Tensor Networks throughout the Basic Theory of STEM. II. Vector Calculus

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Abstract

The current Encyclopaedia is reformulating all of the basic theory of STEM's tensors and other multi-linear arrays in tensor network notation. This gives an index-free formulation, for presenting and conceptualizing about objects, identities and theorems. That is free from various restrictions to the more widely-encountered interior-product index-free formulation. More specifically, the current Article considers tensors with with co-contra distinction. As denoted by a Penrose vertical-variant type of presentation for tensor networks. Article I carried this out for Vector Algebra, while the current Article extends this to Vector Calculus.

We point out that the widespread practise of composing grad , div and curl misses out many higher-derivative combinations. Also freedom from the interior-product formulation changes some of the motivation in deriving identities. Tying this to what occurs in applications such as the Laws of Nature, rather than to the notational limitations of the interior-product formulation. With focus on examples from the 'vection' family of notions, which is well-known via its advection and convection subcases. Advection arises in Fluid Dynamics, While convection and other types of 'vection' such as 'mutual-vection' arise in coupling Fluid Dynamics to other branches of Physics. To the theory of heat's temperature scalar field for convection. And, for 'mutual-vection', e.g. to the magnetic vector field in particular in Astrophysics' (and more generally Plasma Physics') theory of Magnetohydrodynamics.

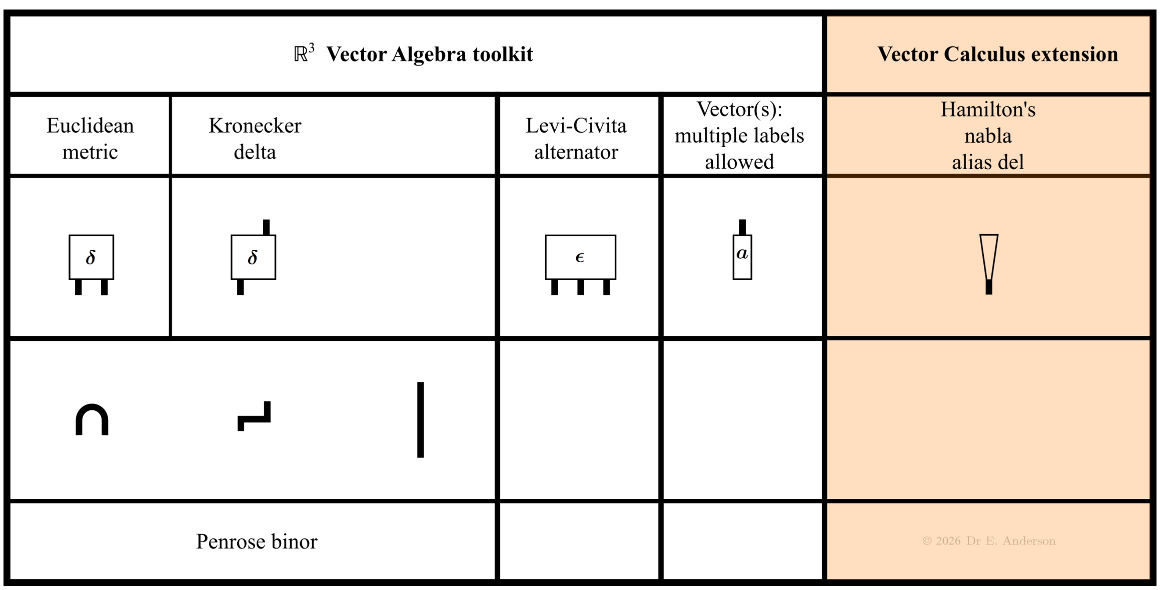

Figure 1:

This Article is ⟨3⟩: accessible to third-year undergraduates.

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1 Introduction

1.1 Vector algebra toolkit

Notational Remark 0 Let us first recap the previous Article [29] on Vector Algebra by writing out its toolkit of elements in tensor network presentation [15, 24, 26, 28, 31, 27, 33, 32] in Fig 1. Using boxes for objects and lines alias *legs* alias *birdtracks* in place of indices. More specifically, we consider tensors with with co-contra distinction [25, 22, 17]. As denoted by a Penrose vertical-variant type of presentation [15, 31, 32, 33] for tensor networks.
Structure 1 Vector Algebra's elements consist of the following (see Fig 1).

1) Vectors $\boldsymbol{a}$, with further labels referring to the bold letter involved and what it means in the model. So abstractly also $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$... And for some specific Physics, $\boldsymbol{u}$ for fluid flow vector field. $\boldsymbol{E}$ and $\boldsymbol{B}$ for electric and magnetic vector fields respectively. And $\boldsymbol{j}$ for electric current vector field.

2) The isotropic tensors [8, 11] supported by $\mathbb{R}^3$. Which are of the 2 following types.

i) The Kronecker delta tensor,[1] whose rank-$(0, 2)$ version is the Euclidean metric. We also display its mixed – rank-$(1, 1)$ version. 2 uses of the mixed version on the metric version returns the $(2, 0)$ version: the corresponding inverse metric. Which is numerically the same as the metric in this case, since the identity matrix is self-inverse!

ii) The Levi–Civita tensor[1] alias alternating tensor alias alternator. We display this in $(0, 3)$ form, so that it can directly manifest its total antisymmetry. Other versions follow from applying the inverse metric to raise whichever combination of the legs.

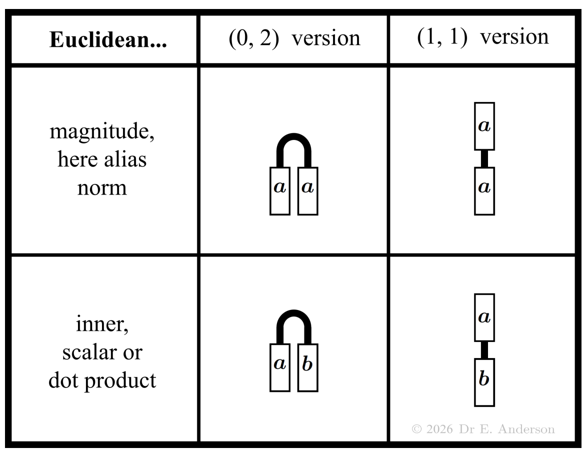

Figure 2:

Notational Remark 1 As regards the $\boldsymbol{\delta}$, there are further notational options at the coordinate-free level, in particular $\mathbb{I}$ standing for the identity matrix. But in any case, we [29] simplify with the Penrose binor [15], leaving us with one less type of boxed object to draw...

Remark 1 See Fig 2 for 2 corresponding formulations of norm and inner product.

Remark 2 It is the displayed alternator that is the 3-d-specific element. n-d sports the n-leg counterpart. With the caveat that the 1-d case is just the zero vector, which is itself just the zero scalar... And the curious and relevant feature that in 2-d the $\boldsymbol{\epsilon}$ and $\boldsymbol{\delta}$ are both 2-leg objects.

[1] These [7] are named after 19th century Mathematician Leopold Kronecker and early 20th century Mathematician Tullio Levi-Civita.

Historical Remark 1 The scalar product is originally due to Grassmann [2]. The vector product first arose fused with the scalar product in the quaternion product, as part of Hamilton's quaternion formulation [1]. Gibbs [5] subsequently isolated the vector product from the quaternion product.

1.2 Extending to the Vector Calculus' toolkit

Structure 1 The new element is the covectorial derivative operator

$$\underline{\boldsymbol{\nabla}} \;=\; \frac{\partial}{\partial \underline{\boldsymbol{x}}} \;.$$

It is a covector because the derivative is with respect to a vector – the (for now Cartesian) coordinate vector – turning its rank upside-down, as follows.

$$\overline{\boldsymbol{x}} \;\longrightarrow\; \frac{\partial}{\partial \overline{\boldsymbol{x}}} \;=\; \underline{\boldsymbol{\nabla}} \;.$$

In components, our derivative operator is

$$\partial_i \;=\; \frac{\partial}{\partial x^i} \;.$$

Naming Remark 1 Our derivative operator is called *del*, with reference to how it is an upside-down Δ . Or *nabla*, after an ancient type of harp.

Historical Remark 2 Prior to its Vector Calculus life, $\boldsymbol{\nabla}$ already featured in Hamilton's study of quaternions.

Notational Remark 2 In tensor network notation, the derivative operator's box is itself nabla-shaped. Were ≥ 1 notion of derivative in play – true in a big follow-up subject: Differential Geometry [23, 17] – then we would label this box with the corresponding type of derivative. But in the current Article, there is just the one, and so we do not bother!

Remark 1 In the $\mathbb{R}^n$ Cartesian setting, the identity in Fig 3.a) holds. Where the commutator – a subcase of Lie bracket – is cast in fireopal. This identity renders it moot whether $\boldsymbol{\nabla}$ acts on vectors or covectors in the context of $\mathbb{R}^n$ in Cartesian coordinates' Vector Calculus. Both it and Subfig b)'s identity for $\mathbb{R}^3$ in general fail in curved spaces and even for curvilinear coordinates still on flat space. Via the metric's coordinate dependence, so that its own derivative arises. And via the generalized alternator carrying a multiplicative factor of the square root of the determinant of the metric: rendering it coordinate-dependent too.

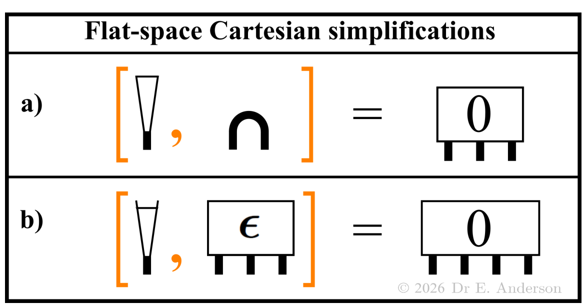

Figure 3:

1.3 grad, div and curl

Structure 1 When nabla acts on a scalar, it is termed the *gradient operator*, which is often denoted by grad . The output is the *gradient* of the corresponding scalar. Not quite as in the gradient of a line, but rather in the sense of the multivariate Calculus generalization of this notion.

Nabla with a forwards-dot is termed the *divergence operator*, which is often denoted by div . Suppose that we act with this dot into a vector. Then since it is a forwards-dot, the derivative acts on this vector. Producing the divergence of this vector.

Nabla with a forwards-cross is termed the *curl operator*, which is often denoted by curl . Suppose that we act with this cross on a vector. Then since it is a forwards-cross, the derivative acts on this vector. Producing the *curl* of this vector.

Remark 2 We take it that the Reader has already seen introductory and conceptual discussions of grad, div and curl . If not, see e.g. [10, 13, 19, 33].

Historical Remark 3 Maxwell [4] tidied up his system of equations for Electromagnetism into quaternionic form. Heaviside [6] introduced the div and curl operators, and reformulated Maxwell's equations in this simpler and subsequently universally-adopted form. This notation turns out to be very useful in other branches of Physics as well [18, 14].

Pointer 1 At least until the advent of higher-rank tensors and higher-dimensional spaces... Though there is also an intervening swathe of theory in which div and curl's generalization in terms of forms and their exterior derivative [12, 16] serves well. And also basic cohomological [20] and Differential Geometric [16] generalizations.

Remark 1 curl exhibits another [29] underline counting problem. This one is resolved slightly differently, since ∇ is a covector; see Fig 4.

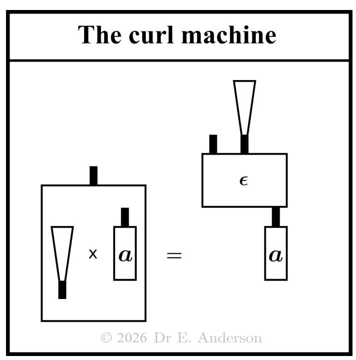

Figure 4:

Remark 2 See within row 1 of Fig 5 for the above 3 differential operators, and row 2 for the corresponding outputs.

1.4 And further options

Structure 1 Vector Calculus is in fact larger than what some basic courses identify. Though in any case standard courses on Tensor Calculus [25, 31] are bigger still.

'Larger' because $\boldsymbol{\delta}$ and $\boldsymbol{\epsilon}$ have 2 and 3 slots respectively. And placing $\boldsymbol{\nabla}$ in the slots not used above is not equivalent. For derivatives act to the right...

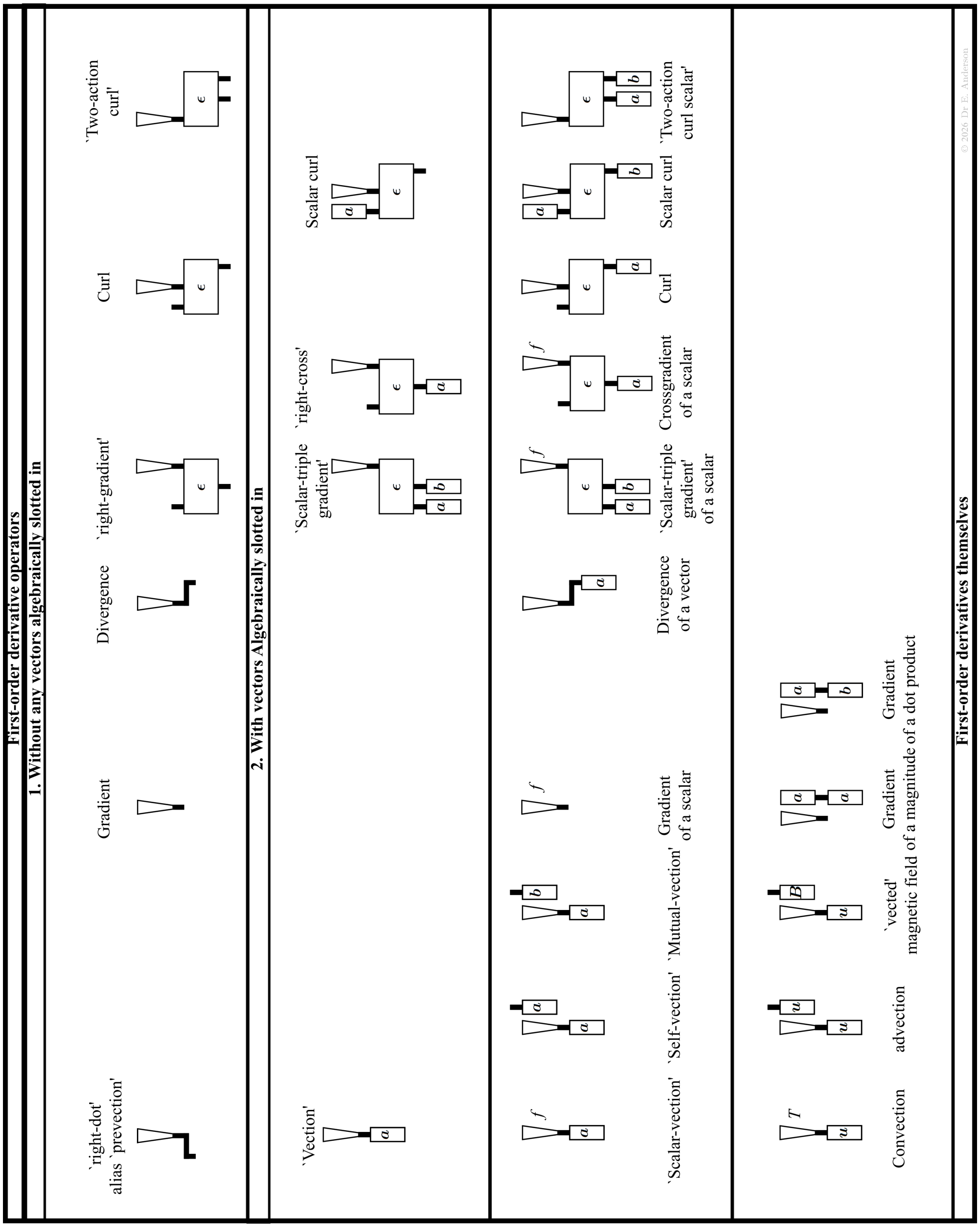

Figure 5:

Firstly suppose that we place $\boldsymbol{\nabla}$ on the second slot of $\boldsymbol{\delta}$: the backwards-pointing dot. This then means that $\boldsymbol{\nabla}$ does not act on the vector on the other side of the dot. This gives the '*vection*' differential operator

$$\overline{\boldsymbol{a}} \cdot \underline{\boldsymbol{\nabla}} \ .$$

Secondly suppose that we place $\boldsymbol{\nabla}$ on the third slot of $\boldsymbol{\epsilon}$: the backwards-pointing cross. This leaves $\boldsymbol{\nabla}$ unable to act on any slot's object.

Finally suppose that we place ∇ on the first slot of ϵ. This leaves ∇ able to act on both slots' object'. Though if only 1 slot is occupied, antisymmetry dictates that we are back to curl, up to a possible minus-sign factor.

See *the rest of* row 1 and 2 for each of the above. With a few more specific common types of 'vection' picked out in the third row. Where T is the temperature scalar field. A further such is that when $\boldsymbol{a}$ is a unit vector, then the 'vection' operator is called the *directional derivative operator in the direction of* $\boldsymbol{a}$ [9].

And 'bigger still' by tensors of all ranks participating in contractions and in being acted upon by derivative operators...

Notational Remark 3 The right and left wrench-shaped Penrose binors merit a brief discussion. The right one is present in the figure for div. It serves to place the second object to the right of the derivative, and so persists. As regards the left one, derivatives act to the right, and down is not right. So we can subsequently phase these out for a vertical binor...

In this way, the humble div already breaks the vertical aesthetic.

[Unless one starts to adorn the nabla box with an arrow of direction of action...]

Exercise 1$^-$ Write out the linearity and product-rule identities for grad, div and curl in tensor network notation. Also extend to 'vection' and the other combinations of derivatives covered in the current subsection.

1.5 Zero-derivative definitions and interpretations

Remark 1 Standard and non-standard such are tabulated in Fig 6.a) and b) respectively.

Naming Remark 1 *Stationary* refers to the Calculus' stationarity condition for finding stationary points. *Irrotational* and *incompressible*, when taken literally, refer to conditions on a fluid flow $\boldsymbol{u}$. While *solenoidal*, when taken literally, refers to a condition on a magnetic field $\boldsymbol{B}$.

Remark 2 Subsec 1.4's extra types of derivatives lead to extra zero cases (Subfig b). For all that most of them immediately decompose into standard cases. Be these from Subfig a) or from how dots and crosses of vectors can be zero.

Remark 3 'Vect' moreover stands alone in giving 3 problems which do not immediately break down into elementary cases. Two are highlighted in yellow.

The first is the 'self-vect-free' problem. I.e. which vector fields do not 'vect' themselves? This is well-determined: 3 equations in 3 unknowns.

The second is the 'mutual-vect' problem. This might be viewed as not well-determined: 3 equations in 6 unknowns. But $\boldsymbol{a}$ could be taken to be a prescribed vectorial function. I.e. given a vector field, what other vector fields are not vected by it?

Another way around this determinedness problem gives our third problem: the *2-way mutual-vect-free problem*. Here the $\boldsymbol{a} \leftrightarrow \boldsymbol{b}$ of our equation is adjoined. I.e. can we find a pair of vector fields that do not 'vect' each other? The system is now interpreted as 6 equations in 6 unknowns: well-determined.

Exercise 2 Investigate these 3 problems' solutions in a) 1-d, b)$^+$ 2-d, and c)$^{++}$ 3-d.

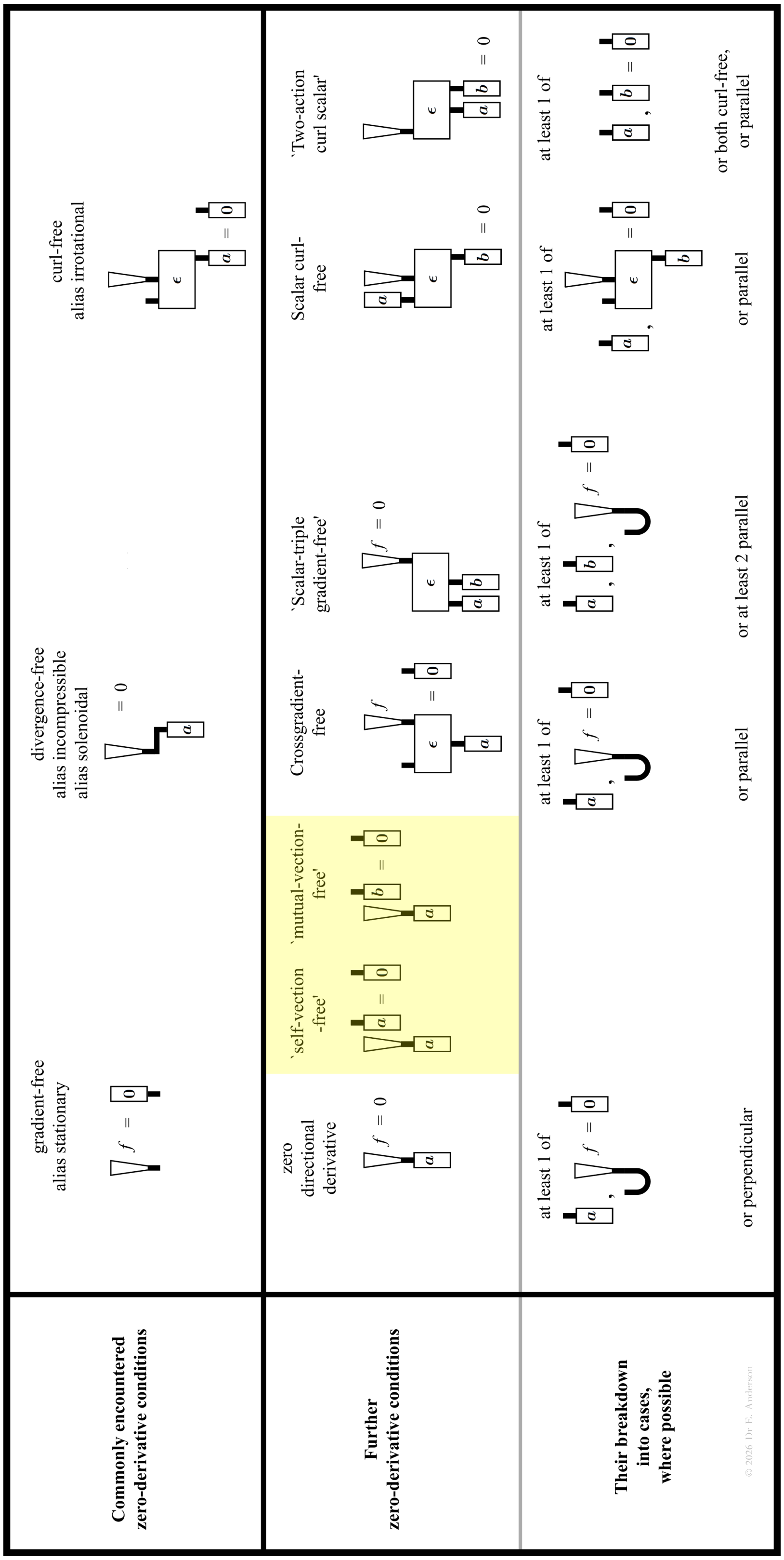
Commonly encountered zero-derivative conditions
gradient-free alias stationary
f = 0
divergence-free alias incompressible alias solenoidal
= 0
curl-free alias irrotational
= 0
Further zero-derivative conditions
zero directional derivative
f = 0
'self-vection -free'
= 0
'mutual-vection-free'
= 0
Crossgradient-free
f
= 0
'Scalar-triple gradient-free'
f = 0
Scalar curl-free
= 0
'Two-action curl scalar'
= 0
Their breakdown into cases, where possible
at least 1 of
f = 0
or perpendicular
at least 1 of
f = 0
or parallel
at least 1 of
f = 0
or at least 2 parallel
at least 1 of
= 0
or parallel
at least 1 of
= 0
or both curl-free, or parallel
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Figure 6:

2 First-order linear identities

2.1 The big integral theorems

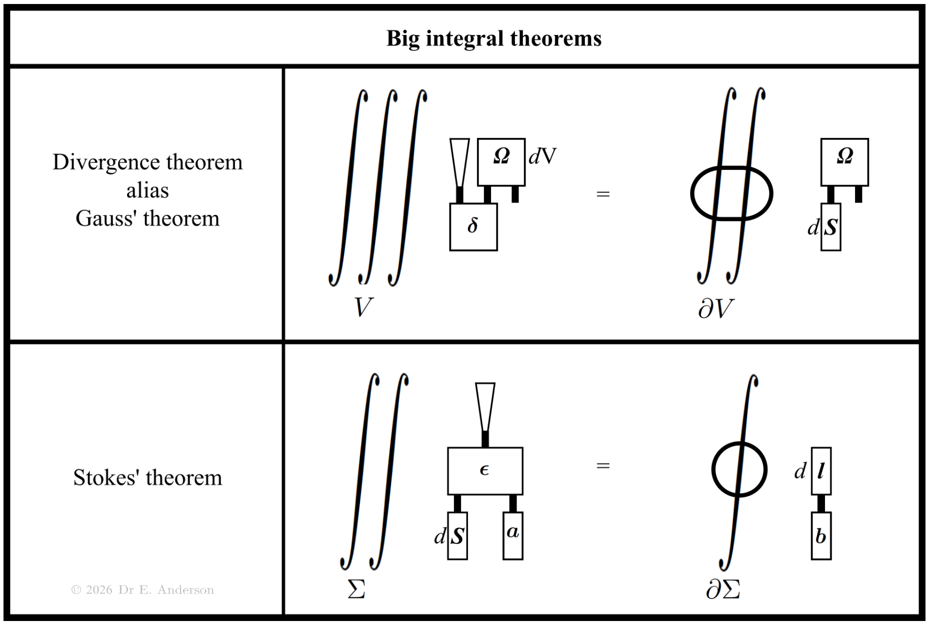

Figure 7:

Remark 1 These are well-known to enter div and curl's local conceptualizations. I.e. flow out of the surface of a small blob and circulation around a small loop.

2.2 Two preliminary identities

Remark 1 Subfig 8.a) establishes that the 2-action curl just reduces to the difference of 2 scalar curls.

Remark 2 Subfig b) is the standard identity for the curl of a cross.

2.3 Advection is not independent

Remark 1] Subfig c) readily follows from applying the Kronecker delta theorem to the RHS's last term.

2.4 'Mutual-vection' is not independent

Remark 1 Subfig d) gives the cross of a curl . This gives the 'mutual-vected' term. In an equation with the following limitations.

a) It is an exchange equation converting the mutual-vected term to a nonstandard term.

b) This nonstandard term cannot be expressed in interior product notation.

Remark 2 b) is remedied by symmetrizing: adding the same equation with the role of the 2 vectors $\boldsymbol{a}, \boldsymbol{b}$ reversed. The product rule then groups up the 2 nonstandard terms into a single interior product term: Subfig e).

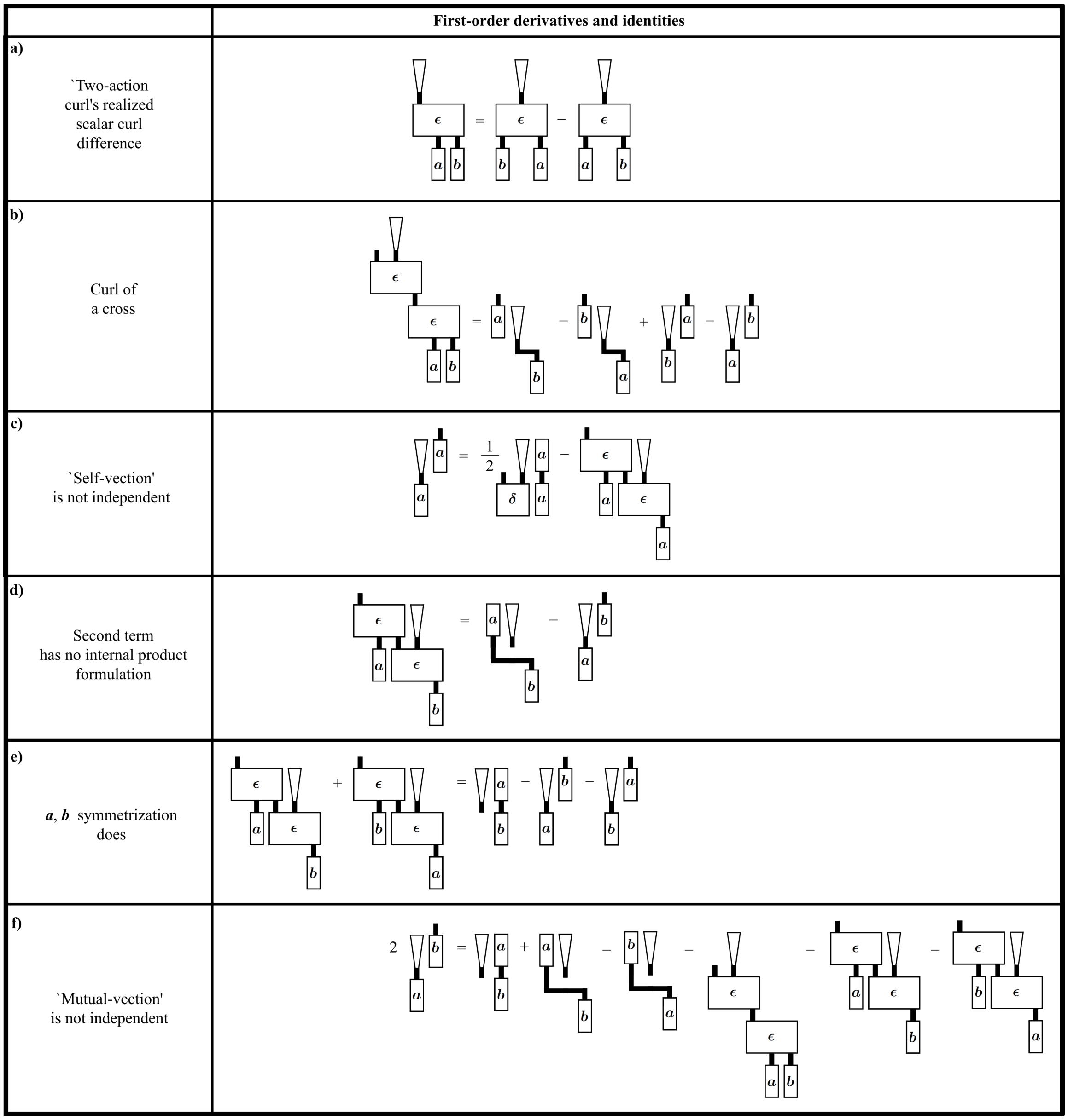

Figure 8:

Remark 3 This still does not isolate $\boldsymbol{a}$ ‘vects’ $\boldsymbol{b}$. To do this, observe that b) contains a different linear combination of these terms. And so can be combined with Subfig e)’s symmetrized sum to make $\boldsymbol{a}$ ‘vects’ $\boldsymbol{b}$. the subject: subfig f).

Remark 4 The tensor network notation moreover frees index-free formulations from having a preference for terms that admit an interior-product formulation. Permitting us to concentrate, rather, on combinations occurring in the Laws of Nature...

As we shall recollect, and cast in tensor network formulation in a subsequent Article [30], ‘vection’ arises from the Balancing Principle on a small blob, in the *material derivative* combination. With the mass-conservation balance contributing a first ‘vected’ scalar: density. And Euler’s momentum balance contributing advection itself. Further ‘vection’ terms arise upon coupling Fluid Dynamics with e.g. the theory of Heat or of Electromagnetism. Returning coupled systems for hot fluids and magnetized fluids respectively. A generic word for ‘magnetized fluids’ is *plasma*, of relevance for instance to stars and to humanity’s attempts to build fusion reactors...

3 Second-order linear identities

3.1 'The boundary of a boundary is zero' [16]

Remark 1 See Fig 9.a-b) for two commonly-encountered versions of this.

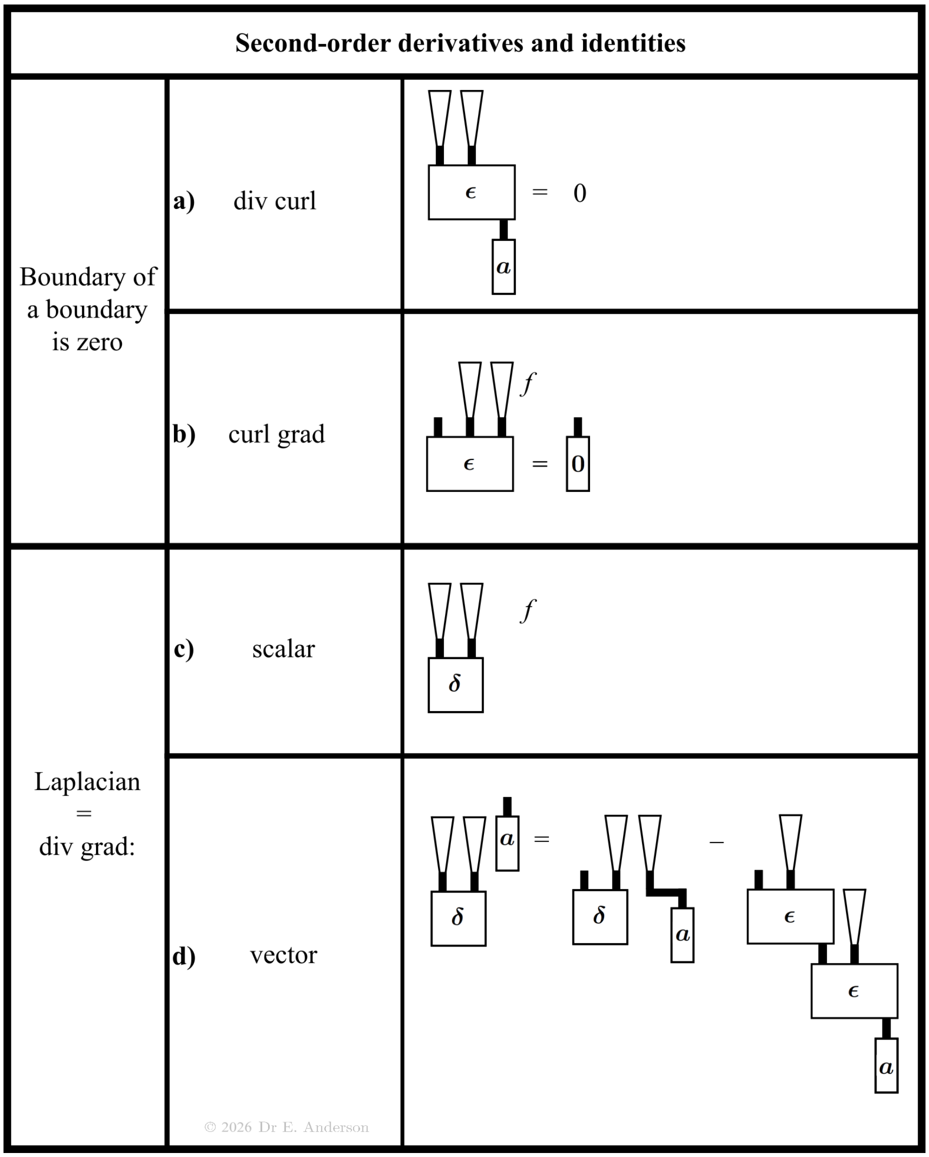

Figure 9:

3.2 The scalar and vector Laplacians

Remark 1 Subfig c)'s case is even more straightforward than the previous subsection's two second-derivative combinations. While Subfig d)'s is another identity that readily follows from using the Kronecker delta theorem on the last term of the RHS...

4 First-order quadratic identities

Remark 1 In each of the follow Exercises, present your final answer in tensor network notation.

Exercise 2 The *vorticity* of a fluid is given by

$$\overline{\boldsymbol{w}} \;=\; \underline{\boldsymbol{\nabla}} \times \overline{\boldsymbol{u}} \,. \tag{1}$$

What does the Kronecker delta theorem equate

$$||\overline{\boldsymbol{w}}||^2$$

to?

Exercise 3 In the regime in which Ampére's law applies in vacuo, the electric current vector is given by

$$\overline{\boldsymbol{j}} \;=\; \frac{1}{\mu_0}\underline{\boldsymbol{\nabla}} \times \overline{\boldsymbol{B}} \,. \tag{2}$$

For (magnetic) permeability in vacuo μ_0 , constant, and magnetic vector field $\boldsymbol{B}$. What does the Kronecker delta theorem equate

$$\boldsymbol{w} \cdot \boldsymbol{j}$$

to?

Exercise 4 In each of the previous, which terms can you isolate as the subject of each equation? In the manner that Subsections 2.3–2.4 isolated the advection and mutual-vection terms, if needs be by using further supporting identities. Free from role-reversed copies of the same object in Exercise 3's case. And free from terms admitting no interior product formulations in both Exercises.

5 Further second-order derivative identities

Exercise 5$^+$ (Very long). Find identities for the following.

a)	$(\overline{\boldsymbol{a}} \cdot \underline{\boldsymbol{\nabla}})$	$(\overline{\boldsymbol{a}} \cdot \underline{\boldsymbol{\nabla}})\, f$
b)	$(\overline{\boldsymbol{a}} \cdot \underline{\boldsymbol{\nabla}})$	$(\overline{\boldsymbol{b}} \cdot \underline{\boldsymbol{\nabla}})\, f$
c)	$(\overline{\boldsymbol{a}} \cdot \underline{\boldsymbol{\nabla}})$	$(\overline{\boldsymbol{a}} \cdot \underline{\boldsymbol{\nabla}})\, \overline{\boldsymbol{b}}$
d)	$(\overline{\boldsymbol{a}} \cdot \underline{\boldsymbol{\nabla}})$	$(\overline{\boldsymbol{b}} \cdot \underline{\boldsymbol{\nabla}})\, \overline{\boldsymbol{c}}$
e)	$(\overline{\boldsymbol{a}} \cdot \underline{\boldsymbol{\nabla}})$	$\overline{\boldsymbol{\nabla}}\, f$
f)	$(\overline{\boldsymbol{a}} \cdot \underline{\boldsymbol{\nabla}})$	$(\underline{\boldsymbol{\nabla}} \times \overline{\boldsymbol{a}})$
g)	$(\overline{\boldsymbol{a}} \cdot \underline{\boldsymbol{\nabla}})$	$(\underline{\boldsymbol{\nabla}} \times \overline{\boldsymbol{b}})$
h)	$(\overline{\boldsymbol{a}} \cdot \underline{\boldsymbol{\nabla}})$	$(\underline{\boldsymbol{\nabla}} \cdot \overline{\boldsymbol{a}})$
i)	$(\overline{\boldsymbol{a}} \cdot \underline{\boldsymbol{\nabla}})$	$(\underline{\boldsymbol{\nabla}} \cdot \overline{\boldsymbol{b}})$
j)	$((\overline{\boldsymbol{\nabla}} f) \cdot \underline{\boldsymbol{\nabla}})$	$\overline{\boldsymbol{a}}$
k)	$((\underline{\boldsymbol{\nabla}} \times \overline{\boldsymbol{a}}) \cdot \underline{\boldsymbol{\nabla}})$	$\overline{\boldsymbol{a}}$
l)	$((\underline{\boldsymbol{\nabla}} \times \overline{\boldsymbol{a}}) \cdot \underline{\boldsymbol{\nabla}})$	$\overline{\boldsymbol{b}}$

Present all answers in tensor network notation!

[Among these, f) includes a 'fluid' vecting its vorticity. And g) is the mutual counterpart of this, for instance a magnetized fluid 'vecting' electric current.]

Open Exercise 6 Extend the previous Exercise to double-derivative expressions involving 1 or 2 uses of yet further types of derivative from the top half of Fig 5.

6 The Helmholtz decomposition in tensor network notation

Remark 1 This [3, 9, 21] is provided in tensor network notation in Fig 10 It has been described as the *Fundamental Theorem of Vector Calculus* (FuToVC for future reference).

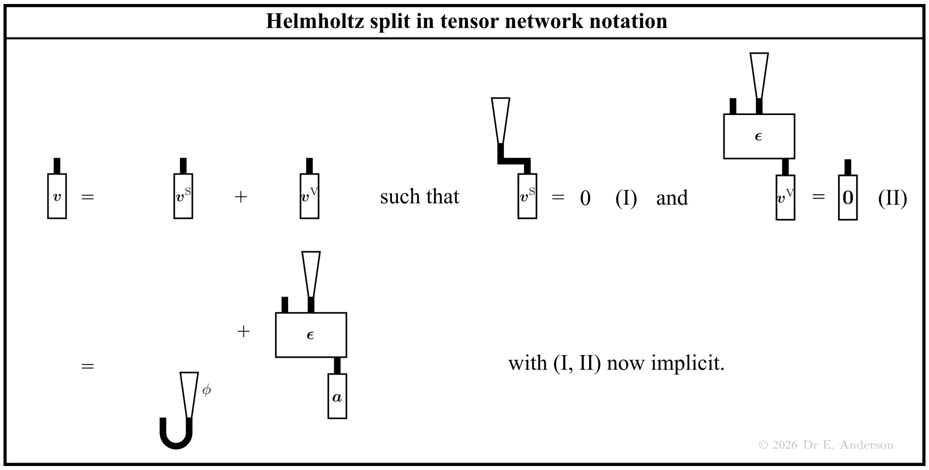

Figure 10:

Exercise 7 Investigate this decomposition's gauge freedom.

[This is a – Exercise, assuming that you have seen some sort of Gauge Theory before...]

Acknowledgements I thank S. Sánchez for teaching me the theory of how to draw accurate families of figures at very low time-cost per new figure. And K. Everard for proofreading an earlier draft.

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