

Tensor Networks throughout the Basic Theory of STEM.

II. Vector Calculus

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Abstract

The current Encyclopaedia is reformulating all of the basic theory of STEM's tensors and other multi-linear arrays in tensor network notation. This gives an index-free formulation, for presenting and conceptualizing about objects, identities and theorems. That is free from various restrictions to the more widely-encountered interior-product index-free formulation. More specifically, the current Article considers tensors with with co-contraction distinction. As denoted by a Penrose vertical-variant type of presentation for tensor networks. Article I carried this out for Vector Algebra, while the current Article extends this to Vector Calculus.

We point out that the widespread practise of composing grad , div and curl misses out many higher-derivative combinations. Also freedom from the interior-product formulation changes some of the motivation in deriving identities. Tying this to what occurs in applications such as the Laws of Nature, rather than to the notational limitations of the interior-product formulation. With focus on examples from the 'vection' family of notions, which is well-known via its advection and convection subcases. Advection arises in Fluid Dynamics, While convection and other types of 'vection' such as 'mutual-vection' arise in coupling Fluid Dynamics to other branches of Physics. To the theory of heat's temperature scalar field for convection. And, for 'mutual-vection', e.g. to the magnetic vector field in particular in Astrophysics' (and more generally Plasma Physics') theory of Magnetohydrodynamics.

$\mathbb{R}^3$ Vector Algebra toolkit				Vector Calculus extension
Euclidean metric	Kronecker delta	Levi-Civita alternator	Vector(s): multiple labels allowed	Hamilton's nabla alias del
				
  				
Penrose binor				© 2026 Dr E. Anderson

Figure 1:

This Article is [\(3\)](#): accessible to third-year undergraduates.

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1 Introduction

1.1 Vector algebra toolkit

Notational Remark 0 Let us first recap the previous Article [31] on Vector Algebra by writing out its toolkit of elements in tensor network presentation [15, 25, 27, 29, 33, 28, 35, 34] in Fig 1. Using boxes for objects and lines alias *legs* alias *birdtracks* in place of indices. More specifically, we consider tensors with with co-contra distinction [26, 23, 17]. As denoted by a Penrose vertical-variant type of presentation [15, 33, 34, 35] for tensor networks.

Structure 1 Vector Algebra's elements consist of the following (see Fig 1).

1) Vectors $\mathbf{a}$, with further labels referring to the bold letter involved and what it means in the model. So abstractly also $\mathbf{b}, \mathbf{c}, \mathbf{d} \dots$. And for some specific Physics, $\mathbf{u}$ for fluid flow vector field. $\mathbf{E}$ and $\mathbf{B}$ for electric and magnetic vector fields respectively. And $\mathbf{j}$ for electric current vector field.

2) The isotropic tensors [8, 11] supported by $\mathbb{R}^3$. Which are of the 2 following types.

i) The Kronecker delta tensor,¹ whose rank-(0, 2) version is the Euclidean metric. We also display its mixed – rank-(1, 1) version. 2 uses of the mixed version on the metric version returns the (2, 0) version: the corresponding inverse metric. Which is numerically the same as the metric in this case, since the identity matrix is self-inverse!

ii) The Levi-Civita tensor¹ alias alternating tensor alias alternator. We display this in (0, 3) form, so that it can directly manifest its total antisymmetry. Other versions follow from applying the inverse metric to raise whichever combination of the legs.

Euclidean...	(0, 2) version	(1, 1) version
magnitude, here alias norm		
inner, scalar or dot product		

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Figure 2:

Notational Remark 1 As regards the δ , there are further notational options at the coordinate-free level, in particular $\mathbb{I}$ standing for the identity matrix. But in any case, we [31] simplify with the Penrose binor [15], leaving us with one less type of boxed object to draw...

Remark 1 See Fig 2 for 2 corresponding formulations of norm and inner product.

Remark 2 It is the displayed alternator that is the 3- d -specific element. n - d sports the n -leg counterpart. With the caveat that the 1- d case is just the zero vector, which is itself just the zero scalar... And the curious and relevant feature that in 2- d the ϵ and δ are both 2-leg objects.

¹These [7] are named after 19th century Mathematician Leopold Kronecker and early 20th century Mathematician Tullio Levi-Civita.

Historical Remark 1 The scalar product is originally due to Grassmann [2]. The vector product first arose fused with the scalar product in the quaternion product, as part of Hamilton’s quaternion formulation [1]. Gibbs [5] subsequently isolated the vector product from the quaternion product.

1.2 Extending to the Vector Calculus’ toolkit

Structure 1 The new element is the covectorial derivative operator

$$\underline{\nabla} = \frac{\partial}{\partial \underline{x}} .$$

It is a covector because the derivative is with respect to a vector – the (for now Cartesian) coordinate vector – turning its rank upside-down, as follows.

$$\bar{x} \longrightarrow \frac{\partial}{\partial \bar{x}} = \underline{\nabla} .$$

In components, our derivative operator is

$$\partial_i = \frac{\partial}{\partial x^i} .$$

Naming Remark 1 Our derivative operator is called *del*, with reference to how it is an upside-down Δ . Or *nabla*, after an ancient type of harp.

Historical Remark 2 Prior to its Vector Calculus life, $\underline{\nabla}$ already featured in Hamilton’s study of quaternions.

Notational Remark 2 In tensor network notation, the derivative operator’s box is itself nabla-shaped. Were ≥ 1 notion of derivative in play – true in a big follow-up subject: Differential Geometry [24, 17] – then we would label this box with the corresponding type of derivative. But in the current Article, there is just the one, and so we do not bother!

Remark 1 In the $\mathbb{R}^n$ Cartesian setting, the identity in Fig 3.a) holds. Where the commutator – a subcase of Lie bracket – is cast in fireopal. This identity renders it moot whether $\underline{\nabla}$ acts on vectors or covectors in the context of $\mathbb{R}^n$ in Cartesian coordinates’ Vector Calculus. Both it and Subfig b)’s identity for $\mathbb{R}^3$ in general fail in curved spaces and even for curvilinear coordinates still on flat space. Via the metric’s coordinate dependence, so that its own derivative arises. And via the generalized alternator carrying a multiplicative factor of the square root of the determinant of the metric: rendering it coordinate-dependent too.

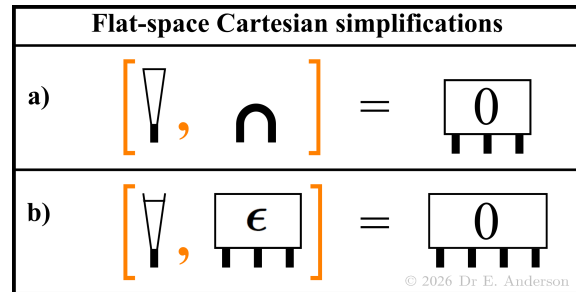


Figure 3:

1.3 grad, div and curl

Structure 1 When nabla acts on a scalar, it is termed the *gradient operator*, which is often denoted by grad . The output is the *gradient* of the corresponding scalar. Not quite as in the gradient of a line, but rather in the sense of the multivariate Calculus generalization of this notion.

Nabla with a forwards-dot is termed the *divergence operator*, which is often denoted by div . Suppose that we act with this dot into a vector. Then since it is a forwards-dot, the derivative acts on this vector. Producing the divergence of this vector.

Nabla with a forwards-cross is termed the *curl operator*, which is often denoted by curl . Suppose that we act with this cross on a vector. Then since it is a forwards-cross, the derivative acts on this vector. Producing the *curl* of this vector.

Remark 2 We take it that the Reader has already seen introductory and conceptual discussions of grad , div and curl . If not, see e.g. [10, 13, 19, 35].

Historical Remark 3 Maxwell [4] tidied up his system of equations for Electromagnetism into quaternionic form. Heaviside [6] introduced the div and curl operators, and reformulated Maxwell's equations in this simpler and subsequently universally-adopted form. This notation turns out to be very useful in other branches of Physics as well [18, 14].

Pointer 1 (3-5) At least until the advent of higher-rank tensors and higher-dimensional spaces... Though there is also an intervening swathe of theory in which div and curl 's generalization in terms of forms and their exterior derivative [12, 16] serves well. And also basic cohomological [20] and Differential Geometric [16] generalizations.

Remark 1 curl exhibits another [31] underline counting problem. This one is resolved slightly differently, since ∇ is a covector; see Fig 4.

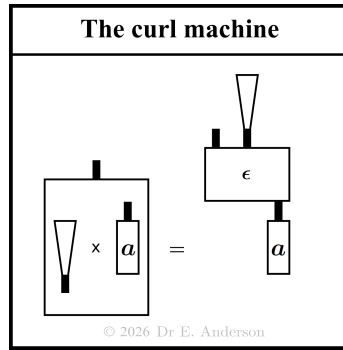


Figure 4:

Remark 2 See within row 1 of Fig 5 for the above 3 differential operators, and row 2 for the corresponding outputs.

1.4 And further options

Structure 1 Vector Calculus is in fact larger than what some basic courses identify. Though in any case standard courses on Tensor Calculus (2-5) [26, 33] are bigger still.

'Larger' because δ and ϵ have 2 and 3 slots respectively. And placing ∇ in the slots not used above is not equivalent. For derivatives act to the right...

Finally suppose that we place ∇ on the first slot of ϵ . This leaves ∇ able to act on both slots' object'. Though if only 1 slot is occupied, antisymmetry dictates that we are back to curl , up to a possible minus-sign factor.

See *the rest of* row 1 and 2 for each of the above. With a few more specific common types of 'vection' picked out in the third row. Where T is the temperature scalar field. A further such is that when $\mathbf{a}$ is a unit vector, then the 'vection' operator is called the *directional derivative operator in the direction of $\mathbf{a}$* [9].

And 'bigger still' by tensors of all ranks participating in contractions and in being acted upon by derivative operators...

Notational Remark 3 The right and left wrench-shaped Penrose binors merit a brief discussion. The right one is present in the figure for div . It serves to place the second object to the right of the derivative, and so persists. As regards the left one, derivatives act to the right, and down is not right. So we can subsequently phase these out for a vertical binor...

In this way, the humble div already breaks the vertical aesthetic.

[Unless one starts to adorn the nabla box with an arrow of direction of action...]

Exercise 1 Write out the linearity and product-rule identities for grad , div and curl in tensor network notation. Also extend to 'vection' and the other combinations of derivatives covered in the current subsection.

1.5 Zero-derivative definitions and interpretations

Remark 1 Standard and non-standard such are tabulated in Fig 6.a) and b) respectively.

Naming Remark 1 *Stationary* refers to the Calculus' stationarity condition for finding stationary points. *Irrrotational* and *incompressible*, when taken literally, refer to conditions on a fluid flow $\mathbf{u}$. While *solenoidal*, when taken literally, refers to a condition on a magnetic field $\mathbf{B}$.

Remark 2 Subsec 1.4's extra types of derivatives lead to extra zero cases (Subfig b). For all that most of them immediately decompose into standard cases. Be these from Subfig a) or from how dots and crosses of vectors can be zero.

Remark 3 'Vect' moreover stands alone in giving 3 problems which do not immediately break down into elementary cases. Two are highlighted in yellow.

The first is the 'self-vect-free' problem. I.e. which vector fields do not 'vect' themselves? This is well-determined: 3 equations in 3 unknowns.

The second is the 'mutual-vect' problem. This might be viewed as not well-determined: 3 equations in 6 unknowns. But $\mathbf{a}$ could be taken to be a prescribed vectorial function. I.e. given a vector field, what other vector fields are not vected by it?

Another way around this determinedness problem gives our third problem: the *2-way mutual-vect-free problem*. Here the $\mathbf{a} \leftrightarrow \mathbf{b}$ of our equation is adjoined. I.e. can we find a pair of vector fields that do not 'vect' each other? The system is now interpreted as 6 equations in 6 unknowns: well-determined.

Exercise 2 Investigate these 3 problems' solutions in a) 1-d, b)⁺ 2-d, and c)⁺⁺ 3-d. You may assume that the input vectors are smooth.

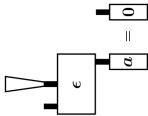
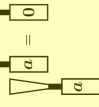
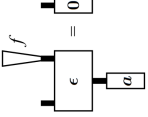
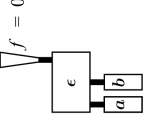
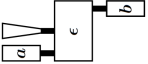
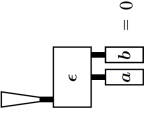
Commonly encountered zero-derivative conditions	gradient-free alias stationary  $\nabla f = 0$ divergence-free alias incompressible alias solenoidal  $\nabla \cdot a = 0$ curl-free alias irrotational  $\nabla \times a = 0$
Further zero-derivative conditions	zero directional derivative  $\nabla f = 0$ 'self-vection-free'  $\nabla a = 0$ 'mutual-vection-free'  $\nabla b = 0$ Crossgradient-free  $\nabla f = 0$ 'Scalar-triple gradient-free'  $\nabla f = 0$ Scalar curl-free  $\nabla \times a = 0$ 'Two-action curl scalar'  $\nabla \times a = 0$
Their breakdown into cases, where possible	at least 1 of $\nabla a, \nabla b, \nabla f = 0$ or perpendicular at least 1 of $\nabla a, \nabla b, \nabla f = 0$ or parallel at least 1 of $\nabla a, \nabla b, \nabla f = 0$ or at least 2 parallel at least 1 of $\nabla a, \nabla b, \nabla f = 0$ or parallel at least 1 of $\nabla a, \nabla b, \nabla f = 0$ or both curl-free, or parallel

Figure 6:

2 First-order linear identities

2.1 The big integral theorems

Big integral theorems	
Divergence theorem alias Gauss' theorem	
Stokes' theorem © 2026 Dr E. Anderson	

Figure 7:

Remark 1 These are well-known to enter div and curl 's local conceptualizations. I.e. flow out of the surface of a small blob and circulation around a small loop.

2.2 Two preliminary identities

Remark 1 Subfig 8.a) establishes that the 2-action curl just reduces to the difference of 2 scalar curls.

Remark 2 Subfig b) is the standard identity for the curl of a cross.

2.3 Advection is not independent

Remark 1] Subfig c) readily follows from applying the Kronecker delta theorem to the RHS's last term.

2.4 'Mutual-vection' is not independent

Remark 1 Subfig d) gives the cross of a curl . This gives the 'mutual-vected' term. In an equation with the following limitations.

- i) It is an exchange equation converting the mutual-vected term to a nonstandard term.
- ii) This nonstandard term cannot be expressed in interior product notation.

Remark 2 ii) is remedied by symmetrizing: adding the same equation with the role of the 2 vectors $\mathbf{a}, \mathbf{b}$ reversed. The product rule then groups up the 2 nonstandard terms into a single interior product term: Subfig e).

	First-order derivatives and identities
a) 'Two-action curl's realized scalar curl difference	
b) Curl of a cross	
c) 'Self-vection' is not independent	
d) Second term has no internal product formulation	
e) a, b symmetrization does	
f) 'Mutual-vection' is not independent	

Figure 8:

Remark 3 This still does not isolate a 'vects' b . To do this, observe that Subfig b)'s identity contains a different linear combination of these terms. And so it and Subfig e)'s symmetrized sum can jointly be used to make a 'vects' b . the subject: Subfig f).

Remark 4 The tensor network notation moreover frees index-free formulations from having a preference for terms that admit an interior-product formulation. Permitting us to concentrate, rather, on combinations occurring in the Laws of Nature...

As we shall recollect, and cast in tensor network formulation in a subsequent Article [32], 'vection' arises from the Balancing Principle on a small blob, in the *material derivative* combination. With the mass-conservation balance contributing a first 'vected' scalar: density. And Euler's momentum balance contributing advection itself. Further 'vection' terms arise upon coupling Fluid Dynamics with e.g. the theory of Heat or of Electromagnetism. Returning coupled systems for hot fluids [\(3\)](#) and magnetized fluids [\(4\)](#) respectively. A generic word for 'magnetized fluids' is *plasma*, of relevance for instance to stars and to humanity's attempts to build fusion reactors...

3 Second-order linear identities

3.1 ‘The boundary of a boundary is zero’ [16]

Remark 1 See Fig 9.a-b) for two commonly-encountered versions of this.

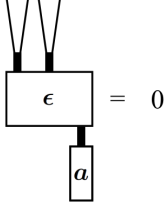
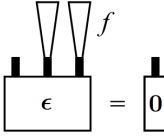
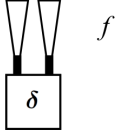
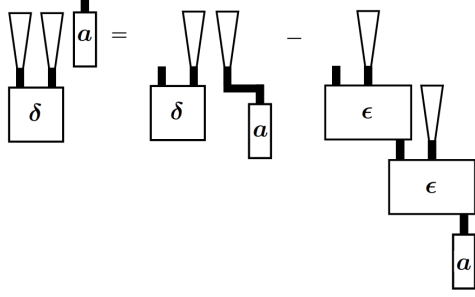
Second-order derivatives and identities		
Boundary of a boundary is zero	a) div curl	 = 0
	b) curl grad	 = 0
Laplacian = div grad:	c) scalar	
	d) vector	
© 2026 Dr E. Anderson		

Figure 9:

3.2 The scalar and vector Laplacians

Remark 1 Subfig c)’s case is even more straightforward than the previous subsection’s two second-derivative combinations. While Subfig d)’s is another identity that readily follows from using the Kronecker delta theorem on the last term of the RHS...

4 First-order quadratic identities

Remark 1 In each of the follow Exercises, present your final answer in tensor network notation.

Exercise 2 The *vorticity* of a fluid is given by

$$\overline{\mathbf{w}} = \underline{\nabla} \times \overline{\mathbf{u}} . \quad (1)$$

What does the Kronecker delta theorem equate

$$||\overline{\mathbf{w}}||^2$$

to?

Exercise 3 (4) In the regime in which Ampère’s law applies in vacuo, the electric current vector is given by

$$\overline{\mathbf{j}} = \frac{1}{\mu_0} \underline{\nabla} \times \overline{\mathbf{B}} . \quad (2)$$

For (magnetic) permeability in vacuo μ_0 , constant, and magnetic vector field $\mathbf{B}$. What does the Kronecker delta theorem equate

$$\mathbf{w} \cdot \mathbf{j}$$

to?

Exercise 4 (4) In each of the previous, which terms can you isolate as the subject of each equation? In the manner that Subsections 2.3–2.4 isolated the advection and mutual-vection terms, if needs be by using further supporting identities. Free from role-reversed copies of the same object in Exercise 3’s case. And free from terms admitting no interior product formulations in both Exercises.

5 Further second-order derivative identities

Exercise 5⁺ (Very long). Find identities for the following.

a)	$(\overline{\mathbf{a}} \cdot \underline{\nabla})$	$(\overline{\mathbf{a}} \cdot \underline{\nabla}) f$
b)	$(\overline{\mathbf{a}} \cdot \underline{\nabla})$	$(\overline{\mathbf{b}} \cdot \underline{\nabla}) f$
c)	$(\overline{\mathbf{a}} \cdot \underline{\nabla})$	$(\overline{\mathbf{a}} \cdot \underline{\nabla}) \overline{\mathbf{b}}$
d)	$(\overline{\mathbf{a}} \cdot \underline{\nabla})$	$(\overline{\mathbf{b}} \cdot \underline{\nabla}) \overline{\mathbf{c}}$
e)	$(\overline{\mathbf{a}} \cdot \underline{\nabla})$	$\overline{\nabla} f$
f)	$(\overline{\mathbf{a}} \cdot \underline{\nabla})$	$(\underline{\nabla} \times \overline{\mathbf{a}})$
g)	$(\overline{\mathbf{a}} \cdot \underline{\nabla})$	$(\underline{\nabla} \times \overline{\mathbf{b}})$
h)	$(\overline{\mathbf{a}} \cdot \underline{\nabla})$	$(\underline{\nabla} \cdot \overline{\mathbf{a}})$
i)	$(\overline{\mathbf{a}} \cdot \underline{\nabla})$	$(\underline{\nabla} \cdot \overline{\mathbf{b}})$
j)	$((\underline{\nabla} f) \cdot \underline{\nabla})$	$\overline{\mathbf{a}}$
k)	$((\underline{\nabla} \times \overline{\mathbf{a}}) \cdot \underline{\nabla})$	$\overline{\mathbf{a}}$
l)	$((\underline{\nabla} \times \overline{\mathbf{a}}) \cdot \underline{\nabla})$	$\overline{\mathbf{b}}$

Present all answers in tensor network notation!

[Among these, f) includes a ‘fluid’ vecting its vorticity. And g) is the mutual counterpart of this, for instance a magnetized fluid ‘vecting’ electric current (4) .]

Open Exercise 6 Extend the previous Exercise to double-derivative expressions involving 1 or 2 uses of yet further types of derivative from the top half of Fig 5.

6 Potentials

6.1 Scalar and vector potentials

Exercise 7 a) Let each input vector field below be smooth and defined on all of $\mathbb{R}^3$. Show that the following hold.

- i) If the curl of a vector is zero, then the vector can be expressed as a gradient of a scalar.
- ii) If the div of an $O(3)$ -vector is zero, then the vector can be expressed as a curl of an $SO(3)$ vector. And likewise with these two groups switched around.
- b) How unique are these outputs?
- c)+ What happens to these results if the input vector field is only defined on some region of $\mathbb{R}^3$?

Naming Remark 2 The above outputs are termed the *scalar potential* and *vector potential* respectively. If i)'s input vector is a force, and the output exists, then this force is said to be *conservative*. Newtonian Gravity, Electrostatics and irrotational fluids provide further context for i). Incompressible fluids provide a context for the first form of ii) above. While the non-existence of magnetic monopoles provides an Electromagnetism context for the second form. Thus 'literally irrotational' and 'literally solenoidal' are not quite aliases, for all that these two adjectives are not usually used literally. Finally, in the context of Electromagnetism, b) becomes the standard gauge choice problem.

7 The Helmholtz decomposition in tensor network notation

Remark 1 This [3, 9, 22] is provided in tensor network notation in Fig 10 It has been described as the *Fundamental Theorem of Vector Calculus* (FuToVC for future reference).

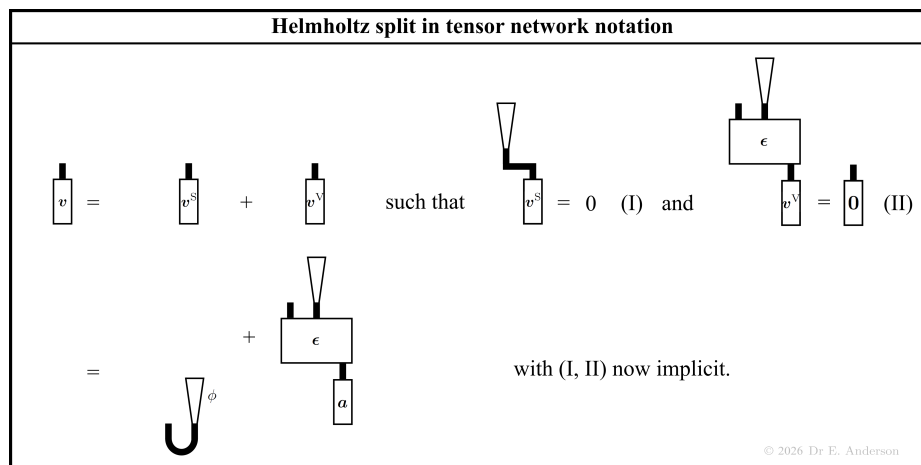


Figure 10:

This is not true unless one of various sets of technical conditions are imposed on the input vector's domain and function space. Over the ages, various function spaces have been used to model this.²

²See e.g. [30] (5) for a recent account of how various well-known references for Electromagnetism fare. Though the current Author's main comment about this is that Mathematical Physics researchers have been using Sobolev spaces as a unified framework for studying the Classical Laws of Nature for at least three decades by now... For all that the best introductory textbooks to Electromagnetism do not do this, on account of being introductory. See indeed e.g. [21] (7) for the Helmholtz decomposition in this context. Certainly the current Author's choice of 'smooth' in the previous subsection is intended for simplicity and not for technical tightness... Finally the current Author is only suggesting this footnote's references for further reading, without making any claims as regards fully checking either of these articles' statements.

Exercise 7 Perform a back of the envelope check on this decomposition's gauge freedom.

[This is a — Exercise, assuming that you have seen some sort of Gauge Theory before...]

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References

- [1] W.R. Hamilton, "On Quaternions, or on a new System of Imaginaries in Algebra", *Phil. Mag.* **25** 489 (1844).
- [2] The scalar product of vectors is due to H. Grassmann (1848) as part of his inception of Linear Algebra. For all that he originally denoted it by a cross!
- [3] It took a number of years for H. von Helmholtz to reach his most general formulation of this decomposition, starting in the 1850s. Subsequent Authors have further generalized it since...
- [4] J.C. Maxwell, *A treatise on Electricity and Magnetism* (Clarendon, Oxford 1873).
- [5] J.W. Gibbs, *Vector Analysis: a Text-Book for the use of Students of Mathematics and Physics* (1881). Written up by E.B. Wilson (Scribner, N.Y. 1901; Y.U.P., New Haven 1913).
- [6] O. Heaviside, "On Operators in Physical Mathematics" *Proc. Roy. Soc.* **54** 105 (1893).
- [7] L.P. Eisenhart, *Riemannian Geometry* (P.U.P., Princeton 1926).
- [8] H. Jeffreys, *Cartesian Tensors* (C.U.P., Cambridge, 1931).
- [9] P.M. Morse and H. Feshbach, *Methods of Theoretical Physics. Part II* (McGraw-Hill, New York 1953).
- [10] M.R. Spiegel, *Vector Analysis (And An Introduction to Tensor Analysis)* (McGraw-Hill, N.Y. 1959).
- [11] B. Spain, *Tensor Calculus. A concise course.* (Oliver and Boyd, Edinburgh 1960; Dover, Mineola N.Y. 2003).
- [12] H. Flanders, *Differential Forms. With Applications to the Physical Sciences* (General Publishing Company, Toronto, Canada 1963; Dover, Mineola N.Y. 1989).
- [13] R.C. Wrede, *Introduction to Vector and Tensor Analysis* (Self-published 1963; Dover, Mineola N.Y. 1972).
- [14] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics* (Pergamon, New York 1965; the original dates back to 1958).
- [15] R. Penrose "Angular Momentum: an approach to Combinatorial Spacetime," in *Quantum Theory and Beyond* ed. T. Bastin (C.U.P., Cambridge 1971); reprinted *Collected Works Vol 2. Works from 1968-1972* (O.U.P., N.Y. 2011).
- [16] C.W. Misner, K. Thorne and J.A. Wheeler, *Gravitation* (Freedman, San Francisco 1973).
- [17] It is more convenient to find GR's spatial configuration metric, the Riemann tensor and the Gauss equation of extrinsic Geometry in reviews such as R.M. Wald, *General Relativity* (U. Chicago P., Chicago 1984).
- [18] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Butterworth-Heinemann, Oxford 1987; the original dates back to 1959).
- [19] H.M. Schey, *Div, Grad, Curl, and All That: An Informal Text on Vector Calculus* (Norton, N.Y. 1997).
- [20] I. Madsen and J. Tornehave, *From Calculus to Cohomology: De Rham Cohomology and Characteristic Classes* (C.U.P., Cambridge 1997).
- [21] C. Amrouche, Christine Bernardi, Monique Dauge and Vivette Girault, "Vector Potentials in Three-dimensional Non-smooth Domains." *Math. Methods Appl. Sci.* **21** 823 (1998), https://perso.univ-rennes1.fr/monique.dauge/publis/ABDG_VPot.pdf.
- [22] I.s. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products* (A.P., San Diego, CA 2000).
- [23] W. Rindler, *Relativity. Special, General and Cosmological* (O.U.P., Oxford 2001).
- [24] B. O'Neill, *Elementary Differential Geometry* Revised 2nd ed. (Elsevier, Amsterdam 2002; the first edition dates back to 1966).

- [25] P. Cvitanović, *Group Theory, Birdtracks, Lie's, and Exceptional Groups* (P.U.P., Princeton NJ 2008).
- [26] D.C. Kay, *Tensor Calculus* (McGraw–Hill, N.Y. 2011).
- [27] M. Abadi et al, "TensorFlow: a System for Large-Scale Machine Learning", arxiv:1605:08965.
- [28] E. Anderson, "The Smallest Shape Spaces. I. Shape Theory Posed, with Example of 3 Points on the Line", arXiv:1711.10054. Updated preprint: wordpress.com/page/conceptsofshape.space/1225 .
- [29] C. Roberts et al, "TensorNetwork: A Library for Physics and Machine Learning", arxiv:1905:01330.
- [30] O.D. Johns, "Helmholtz Theorem and Uniqueness", arxiv.org/pdf/2310.20055 .
- [31] E. Anderson, "Tensor Networks throughout the Basic Theory of STEM. I. Vector Algebra", Online Encyclopaedia of Tensors and other Multi-Linear Arrays, institute-theory-stem.org/oetoma-tensor-networks-in-vector-algebra/ (2026).
- [32] "V. Nonrelativistic Classical Physics", Online Encyclopaedia of Tensors and other Multi-Linear Arrays, institute-theory-stem.org/oetoma-tensor-networks-in-vector-algebra/ (forthcoming 2026).
- [33] *Online Encyclopaedia of Tensors and other Multi-Linear Arrays*, institute-theory-stem.org/online-encyclopaedia-of-tensors-and-other-multi-linear-arrays/ .
- [34] *The Structure of Flat Geometry*, forthcoming (2026); see its Homepage conceptsofshape.space/geometry/ for news and related articles.
- [35] *Linear Mathematics*, forthcoming (2028); see its Homepage conceptsofshape.space/linear-mathematics/ for news and further samples.