - 1 = 1 - k .$$

Inverting to

$$k = 1 - \chi .$$

Example 1 Adding a single twist returns $\chi = 0$. Adding any further number of twists, or any number of handles, return negative χ .

Chapter 5

The Gauss–Bonnet theorem

5.1 Area–Euler for on maximally-symmetric spaces

5.1.1 The sphere revisited

Remark 1 As mentioned in Chapter 3, for a whole sphere,

$$\chi\left(\mathbb{S}^2\right) \;=\; 2\;, \quad \text{so} \quad g\left(\mathbb{S}^2\right) \;=\; 0\;. \tag{5.1}$$

So we can rewrite theorem S-CWB as follows.

Theorem S-CWB-T

$$A\left(\mathbb{S}^2\right) \;=\; 2\,\pi\,\chi\left(\mathbb{S}^2\right) \;=\; 4\,\pi\,\left(1 \;-\; g\left(\mathbb{S}^2\right)\right)\;. \tag{5.2}$$

Remark 2 Also great-circle triangles and polygons are homeomorphic to discs. So

$$\begin{array}{ccccc} \chi(\triangle) & = & \chi\left(\mathbb{D}^2\right) & = & \chi\left(P^N\right) \\ & & \| & & \\ & & 1 & & . \end{array} \tag{5.3}$$

And we can rewrite theorems S-3 and S-N as follows.

Theorem S-3-T For a geodesic-edged triangle on a sphere,

$$A \;+\; E \;=\; 2\,\pi\,\chi\left(\mathbb{D}^2\right) \;=\; 4\,\pi\,\left(1 \;-\; g\left(\mathbb{D}^2\right)\right)\;.$$

Naming Remark 1 This is a *global* version [44, 46] of Gauss–Bonnet, in contrast to the internal-angle formulation's more *local* version.

Theorem S-N-T For a geodesic-edged polygon P_N on a sphere,

$$A_N \;+\; E_N \;=\; 2\,\pi\,\chi\left(\mathbb{D}^2\right) \;=\; 4\,\pi\,\left(1 \;-\; g\left(\mathbb{D}^2\right)\right)\;.$$

5.2 Gaussian curvature formulations

5.2.1 CWB-surface version

Theorem G-CWB For a CWB surface $\mathfrak{S}$,

$$K_{\mathrm{TG}}(\mathfrak{S}) \;=\; 2\,\pi\,\chi\,(\mathfrak{S}) \tag{5.4}$$

Remark 1 For a sphere, use of K_{TG} in place of area is consistent by Sec *.

Remark 2 The above use of A_N extends to other maximally-symmetric spaces: of constant curvature. Though the sign of the excess may differ; see Sec 5.5 for two cases of this.

5.2.2 Geodesic-edged triangle on a surface

Theorem G-3-I For a geodesic-edged triangle on a surface $\mathfrak{S}$,

$$K_{\mathrm{TG}}(\mathfrak{S}) \;=\; I \,-\, \pi \;. \tag{5.5}$$

<u>Style of proof 1</u> Compute the LHS in geodesic polar coordinates [93].

Remark 1 Gauss [2] was the first to prove this result.

Remark 2 We can furthermore convert to external angles. In the case of the sphere, this is as before. While for a general surface, this now brings out the Euler characteristic, or equivalently the genus, as follows.

Theorem G-3-T) For a geodesic-edged triangle $\triangle$ in a surface $\mathfrak{S}$.

$$K_{\mathrm{TG}}(\triangle) \,+\, E_N \;=\; 2\,\pi\,\chi(\triangle) \;. \tag{5.6}$$

5.2.3 Geodesic-edged N-a-gon on a surface

Remark 1 The above is then readily amenable to extension to geodesic-edged N-a-gons, by the same triangulation procedure as we used for spheres. In this way, we arrive at the following.

Theorem G-N-T For a geodesic-edged N-a-gon P_N on a surface $\mathfrak{S}$,

$$K_{\mathrm{TG}}\,(\mathrm{P}_N) \,+\, E_N \;=\; 2\,\pi\,\chi\,(\mathrm{P}_N) \;. \tag{5.7}$$

5.2.4 The CWB case on a surface

<u>Style of proof 1</u> We can finally arrive at the CWB case by triangulating the whole surface. Once again, the external angles cancel.

5.3 Gaussian curvature formulations for arbitrary polygons

5.3.1 Arbitrary triangle on a surface

Theorem A-3-I For an 'arbitrary' triangle[1] $\triangle$ on a surface $\mathfrak{S}$

$$K_{\mathrm{TG}}(\triangle) \,+\, \kappa_{\mathrm{Tg}}(\partial\triangle) \;=\; I \,-\, \pi \;. \tag{5.8}$$

<u>Style of proof 1</u> Cut our arbitrary triangle into 3 triangles, each of which meets the following description Item 2 of row 3 of Fig 5.1.

$$\text{It contains } 2 \text{ geodesic sides and } 1 \text{ non-geodesic side.} \;. \tag{5.9}$$

We then evoke geodesic polar coordinates to calculate *both* LHS terms. This returns the desired RHS.

[1] This is a curvilinear triangle. We do assume that its boundary is a simple curve. That it is smooth, aside from at its vertices, which play the role of corners. And that it has 1-sided derivatives at each corner. These last two conditions stop our triangle from being too 'wavy'. These modelling conditions match [42].

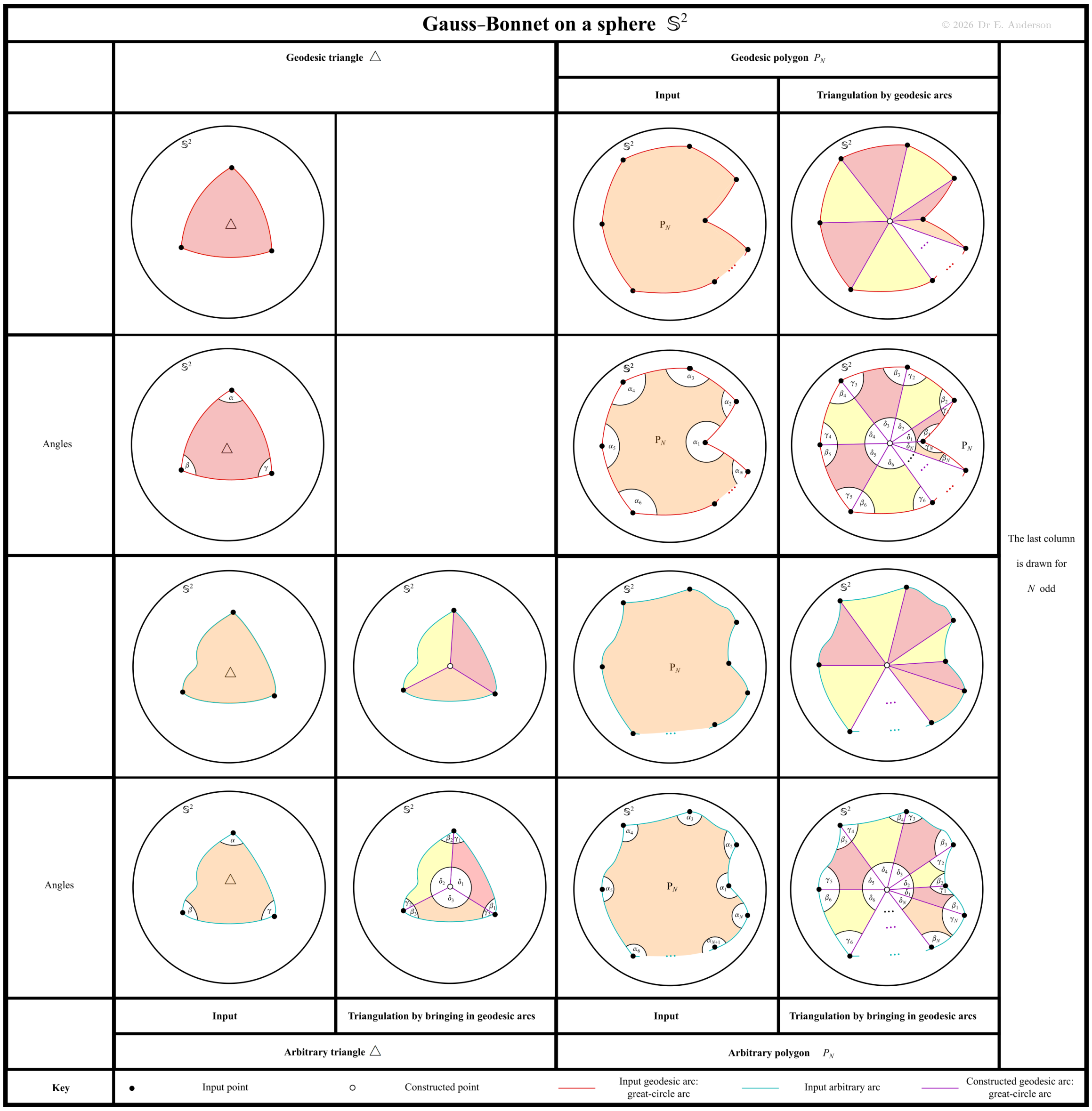

Figure 5.1:

Style of proof 2 ([21, 22, 46] going back to Darboux [11].) Use Vector Calculus, in particular *Green's theorem in the plane*. Or, better, *Green's theorem in a patch of surface*, whose image forms the region $\mathfrak{R}$,

$$\oint_{\partial\mathfrak{R}} (P \, \mathrm{d}u \; + \; Q \, \mathrm{d}v) \;\; = \;\; \iint_{\mathfrak{R}} (Q_{,u} \; - \; P_{,v}) \frac{\mathrm{d}A}{\sqrt{E\,G \; - \; F^2}} \; . \tag{5.10}$$

So starting with the total geodesic curvature and applying Liouville's formula (**??**),

$$\oint_{\partial\mathfrak{R}} \kappa_{\mathrm{g}} \, \mathrm{d}s \;\; = \;\; \oint_{\partial\mathfrak{R}} \left[\, \mathrm{d}\psi \; + \; \left(\kappa_{\mathrm{g}_v} \cos \psi \; + \; \kappa_{\mathrm{g}_u} \sin \psi \right) \mathrm{d}s \, \right]$$

$$= \;\; \oint_{\partial\mathfrak{R}} \mathrm{d}\psi \; + \; \iint_{\mathfrak{R}} \left[\left(\kappa_{\mathrm{g}_u} \sqrt{G} \right)_{,u} \; - \; \left(\kappa_{\mathrm{g}_v} \sqrt{E} \right)_{,v} \right] \mathrm{d}u \, \mathrm{d}v \;\; = \;\; \oint_{\partial\mathfrak{R}} \mathrm{d}\psi \; + \; \iint_{\mathfrak{R}} K_{\mathrm{G}} \, \mathrm{d}S \; . \tag{5.11}$$

Where the second equality uses linearity of integration and (**??**, **??**). While the third makes use of (5.10) and the fourth form (**??**) for the Gaussian curvature in reverse.

This second approach firstly clarifies that the total geodesic curvature is the correct surface term to pair with the total Gaussian curvature integral. Secondly, the first term on the RHS picks up contributions from discontinuities at corners, sending us back to form A-N-I.

Style of proof 3 Lean upon [44] the *turning-tangents theorem* [85]. It is however more convenient for us to postpone detailing this to the sister Review [123].

Remark 1 Generalizing the second item in row 3 of Fig 5.1 to arbitrary surfaces yields the following.

Theorem A-3 For an arbitrary triangle $\triangle$ on a surface $\mathfrak{S}$,

$$K_{\mathrm{TG}}(\triangle) \ + \ \kappa_{\mathrm{Tg}}(\partial\triangle) \ + \ E \ = \ 2\,\pi\,\chi\,(\triangle) \ . \tag{5.12}$$

5.3.2 Arbitrary polygon on a surface

Theorem A-N-I For an arbitrary polygon P_N on a surface $\mathfrak{S}$

$$K_{\mathrm{TG}}\,(P_N) \ + \ \kappa_{\mathrm{Tg}}\,(\partial P_N) \ = \ I \ - \ T\,\pi \ .$$

Theorem A-N For an arbitrary N-a-gon P_N on a surface,

$$K_{\mathrm{TG}}(\triangle) \ + \ \kappa_{\mathrm{Tg}}(\partial\triangle) \ + \ E_N \ = \ 2\,\pi\,\chi\,(\triangle) \tag{5.13}$$

Style of proof 1 One approach to this is to triangulate our arbitrary N-a-gon with N triangles of the form (5.9); see the fourth item on row 3 of Fig 5.1.

Remark 1 Bonnet was the first to offer a proof of the local version of A-N-I for a sphere.

5.4 Intrinsic-curvature formulations

By Gauss' outstanding theorem (**??**), we can furthermore place an intrinsic curvature interpretation on the RHS of the Gauss–Bonnet theorem, as follows.

Theorem S-CWB-R

$$R_{\mathrm{T}}(\mathfrak{S})/2 \ = \ 2\,\pi\,\chi\,(\mathfrak{S}) \ . \tag{5.14}$$

5.4.1 Geodesic-sided polygons on a surface

Theorem G-3-R) For a geodesic-sided triangle on a surface,

$$R_{\mathrm{T}}(\triangle)/2 \ + \ E \ = \ 2\,\pi\,\chi(\triangle) \ . \tag{5.15}$$

Theorem G-N-R For a geodesic-sided N-a-gon on a surface,

$$R_{\mathrm{T}}\,(P_N)\,/2 \ + \ E_N \ = \ 2\,\pi\,\chi\,(P_N) \ . \tag{5.16}$$

Remark 3 Isolating the Euler piece gives the following alternative formulations.

5.4.2 Arbitrary polygons on a surface

Theorem A-3-I For an arbitrary triangle,

$$R_{\mathrm{T}}(\triangle)/2 \ + \ \kappa_{\mathrm{Tg}}(\partial\triangle) \ = \ I \ - \ \pi \ .$$

Theorem A-N-I For an arbitrary polygon,

$$R_{\mathrm{T}}\,(P_N)\,/2 \ + \ \kappa_{\mathrm{Tg}}\,(\partial P_N) \ = \ I_N \ - \ T\,\pi \ .$$

Theorem A-3) For an arbitrary triangle $\triangle$ on a surface,

$$R_{\mathrm{T}}(\triangle)/2 \;+\; \kappa_{\mathrm{Tg}}(\partial\triangle) \;+\; E \;\;=\;\; 2\,\pi\,\chi\,(\triangle) \;\; . \tag{5.17}$$

Theorem A-N-T For an arbitrary N-a-gon P_N on a surface,

$$R_{\mathrm{T}}\,(P_N)\,/2 \;+\; \kappa_{\mathrm{Tg}}\,(\partial P_N) \;+\; E_N \;\;=\;\; 2\,\pi\,\chi\,(P_N) \;\; . \tag{5.18}$$

5.5 Special cases

Remark 1 The idea here is to view Fig 5.2 as a master equation. And then obtain the previous smaller equations (and more) by considering under which circumstances ≥ 1 terms are zero.

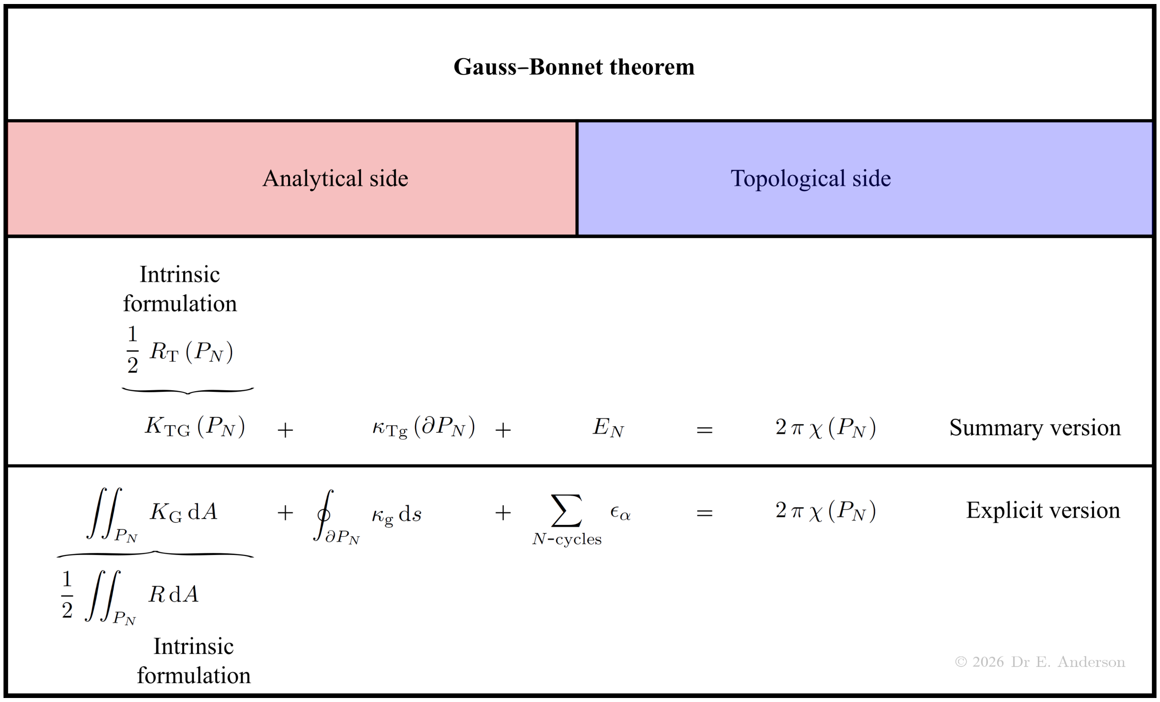

Figure 5.2:

5.5.1 Gauss–Bonnet in flat space

Remark 1 For a triangle, this reduces to the elementary result

$$I \;=\; \pi \; . \tag{5.19}$$

While for an N-a-gon,

$$I \;\;=\;\; T\,\pi \; . \tag{5.20}$$

Remark 2 Versions with curves instead of lines, thus imbuing curvilinear polygons with geodesic curvature, are less trivial but also not commonly considered.

$$\kappa_{\mathrm{Tg}}\,(\partial P_N) \;\;=\;\; I_N \;-\; T\,\pi \; . \tag{5.21}$$

With triangle subcase

$$\kappa_{\mathrm{Tg}}\,(\partial\triangle) \;\;=\;\; I \;-\; \pi \; . \tag{5.22}$$

5.5.2 Gauss–Bonnet in hyperbolic space $\mathbb{H}^2$

Remark 1 This looks a great deal like the $\mathbb{S}^2$ case, except that the angle sums are now smaller than flat space's rather than larger. I.e. E is negative. Or equivalently $D := \quad - E$ is positive: now a *hyperbolic deficit* rather than a spherical excess... Proving the geodesic-sided triangle angle sum is now however somewhat more involved.

Exercise 1$^+$ Do this.

[This Exercise is only a '+' if you do not look up specific methods.] After you have an answer, however, do see what specific methods you find in the literature...

5.5.3 Gauss–Bonnet on the torus $\mathbb{T}^2$

Example 1 For the torus $\mathbb{T}^2$ [21] ,

$$g\left(\mathbb{T}^2\right) \ = \ 1 \quad \Rightarrow \quad \chi\left(\mathbb{T}^2\right) \ = \ 0 \ .$$

So

$$K_{\mathrm{TG}}\left(\mathbb{T}^2\right) \ = \ 0 \ . \tag{5.23}$$

5.5.4 Geodesic sides

These knock out the total geodesic curvature boundary integral term...

5.5.5 (On average) flat regions

Flat regions knock out the Gaussian curvature integral.

So do on-average flat regions. How much extra scope do these offer?

5.5.6 CWB subcase

This knocks out both the total geodesic curvature boundary integral term and the boundary angle-sum terms!

5.5.7 Open surface analogue

Cohn-Vossen's inequality

$$K_{\mathrm{TG}}\left(\mathrm{P}_N\right) \ \leq \ 2\,\pi\,\chi\left(\mathrm{P}_N\right) \ .$$

Remark 1 Further special cases shall be included in subsequent versions.

5.5.8 A first outline of modelling assumptions and caveats with the proofs

Caveat family 1. Coordinate patch assumptions. Such as existence of an orthogonal, isothermal or geodesic-polar coordinate patch.

Caveat family 2. Convexity subtleties concerning one's polygon.

Caveat family 3. Triangulation technicalities. Or technicalities corresponding to generalizations of triangulations.

Caveat family 4. A further technicality is that some authors prove the theorem specifically for Riemann surfaces.

Remark 2 Proofs prior to the 1930s were made without a good feel for these matters. Amounting to either limiting the cases covered or leaving holes in the proofs.

Chapter 6

Homological interpretations of Gauss–Bonnet

6.1 Homology groups and Betti numbers

Definition 1 Suppose that we are given a p-manifold $\mathfrak{M}^p$. Then the nth *homology group* is

$$H_n(\mathfrak{M}^p) = \frac{Z_n(\mathfrak{M}^p)}{B_n(\mathfrak{M}^p)} = \frac{\mathfrak{Ker}\,\mathrm{d}_n}{\mathfrak{Im}\,\mathrm{d}_{n+1}} .$$

With reference to the *chain complex*

$$0 \longrightarrow C_p \longrightarrow \ldots \longrightarrow C_{n+1} \xrightarrow{\mathrm{d}_{n+1}} C_n \xrightarrow{\mathrm{d}_n} C_{n-1} \xrightarrow{\mathrm{d}_{n-1}} \ldots \longrightarrow C_0 \longrightarrow 0 .$$

And the *boundary maps* d_n such that

$$d_n \circ d_{n+1} = 0 \ : \ \text{the boundary of a boundary is zero} .$$

And where $B_n(\mathfrak{M}^p)$ is the *boundary group* and $Z_n(\mathfrak{M}^p)$ is the *cycle group*.

Remark 1

$$B_n(\mathfrak{M}^p) \lhd Z_n(\mathfrak{M}^p) \ : \ \text{a normal subgroup} .$$

So $H_n(\mathfrak{M}^p)$ is indeed a well-defined quotient group.

Definition 2 The nth *Betti number* is

$$\beta_n := \mathrm{rank}\,(H_n(\mathfrak{M}^p)) .$$

Historical Remark 1 Betti was the first to compute these objects [7]. Poincaré [10] named the Betti numbers in his honour.

The above Group-Theoretic conceptualization is not however how Betti or Poincaré conceived of these objects. It is rather Emmy Noether's [14] reformulation thereof. Which has become the point of view taken since the 1930s. As part of the inception of the subject subsequently known as Algebraic Topology [126, 25, 135, 136]...

6.2 The second alternating sum

6.2.1 For 2-surfaces

Remark 1 View vertices as 0-simplices, numbering

$$S_0 = V .$$

Edges as 1-simplices, numbering

$$S_1 = E .$$

And faces as 2-simplices, numbering

$$S_2 = F .$$

Remark 2 The Euler characteristic can thus be reformulated as

$$\chi(\mathfrak{S}_\tau) = \sum_{n = 0^2} S_n . \tag{6.1}$$

I.e. the alternating sum over the numbers of simplices of the 3 smallest dimensions.

Theorem 1 'Euler–Poincaré' For a polyhedrally-split 2-surface $\mathfrak{S}$, the Euler characteristic is

$$V - E + F = \chi(\mathfrak{S}) = b_0 - b_1 + b_2 .$$

That is,

$$\sum_{n = 0}^{2} (-1)^n S_n = \chi(\mathfrak{S}) = \sum_{n = 0}^{2} (-1)^n \beta_n . \tag{6.2}$$

I.e. it is *the alternating sum of simplex counts or of Betti numbers*.

Remark 3 A further interpretation of the unit–genus split of χ for a surface $\mathfrak{S}$ is then afforded. Via

$$\beta_0 = \#(\text{ connected components }) = 1 \text{ for } \mathfrak{S} \text{ connected} .$$

And

$$\beta_2 = 1 \text{ for } \mathfrak{S} \text{ CWB and oriented} .$$

Leaving

$$\beta_1 = 2\,g$$

as the sole pure-genus contribution.

Thus as a Betti number expansion,

$$\chi(\mathfrak{S}) = 2\,(1 - g) = 1 - 2\,g + 1 .$$

In this way, g is not merely [96] a convenient adapted variable, but rather has higher significance: Homologically rooted.

6.2.2 More generally

Remark 1 Remark 1 of the previous Subsec readily extends to simplices of the p smallest dimensions as supported by a *p*-surface $\mathfrak{S}^p$. Returning

$$\chi(\mathfrak{S}^p) = \sum_{n = 0}^{p} S_n . \tag{6.3}$$

Historical Remark 2 Schläfli [5] considered this as part of the world's first systematic study of what came to be called *polytopes*. I.e. the p-dimensional analogue of polyhedra. This retains cut-up independence, being solely $\mathfrak{S}^p{}_\tau$ -dependent.

Theorem 2 'Euler–Schläfli–Poincaré'

$$\sum_{n\,=\,0}^{2} (-1)^p\, S_n \;=\; \chi\,(\mathfrak{S}^p) \;=\; \sum_{n\,=\,0}^{2} (-1)^p\, \beta_n\ . \tag{6.4}$$

Remark 3 This in part rests on how simplex dimensions and the number of homology groups are jointly bounded as follows. $\mathfrak{S}^p$ does not support either of these beyond p .

Naming Remark 1 Both of the above Theorems usually go by the same name: Euler–Poincaré. This is quite possibly because the general version is no harder to prove than the little version, as follows.

6.2.3 Proof of Theorem 2

Style of proof 1 1) By the rank–nullity theorem [117, 120, 121, 122] of Linear Algebra,

$$\dim\,(\,C_n\,(\mathfrak{m}^p)\,) \;=\; \dim\,(\,Z_n\,(\mathfrak{m}^p)\,) \;+\; \dim\,(\,B_{n\,-\,1}\,(\mathfrak{m}^p)\,)\ .$$

A shorthand for which is

$$c_n \;=\; z_n \;+\; b_{n\,-\,1}\ . \tag{6.5}$$

Furthermore, at the top end of the chain complex,

$$0 \;=\; \dim\,(0) \;=\; z_{p\,+\,1} + b_p \;=\; 0 + b_p \;\Rightarrow\; b_p \;=\; 0 \qquad \text{(B)}\ .$$

While at the bottom end of the chain complex,

$$0 \;=\; \dim\,(0) \;=\; z_0 + b_{-1} \;=\; z_0 + 0 \;\Rightarrow\; b_p \;=\; 0 \qquad \text{(Z)}\ .$$

2) From our quotient group,

$$\dim\,(\,H_n\,(\mathfrak{m}^p)\,) \;=\; \dim\,(\,Z_n\,(\mathfrak{m}^p)\,) \;-\; \dim\,(\,B_n\,(\mathfrak{m}^p)\,)\ .$$

A shorthand for which is

$$\beta_n \;=\; z_n \;-\; b_n\ . \tag{6.6}$$

Next treat (6.5, 6.6) as a pair of simultaneous linear equations. For a 'useful unknown' c_n and a 'nuisance variable' z_n . So that this system of equations is to be half-solved for c_n by eliminating z_n , as follows.

$$c_n \;=\; \beta_n \;+\; b_n \;+\; b_{n\,+\,1}\ .$$

3) The C_n are a model for our simplices, whose counts are S_n . Thus

$$\sum_{n\,=\,0}^{p} (-1)^n\, S_n \;=\; \sum_{n\,=\,0}^{p} (-1)^n\, c_n \;=\; \sum_{n\,=\,0}^{p} (-1)^n\, (\,\beta_n \;+\; b_n \;+\; b_{n\,+\,1}\,)$$

$$=\; \sum_{n\,=\,0}^{p} (-1)^n\, \beta_n \;+\; \sum_{n\,=\,0}^{p} (-1)^n\, (\,b_n \;+\; b_{n\,+\,1}\,)\ .$$

But this last term is a telescoping sum T_p . Whose sole non-cancelling terms are its first and last,

$$T_p \;=\; b_p \;-\; b_{-1} \;=\; 0 \;-\; 0 \;=\; 0\ .$$

Where the first zero is by (B), while the second is by an intermediary step in (Z). $\square$

6.2.4 The role of even dimensions

Remark 1 p-dimensional CWB manifolds (whether or not viewed as a p-surface) enjoy *Poincaré duality* [136].

$$\beta_n = \beta_{p-n} .$$

Thus for p odd, the Betti alternating sum's terms cancel off in pairs! By which the homological reformulation swiftly tells us that

$$\chi(\mathfrak{m}^p) = 0 \text{ for odd-}p \text{ CWB manifolds } \mathfrak{m}^p .$$

Thus

$$\chi(\mathfrak{m}^p) \text{ can only play a nontrivial role for even-}p \text{ among the CWB manifolds } \mathfrak{m}^p .$$

Exercise 1$^-$ Interpret this statement for $p = 1$ by use of simple-graph models for the CWB 1-manifolds.

Pointers 1-2 Thus in particular χ plays no role for CWB 3-manifolds [140, 131, 137, 139].

Which turn out to be 1 of the 2 Goldilocks cases for manifolds.

The other is the CWB 4-manifolds [138, 141]. For which χ does play a role.

Indeed, dimension 4 shall be emphasized in Chapter **??** as both the most immediate and the most interesting even-dimensional successor to dimension 2 ...

6.3 Gauss–Bonnet–Euler–Betti

Corollary 1 The further formulation of global Gauss–Bonnet in Fig 6.1 is thus afforded.

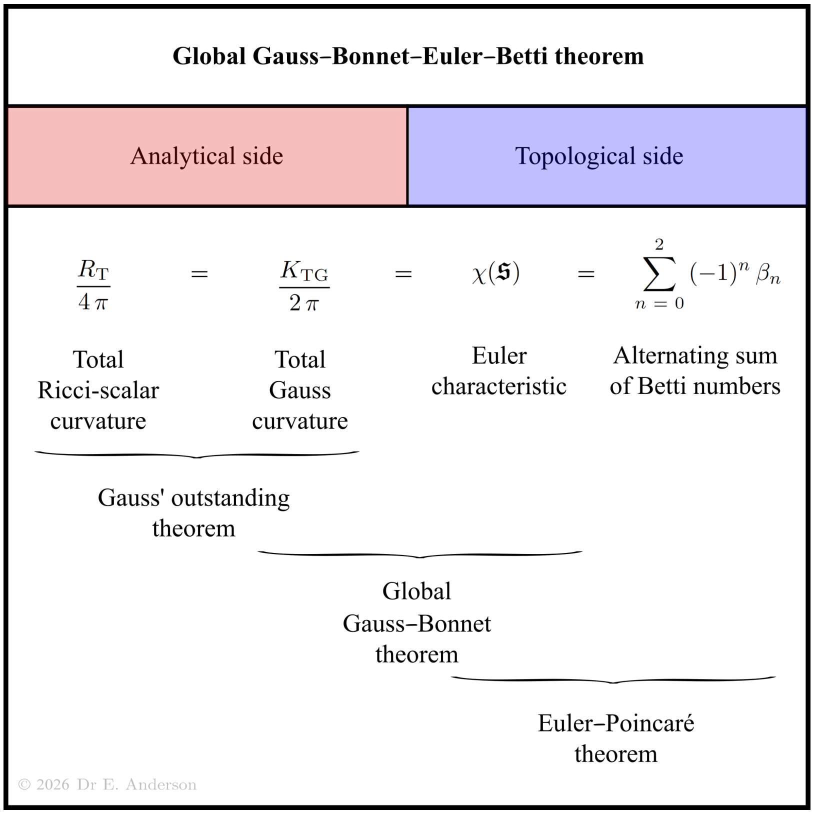

Figure 6.1:

Naming Remark 2 We do not call this Gauss–Bonnet–(Euler–)Poincaré because Gauss–Bonnet–Poincaré(–Hopf) is used for the distinct theorem in the next Chapter. With only some Authors including Hopf's name. So, given that Euler could be co-credited for various global versions of Gauss–Bonnet, we elect to name the current Chapter's theorem via its distinctive feature of containing the Betti numbers...

6.4 Hodge-Theoretic, i.e. harmonic-forms reinterpretation

Structure 1 Betti numbers, i.e. homology group ranks, can be further reinterpreted as follows. The dimensions of arenas of harmonic forms on $\mathfrak{M}^p$.

We work with differential operators on antisymmetric wedge-product spaces of forms,

$$D \ : \ \bigwedge_{i\,=\,1}^{m} \mathfrak{M}^p \ \longrightarrow \ \bigwedge_{i\,=\,1}^{m} \mathfrak{M}^p \ .$$

For some differential operator D , for now the Laplacian. To which some objects with the epithet 'harmonic' are well-known to be associated... We consider in particular

$$\mathrm{dim}\,\mathfrak{Ker}\,D \ .$$

Our choice of differential operator is to be constructed from the bundle in question. Which for now for the usual Gauss–Bonnet is just the tangent bundle. We will further explain this bundle's involvement 2 Chapters along. With generalizations in Chapter **??** .

We then use that

$$b_n \ = \ \mathrm{dim}\,(\,Harm_n\,(\mathfrak{S}^p)\,) \ .$$

Pointer 1 The Hodge interpretation points towards the Gauss–Bonnet theorem being a simple toy version of the much more powerful *Atiyah–Singer theorem*. This uses

$$(\,\mathrm{nullity}\,)\ , \quad \nu \ = \ \mathrm{dim}\,\mathfrak{Ker}\,D$$

Alongside

$$(\,\mathrm{conullity}\,)\ , \quad \nu^* \ = \ \mathrm{dim}\,\mathfrak{Coker}\,D \ = \ \mathrm{dim}\,\mathfrak{Ker}\,D^{\dagger} \ .$$

Some *higher curvature form*, which we further describe in terms of characteristic classes once bundles, and these, are in play, in Chapter **??**. And other *elliptic differential operators* in place of the Laplacian.

Pointer 2 But even the rank–nullity theorem of Linear Algebra can be viewed in terms of kernel and cokernel of a Linear-Algebraic operator [119, 122], And thus as a toy index theorem [123] ...

Remark 1 This is not however the simplest way of seeing that the Gauss–Bonnet theorem is of index type. A simpler story – that of Poincaré and Hopf – is the subject of the next Chapter...

Acknowledgments I thank S. Sánchez for discussions on toy index theorems. And both S. Sánchez and A. Ford for discussions on truer-naming, truer-denotation, and Graph Theory and its generalization to simplicial complexes. And K. Everard, Chris Isham Niall ó Murchadha, Jimmy York Jr and Malcolm MacCallum for previous discussions and/or mentoring. And N. Shepherd-Barron, G. Paternain, M. Dafermos, and K.V. Kuchař for inspiring lectures on extrinsic curvature when I was a young student.

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