Tensor Networks throughout the Basic Theory of STEM.

I. Vector Algebra

Edward Anderson[*]

Abstract

We introduce and motivate the program of using tensor network notation throughout the basic theory of STEM. We specifically use the 'Penrose' vertical variant of this. For tensors and other multi-linear arrays that encode the upstairs–downstairs index distinction. As generalized to our rainbow vertical variant that handles multiple conceptual types of index using a systematic colour palette.

When an application has a large cluster of conceptual types of index, we factorize into multiple candy-patternings of a base colour. And when an application involves a few copies of a family of conceptual types, we factorize the palette into say the intense copy and the pastel copy. These Visualization features are rather helpful in quickly alerting Readers to which copy, and/or which conceptual-type cluster, each object belongs to. This paragraph's details are mentioned since for instance there are large clusters in Lie Theory, which subject has basic applications to somewhat over half of the basic STEM theory courses. And some of its applications involve multiple copies, such as the spatial copy versus the spacetime copy.

In this first Article, we recast Vector Algebra in tensor network notation. With particular emphasis on the usual 3-d vector product, and composites formed from it and the scalar product.

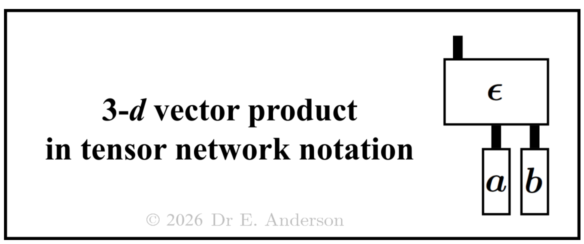

Figure 1:

This Article is ⟨3⟩: accessible to third-year undergraduates.

Cite as E. Anderson, "Tensor Networks throughout the Basic Theory of STEM. I. Vector Algebra", Online Encyclopaedia of Tensors and other Multi-Linear Arrays, institute-theory-stem.org/oetoma-tensor-networks-in-vector-algebra/ (2026).

Date-stamp: v1 06-06-2026.

* Dr.E.Anderson.Maths.Physics *at* protonmail.com . Institute for the Theory of STEM

1 Introduction

Notational Remark 1 The current Encyclopaedia's [36] investigates how the Combinatorial and Visual tensor networks notation [7, 22, 28, 31, 38, 37] fares throughout the basic theory of STEM. 'Birdtracks' is an occasional alias for this notation in the Lie Theory and Physics literature. With reference to the lines – a coordinate-free rendition of indices – that join the boxes that denote the tensors and other multi-linear arrays.

Remark 1 Our fairly limited time resources lead to us concentrating on, at least at first, the slightly more advanced case with co-contra distinction [25, 19, 12]. Aliases for which include *downstairs-upstairs formulation* and (a, b)-*tensor rank formulation*.

Motivation 1 There is plenty of folklore and pedagogy behind it being beneficial to conceptualize using index-free notation. For all that almost everyone[1] then calculates results using indices. There are also various reasons why dot-product type index-free notation is not flexible enough to furnish index-free notation, even for the purposes of the basic theory of STEM. The tensor network notation then serves as a more flexible alternative...

Notational Remark 2 From a Visual point of view, we consider the *vertical variant*. In which the default is for lines to be vertical. A reasonable case can be made to further call this *Penrose's vertical variant*. For all that he, as the at-least-public originator of such notations, used other variants as well.

Motivation 2 This kind of Visualization, and associated Combinatorics, on the one hand has been found to be useful in and consequently well-developed for, multiple advanced pockets of STEM theory. On the other hand, it has not been systematically considered in the most basic university-level STEM courses.

A lesser pedagogical reason for this is that this would be useful to people entering the research fields that prominently use it.

A greater pedagogical reason is that this is a strong conceptual toolkit, that the basic subjects themelves benefit from... So why just develop it for a few corners of research with between several and a few hundred people working in each. When it already offers useful content to subjects such as Vector Algebra and Vector Calculus. Linear Algebra. The Cartesian Tensors of Physics in $\mathbb{R}^3$ space and their Lorentzian counterpart in the Minkowski spacetime of Special Relativity. A first course on surfaces, a first course on Differential Geometry and a first course on Lie Theory? I.e. in courses that have between tens of millions and hundreds of thousands of new students per year... The current Series of Articles fills in this gap.

[1] Cvitanović [22] however pioneered systematic use of index-free calculations using birdtracks.

2 The vector product

Structure 1 The standard presentation of the 3-d vector product

$$\underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}}$$

has an underline counting problem. For despite sporting 2 indices, it is not a 2-index object but a 1-index object.

2.1 Indexed formulation without co-contra distinction

Remark 1 In this setting, in components,

$$(\underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}})_i \quad = \quad \epsilon_{i\,j\,k}\, a_j\, b_k \ . \tag{1}$$

This encodes the vector product's antisymmetry as part of the totally antisymmetry of a more widely useful object. Namely the alternator $=$ alternating tensor $=$ Levi–Civita tensor, $\boldsymbol{\epsilon}$ [2, 10, 1].

Remark 2 The alternator's 'pseudo-tensoriality' furthermore takes over the cross-product's 'axial vector' role. With reference to how its transformation law behaves under reflections. Which can also be thought of as a subcase of needing to specify the tranformation group in setting up a tensor [34, 35]. So if one's $\mathbb{R}^n$ object obeys a tensor transformation law under precisely the $SO(n)$ transformations, then it is what more basic texts call a pseudo-tensor. But if it obeys a tensor transformation law under precisely the $O(n)$ transformations – as before but now also with a reflection generator – then it is what more basic texts just call a tensor...

The object might also get called a pseudo-tensor if it obeys a G-tensor transformation law under a super-group of $SO(n)$ but not of $O(n)$: no reflection generator. If such a distinction is to play any role, however, calling it a G-tensor is both sharper and recommendable. In this way, the terms 'axial vector', 'pseudo-vector' and 'pseudo-tensor' from basic texts do not have a long shelf-life as regards delving into the theory of tensors...

Remark 3 To clarify why the vector product has a 3-d-specific definition, let us temporarily forget that its output behaves differently under reflections than its inputs. And firstly view it as a machine for turning 2 vectors into a further vector. In 3-d , and only in 3-d , the alternator has the right number of components to do this...

Whereas in 1-d it is zero, in 2-d it returns a scalar and in 4-d it returns a 2-tensor. Might one just use some other 3-tensor in say 4-d ? Such a product could be defined, but the alternator has some significant properties that this 3-tensor will not possess. For alternators in whichever dimension are part of the solution to the isotropic tensors problem. Isotropic tensors are furthermore rare [2]. The only other way to form them is concatenating Kronecker deltas, but this only produces even-rank tensors, and 3 is clearly not even.

Secondly, one can view the vector product as a form; *forms* [3, 10] are more generally totally-antisymmetric tensors. For instance, in 4-d , the corresponding rank-4 alternator takes the 2 input vectors and turns them into a 2-form. While in 2-d , the corresponding rank-2 alternator turns them into a 0-form, i.e. a scalar. So now we have a Mathematically-meaningful generalization to other dimensions at the cost of the output's rank being dimension-dependent.

What is it about 3-d that causes the input and output ranks of the objects to match up? The theory of forms [3] enjoys a notion of duality, by which p-forms in n-d are dual to $(\,n\,-\,p\,)$-forms. The 3-d vector product can then be viewed in two steps: pass from 2 input vectors to a 2 form and then pass to its dual. This recipe only sends a pair of vectors to a vector in 3-d . In more detailed treatments, this notion of 'forms duality' is built up into so-called *Hodge duality* [26].

2.2 Indexed formulations with co-contra distinction

Remark 1 Two choices are encountered here. Should the inputs be vectors or covectors? And should the output be a vector or a covector? The version that we consider below picks vectors in all cases. This corresponds to using the rank-$(\,1,\,2\,)$ version of the alternator in our conversion process.

Remark 2 In components,

$$(\,\underline{\boldsymbol{a}}\ \times\ \underline{\boldsymbol{b}}\,)^{i} \quad = \quad \epsilon^{i}{}_{j\,k}\,a^{j}\,b^{k}\ . \tag{2}$$

2.3 Index-free formulations using interior products

Remark 1 Let us first rephrase Remark 1 of Sec 2's equation's LHS as

$$\underline{(\,\underline{\boldsymbol{a}}\ \times\ \underline{\boldsymbol{b}}\,)}\ . \tag{3}$$

Or, in co-contra distinguishing notation,

$$\overline{(\,\overline{\boldsymbol{a}}\ \times\ \overline{\boldsymbol{b}}\,)}\ . \tag{4}$$

Which can be given an input functional dependence interpretation on the small underlines or overlines. While the large underline or overline denotes the nature of our machine's output...

Notational Remark 3 Let us next consider some index-free presentations of interior product type for the RHS of (2).

1)

$$\overline{(\,\underline{\boldsymbol{a}}\ \times\ \underline{\boldsymbol{b}}\,)} \quad = \quad \underline{\underline{\overline{\boldsymbol{\epsilon}}}}\,\cdot\ \cdot\,\overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}} \tag{5}$$

Some problems that this has are as follows. It is unclear in which order the overline and underline indices feature. It is also unclear whether it is the upper or lower underline that contracts into $\boldsymbol{a}$.

The pictorial first RHS in row 2 of Fig 2.a) clarifies by a minor use of colour that $\boldsymbol{a}$ is contracted into the upper underline. And the second RHS additionally that the overline leads, followed by the upper underline and then finally the lower underline.

2) The alternative RHS for (5)

$$\overline{\boldsymbol{a}}\,\cdot\,\underline{\underline{\overline{\boldsymbol{\epsilon}}}}\,\cdot\,\overline{\boldsymbol{b}}$$

frees us from concatenating interior products. This kind of approach rests on adjacency determining what is contracted with what. Now without use of colour, as per the first row of Subfig a). The limitation on doing this is that it is often not possible to arrange for adjacency. We shall be seeing a number of naturally-ocurring basic examples of this over the course of this Series of articles...

3) The alternative RHS for (5)

$$\underline{\underline{\overline{\boldsymbol{\epsilon}}}}\ \circ\ \overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}$$

rolls up 1)'s 2 dots into a large dot that signifies double-dot. This quite widely features in the General Relativity literature, where the configurations [12, 29] are 3-metrics and thus carry twice the indices that configurations usually do.

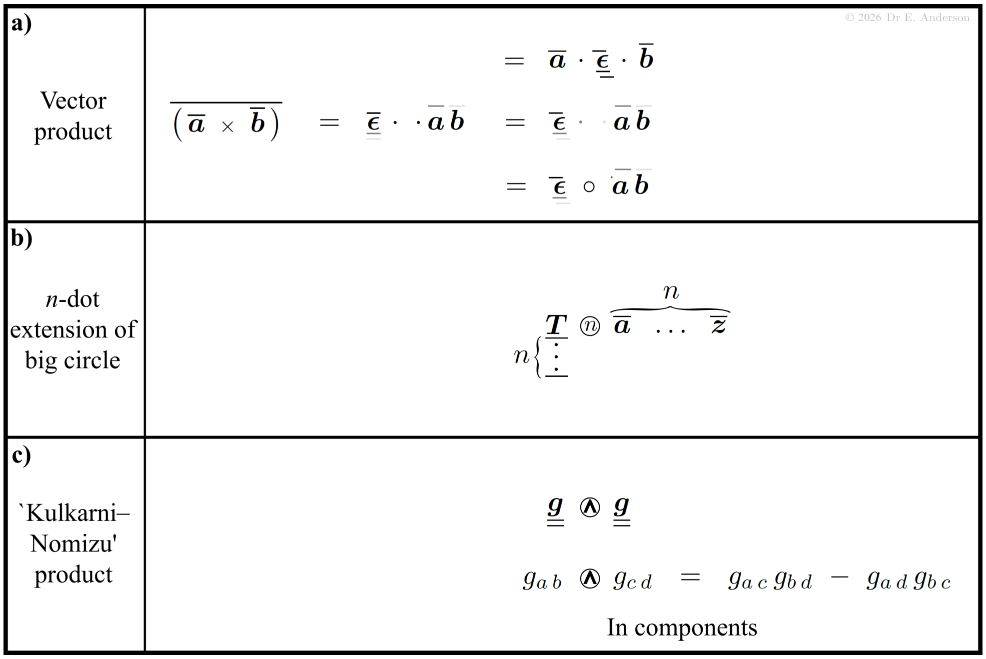

Figure 2:

One can extend this to n-tuple dots by placing the corresponding n inside the large dot (Subfig b). For all that this extension is much more rarely used. A problem with this and the previous paragraph is as follows. On the one hand, 1)'s concatenated dots can themselves be matchingly colour-coded (also displayed in the second RHS of row 2). But on the other hand, arranging for a big dot to be matchingly colour-coded is increasingly contrived and presentationally cumbersome with increasing n .

Even Representation-Theoretic labels have been placed within the big dot. The 'Kulkarni–Nomizu product' (Subfig c) is of this nature, corresponding to the Riemann curvature tensor's symmetries being imposed. By which it is already implicit in earlier work of Gauss and Riemann [12]. And, in an another way, in also easlier work of Schouten [21].

2.4 Tensor network formulation

Notational Remark 4 *Tensor network notation supercedes all of the above modified notations by use of rather more versatile and combineable symbols.* Rows 1 and 2 of Subfig a) can be viewed as prototypes of some tensor network features. See row 2 column 1 of Fig 3 for this and for a birdtack counterpart of (4). Though once the RHS notation has been ascertained, we subsequently only use this more precise and well-known formulation...

3 Some Vector Algebra identities

3.1 Plain and triple products

Structure 2 The *scalar product* of 2 vectors, alias *inner product* or *dot product*, is given in tensor network notation in the first row of Fig 3. While more general such can be defined using a metric or bilinear form [38], this most usually encountered scalar product is built out of the Euclidean metric. Euclidean distances and angles can straightforwardly be constructed out of this [37]. While double use of a vector input returns the Euclidean norm. The Euclidean metric encodes the identity matrix, and is often written in components as

$$\delta_{i\,j} \ :$$

standing for *Kronecker delta tensor*.

Structure 3 The *scalar triple product* is given in row 3. An alias is *volume form*: the top form supported in $3\text{-}d$. This corresponds to the rank $3 - 3 = 0$ form case of duality in $3\text{-}d$.

Remark 1 Double use of a vector input in the cross product is however subjecting an antisymmetric object – $\boldsymbol{\epsilon}$ – to contraction with a symmetric one: the equal pair of vectors. Which annihilates our object down to the zero object of matching rank. See row 2 (and this plays out again for row 3, for all that we leave this depiction to you).

Structure 4 The *vector triple product* concatenates 2 $\boldsymbol{\epsilon}$ (row 4 column 1). The RHS readily follows from the Kronecker delta theorem (see the next Section). Only 1 double use of a vector is nontrivial, as depicted in the second column.

Structure 5 The last row gives a version of the *Jacobi identity*. This turns out to be fundamental in setting up Lie algebras [6].

Notational Remark 5 In the previous item, the jagged bar through 3 of the indices is the tensor networks way of denoting antisymmetrization.

In components, a longhand version of this is

$$\left(\epsilon^{i}{}_{j\,m}\, \epsilon^{m}{}_{k\,l} \ + \ \epsilon^{i}{}_{k\,m}\, \epsilon^{m}{}_{l\,j} \ + \ \epsilon^{i}{}_{l\,m}\, \epsilon^{m}{}_{j\,k} \right) a^{j}\, b^{k}\, c^{l} \ \ = \ \ 0^{i} \ .$$

A briefer formulation for the terms in big round brackets is

$$\epsilon^{i}{}_{m\,[\,j}\, \epsilon^{m}{}_{k\,l\,]} \ .$$

Where we have now introduced *antisymmetric Bach brackets*, $[\;]$: the index counterpart of the jagged line... This is in fact minus the expression in big round brackets, chosen to have contiguous indices enter our Bach brackets.

Without performing such a sign change, one could write

$$\epsilon^{i}{}_{[\,j\,|\,m\,|}\, \epsilon^{m}{}_{k\,l\,]} \ .$$

Where now $|\;|$ denotes a ‘temporary interruption’ to the indices entering our Back bracket!

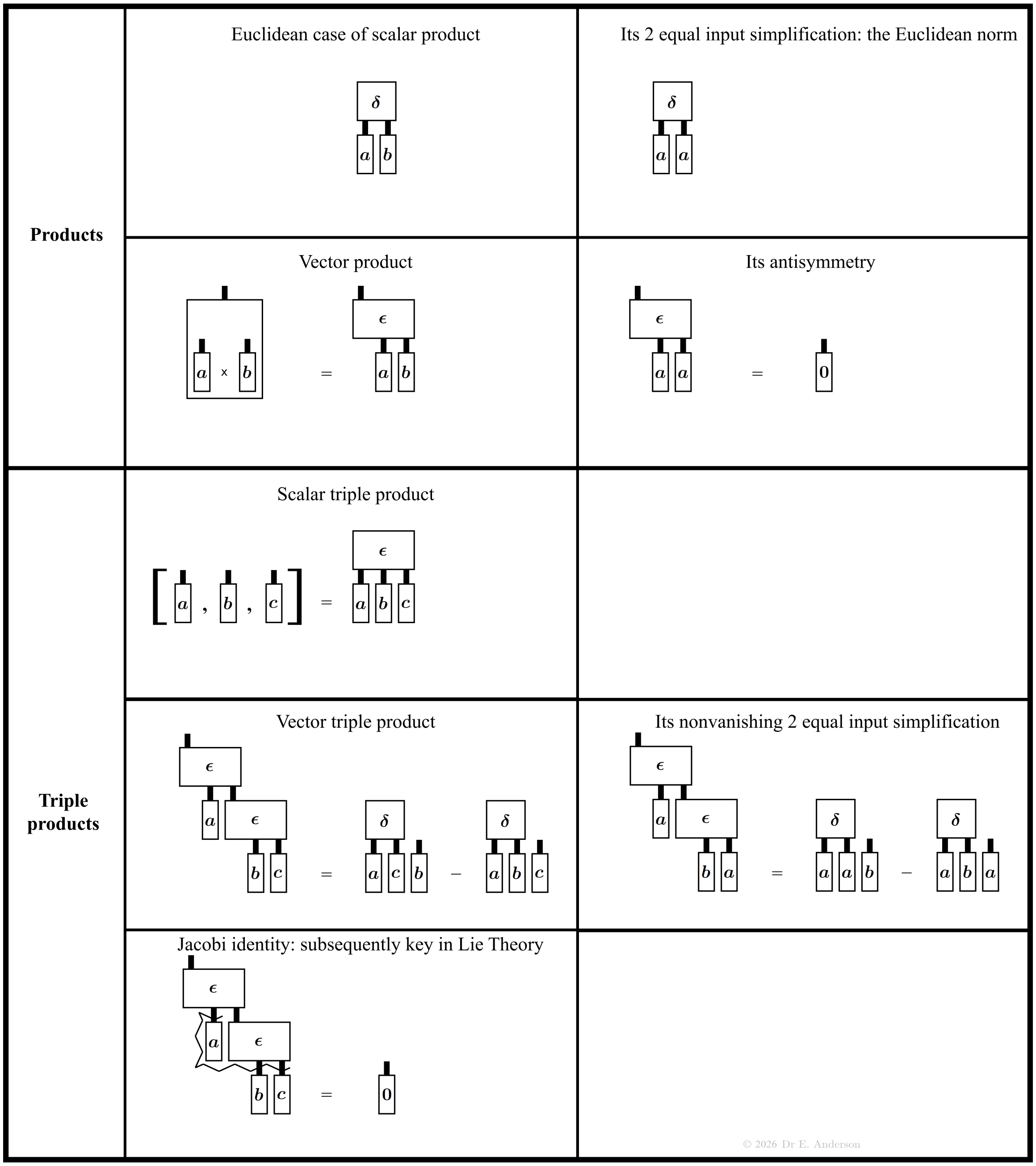
Products
Euclidean case of scalar product
Its 2 equal input simplification: the Euclidean norm
Vector product
Its antisymmetry
Triple products
Scalar triple product
Vector triple product
Its nonvanishing 2 equal input simplification
Jacobi identity: subsequently key in Lie Theory
© 2026 Dr E. Anderson

Figure 3:

3.2 Some quadruple to octuple products

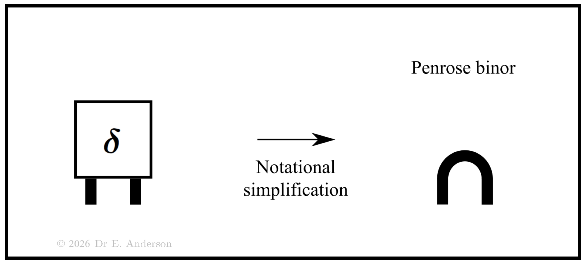

Figure 4:

Structure 6 The '*Cauchy–Binet*' *scalar quadruple product* (row 1 of Fig 5) concatenates 2 $\boldsymbol{\epsilon}$ via a $\boldsymbol{\delta}$. A $\boldsymbol{\delta}$ can however just be replaced by a birdtrack: the *Penrose binor* in Fig 4. This accounts for close similarities in RHS with the vector triple product. The remaining difference is in contracting the latter's free indices with one further vector in the former.

Exercise 1 *The Lagrange identity* is row 2's 2 pairs of equal entries version of Cauchy–Binet. Which Trigonometric identity is this a disguised form of?

Structure 7 The *vector quadruple product* (row 3) concatenates 3 $\boldsymbol{\epsilon}$. So the Kronecker delta theorem sends it to a combination of single-$\boldsymbol{\epsilon}$ factor terms.

Exercise 2 a) What single pair of equal entries version does Cauchy–Binet support?

b) And the vector quadruple product?

c) What quadruple product formula do we get if we concatenate 3 $\boldsymbol{\epsilon}$ in a straight rather than bent 3-path?

Exercise 3 What scalar- and vector-quintuple products of vectors are supported in 3-d?

Remark 1 The binor trick renders our second hexuple scalar product a bent 4-path of $\boldsymbol{\epsilon}$. To the first's 3-star of $\boldsymbol{\epsilon}$. This corresponds to the first ambiguity among tree graphs.

Exercise 4 Work throughout in 3-d .

a) What is the first tree not realized as a network of $\boldsymbol{\epsilon}$?

b) Work out the RHS for the scalar and vector octuple products of vectors that are built out of the quadruped tree of $\boldsymbol{\epsilon}$.

Exercise 5 a) In 2-d , many applications use the component of the 3-d vector product that runs along a fiducial third dimension [37]. Which of the above triple to hexuple products have a counterpart of this kind?

b) Using instead the 2-component ϵ intrinsic to 2-d , which triple to hexuple vector products are supported?

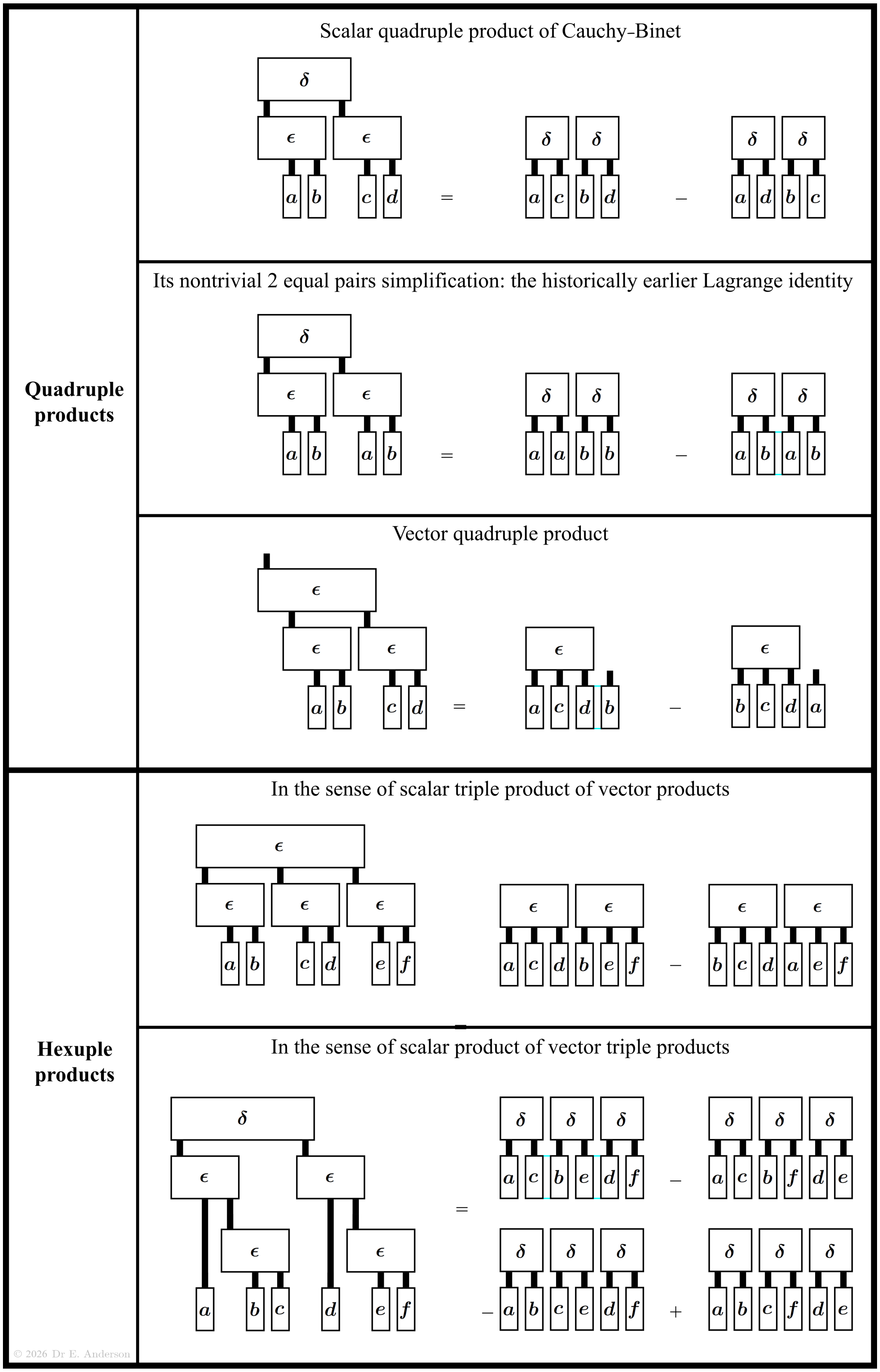
Quadruple products
Scalar quadruple product of Cauchy-Binet
Its nontrivial 2 equal pairs simplification: the historically earlier Lagrange identity
Vector quadruple product
Hexuple products
In the sense of scalar triple product of vector products
In the sense of scalar product of vector triple products
© 2026 Dr E. Anderson

Figure 5:

4 The Kronecker delta Theorem

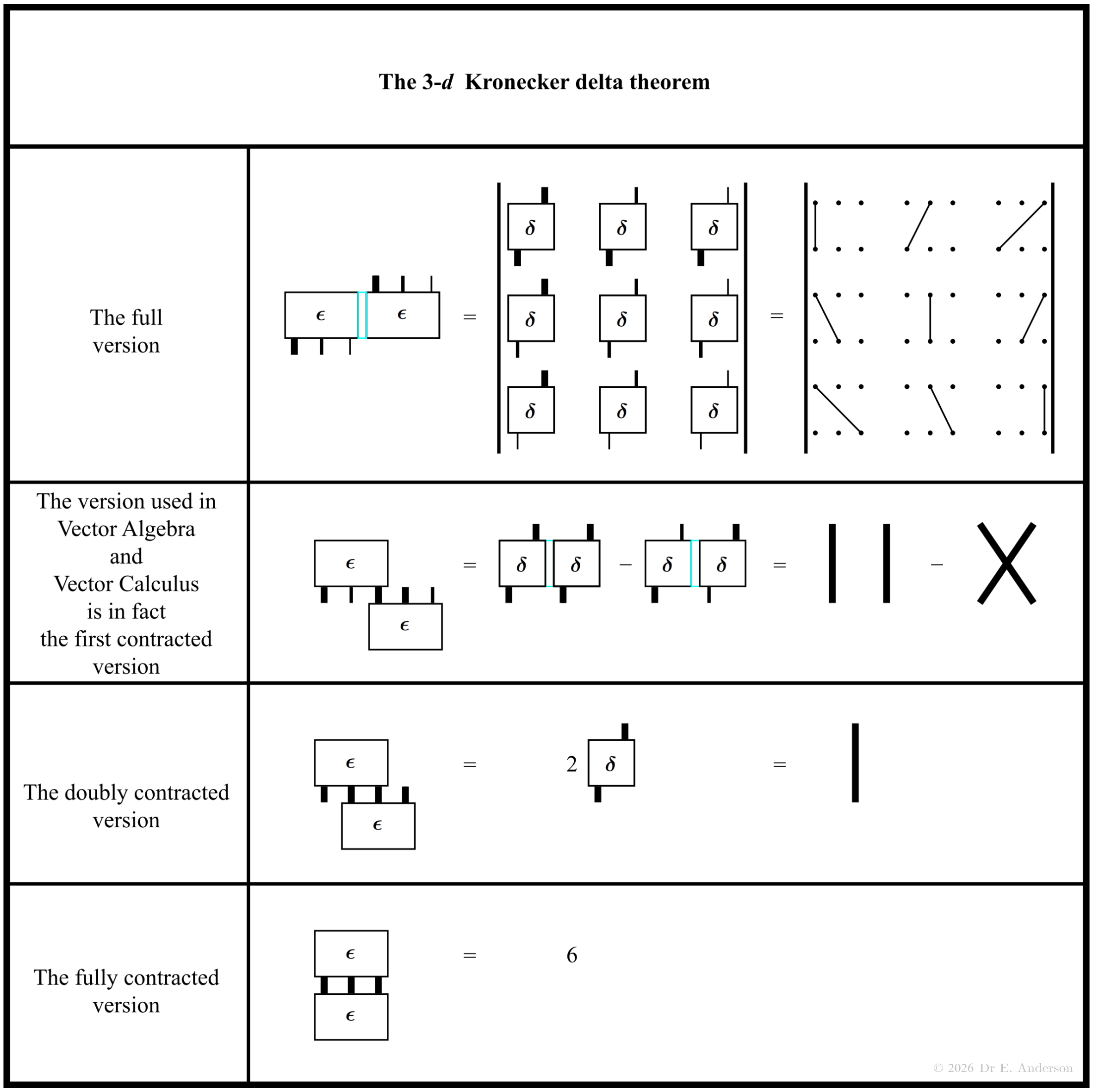

Figure 6:

Remark 1 In the previous section, the vectors themselves serve to label which indices go where. In their absence, there is an identification problem: where does each index in the LHS end up in the RHS? This is compounded when a sum of terms in present. Line thickness can track this for us, with the 3-d Kronecker delta theorem and some contractions in Fig 6 serving as example. This theorem is straightforwardly proven by exhaustion. The second row of this is furthermore the key identity in evaluating the previous Section's RHSs. Toward the n-d generalization, the fourth row's 6 arises as $3!$

Notational Remark 6 The second and third rows reuse Penrose binors. The first row's determinant does as well: a third iteration of the determinant could jettison the grid of dots, leaving just binor lines. Though in n-d for large enough n, the inclinations of such dot-unsupported binors would become hard to distinguish. So there is a longer-term reason to keep the grid of dots in this case.

Remark 2 We leave the 2-d Kronecker delta theorem to you, in connection with Exercise 5.

5 Conclusion

5.1 Synopsis and immediate sequel

Remark 1 We gave a conceptual discussion of the vector product and notation for it. Having settled on vertical tensor network notation, we then considered triple through to hexuple products. So as to have further examples, sources of ambiguity and notational features.

Pointer 1 The second Article shall cover Vector Calculus likewise. It may be useful to think of this first pair of articles as how first-year Vector Algebra and Vector Calculus might be recollected and reappraised in the early stages of a second- or third-year course that considers some generalization of it. Such as a Special Relativity course [19]. Or a Tensor Calculus course [25], or a Physics course that makes heavy use of Tensor Calculus. Or a first Geometry of Surfaces course [23], a first Differential Geometry course [20], or a course that in part introduces Cohomology [17].

5.2 The rainbow vertical variant

Motivation 3 In many subsequent Articles, we need to consider objects carrying multiple conceptual types of index.

Notational Remark 6 We encode these using a systematic colour palette...

Some small examples include the following.

S.1) Spatial versus label versus configuration indices in Mechanics [30, 37].

S.2) Spatial versus spacetime versus internal indices in Relativistic Physics [12, 16], and also primed versus unprimed frame indices.

S.3) ⟨4⟩ Bitensors such as 2- and n-point functions, including Greens' functions, propagators and correlators.

S.4) Surface versus ambient-space indices in Differential Geometry [23, 12]. And also base space versus fibre and total space indices [13].

S.5) Bases versus components in Linear Algebra [38].

S.6) Group indices versus acted-upon objects' indices, and on occasion also $\geq$ 1 key subgroups' indices.

Larger instances include the following.

L.1) ⟨4-5⟩ Types of spinorial index have greater diversity than their tensorial counterparts [15, 12] and as covered by much of Penrose's own application of tensor networks.]

L.2) ⟨5⟩ First-class versus second-class, and primary versus secondary (and tertiary and n-ary), in Dirac's context of Constrained Dynamical Systems [5, 14, 32] . Alongside compositions such as first-class secondary, or with linear versus nonlinear distinction. Many of which are partnered by their own type of observables index [33]. With first integral types and conserved quantities adding various somewhat more basic indices to this Principles of Dynamics repertoire.

L.0) ⟨5⟩ Lie Theory [24, 11] has many facets which contribute their own indices.

⟨6⟩ In fact everything in Dirac's context is a slight variant –Poisson algebra– of Lie Theory's local structure: Lie algebra. Indeed, Lie brackets have their own notions of class, -ary and observable [33]. For instance when applied to Geometry, the role of observables is taken over by Geometrical invariants [33]. When applied to Physics, Dirac's version arises in the 'spatial' (canonical) copy, but there is also a spacetime copy of almost everything. And further maps that convert between these 2 copies [29].

Indeed, over 50 conceptual types of index turn out to be necessary to model [33, 32] the observationally-accepted Fundamental Physics laws. Thus explaining the scope of Notational Remark 7 below.

⟨3⟩ Even the 4 reductions approach to the N-body problem requires over 30 [30, 37]. And so is being used as a test run for the Fundamental Physics case, both at the conceptual level and at the level of notational development. This is already pointing out many gaps in previous generations of N-body and Shape Statistics reviews [30]. And exhibiting major intertwining with Flat Geometry, dozens of new theorems and proofs included [37].

Notational Remark 7 When an application has a large cluster of variants of a conceptual type of index, we factorize into multiple candy-patternings in the same colour. And when an application involves a few copies of a family of conceptual types, we factorize the palette into say the intense copy and the pastel copy. These Visualization features are rather helpful in quickly alerting Readers to which copy, and/or which conceptual-type cluster, each object belongs to.

For it is quickly unsightly to extend the Physicsists' common practise of denoting each conceptual type of index by a different font. And quite quickly downright impractical: few people know over 50 font types or can quickly figure out which letters belong to which fonts...

5.3 Completing our main motivations

Motivation 4 Discrete Mathematics has its own uses of matrices and higher arrays. Such as graph adjacency and incidence matrices, subset and poset counterparts, and permutation, binary and stochastic matrices [38, 4]. While Numerical Analysis' matrix decompositions [18, 27] provide a further source of rainbows. And many Machine Learning processes are amenable to both Visualization and computational organization using tensor networks notions [28, 31]. By which large student number courses in the 'Discrete half' of Basic STEM theory – alongside their main client Computer Science, old topics or new – also stand to benefit from a systematic treatment of tensor networks.

Acknowledgements I thank S. Sánchez for teaching me the theory of how to draw accurate families of figures at very low time-cost per new figure. For all that I was left to develop multi-linear figure-making in my own time... I also thank K. Everard for multi-linear discussions, and R.N.C. Pfeifer for discussions specifically about tensor networks. I finally thank M.A.H. MacCallum, N. ó Murchadha and J.W. York Jr. for discussions about tensors while I still was a student. And J. Saxl and C.J. Isham for comparable-level discussions on Algebra, Lie Theory and fibre bundles.

References

[1] M.R. Spiegel, *Vector Analysis (And An Introduction to Tensor Analysis)* (McGraw-Hill, N.Y. 1959).

[2] B. Spain, *Tensor Calculus. A concise course.* (Oliver and Boyd, Edinburgh 1960; Dover, Mineola N.Y. 2003).

[3] H. Flanders, *Differential Forms. With Applications to the Physical Sciences* (General Publishing Company, Toronto, Canada 1963; Dover, Mineola N.Y. 1989).

[4] M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities* (Prindle, Boston 1964; Dover, Mineola N.Y. 1992).

[5] P.A.M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York 1964).

[6] J-P. Serre, *Lie Algebras and Lie Groups*, (Springer, 1965).

[7] R. Penrose "Angular Momentum: an approach to Combinatorial Spacetime," in *Quantum Theory and Beyond* ed. T. Bastin (C.U.P., Cambridge 1971); reprinted *Collected Works Vol 2. Works from 1968-1972* (O.U.P., N.Y. 2011).

[8] R.S. Kulkarni, “On the Bianchi identities". Math. Ann., **199** 175 (1972).

[9] K. Nomizu, “On the Decomposition of Generalized Curvature Tensor Fields. Codazzi, Ricci, Bianchi and Weyl revisited", In *Differential Geometry. In Honor of Kentaro Yano*, (Kinokuniya, Tokyo, 1972).

[10] C.W. Misner, K. Thorne and J.A Wheeler, *Gravitation* (Freedman, San Francisco 1973).

[11] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications.* (Wiley, New York 1974; Dover, Mineola N.Y. 2002).

[12] It is more convenient to find GR’s spatial configuration metric, the Riemann tensor and the Gauss equation of extrinsic Geometry in reviews such as R.M. Wald, *General Relativity* (U. Chicago P., Chicago 1984).

[13] M. Nakahara, *Geometry, Topology and Physics* (I.O.P., London 1990).

[14] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (P.U.P, Princeton 1992).

[15] S. Weinberg, *The Quantum Theory of Fields. Vol I. Foundations.* (C.U.P., Cambridge 1995).

[16] *Vol II. Modern Applications.* (C.U.P., Cambridge 1995).

[17] I. Madsen and J. Tornehave, *From Calculus to Cohomology: De Rham Cohomology and Characteristic Classes* (C.U.P., Cambridge 1997).

[18] C.D. Meyer, *Matrix Analysis and Applied Linear Algebra* (S.I.A.M. 2000).

[19] W. Rindler, *Relativity. Special, General and Cosmological* (O.U.P., Oxford 2001).

[20] B. O’Neill, *Elementary Differential Geometry* Revised 2nd ed. (Elsevier, Amsterdam 2002; the first edition dates back to 1966).

[21] It is more convenient to find the Schouten tensor in reviews such as A.L. Besse, *Einstein Manifolds* (Springer-Verlag, 2007).

[22] P. Cvitanović, *Group Theory, Birdtracks, Lie’s, and Exceptional Groups* (P.U.P., Princeton NJ 2008).

[23] A.N. Pressley, *Elementary Differential Geometry* (Springer, London 2010).

[24] J. Stillwell, *Naive Lie Theory* (Springer, N.Y. 2010).

[25] D.C. Kay, *Tensor Calculus* (McGraw–Hill, N.Y. 2011).

[26] T. Frankel, *The Geometry of Physics: An Introduction* (C.U.P., Cambridge 2011).

[27] G.H. Golub and C.F. Van Loan, *Matrix Computations* 4th ed. (Johns Hopkins, Baltimore MD, 2013).

[28] M. Abadi et al, “TensorFlow: a System for Large-Scale Machine Learning", arxiv:1605:08965.

[29] E. Anderson, *The Problem of Time*, Fundam. Theor. Phys. **190** (2017) DOI: 10.1007/978-3-319-58848-3 , alias *The Problem of Time. Quantum Mechanics versus General Relativity*, (Springer, New York 2017).

[30] “The Smallest Shape Spaces. I. Shape Theory Posed, with Example of 3 Points on the Line", arXiv:1711.10054. Updated preprint: wordpress.com/page/conceptsofshape.space/1225 .

[31] C. Roberts et al, “TensorNetwork: A Library for Physics and Machine Learning", arxiv:1905:01330.

[32] E. Anderson, “Lie Theory suffices for Local Classical Resolution of Problem of Time. 1. Closure, as implemented by Lie brackets and Lie’s Algorithm, is Central”,
conceptsofshape.files.wordpress.com/2020/10/lie-pot-1-v2-15-10-2020.pdf/ (2020).

[33] “Lie Theory Suffices to Resolve the Problem of Time", in *Time and Science* **3** ed. P. Harris and R. Lestienne (World Scientific, Singapore 2023).

[34] “What is a Vector?" conceptsofshape.space/linear-mathematics-sample-chapter-1/ (2023).

[35] “What is a Tensor?" conceptsofshape.space/linear-mathematics-sample-chapter-tensors-1/ (2026).

[36] *Online Encyclopaedia of Tensors and other Multi-Linear Arrays*, institute-theory-stem.org/online-encyclopaedia-of-tensors-and-other-multi-linear-arrays/ .

[37] E. Anderson, *The Structure of Flat Geometry*, forthcoming (2026); see its Homepage conceptsofshape.space/geometry/ for news and related articles.

[38] *Linear Mathematics*, forthcoming (2028); see its Homepage conceptsofshape.space/linear-mathematics/ for news and further samples.