The Desargues Incidence Graph

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Abstract

Desargues' Theorem is one of the two main structural theorems of Projective Geometry. Projective Geometry is furthermore Geometry pared down to the study of incidence, and incidence graphs are themselves meaningful. This leads us to study here the Desargues incidence graph, which we arrive at in a bipartite presentation. We then recast it as a generalized Petersen graph, in a regular-iscosagon manifestly-Hamiltonian presentation and in a minimum-crossing presentation with various optimized features. Further Graph Theory Drawing and Visualization that we employ includes shells, grids and other tessellations.

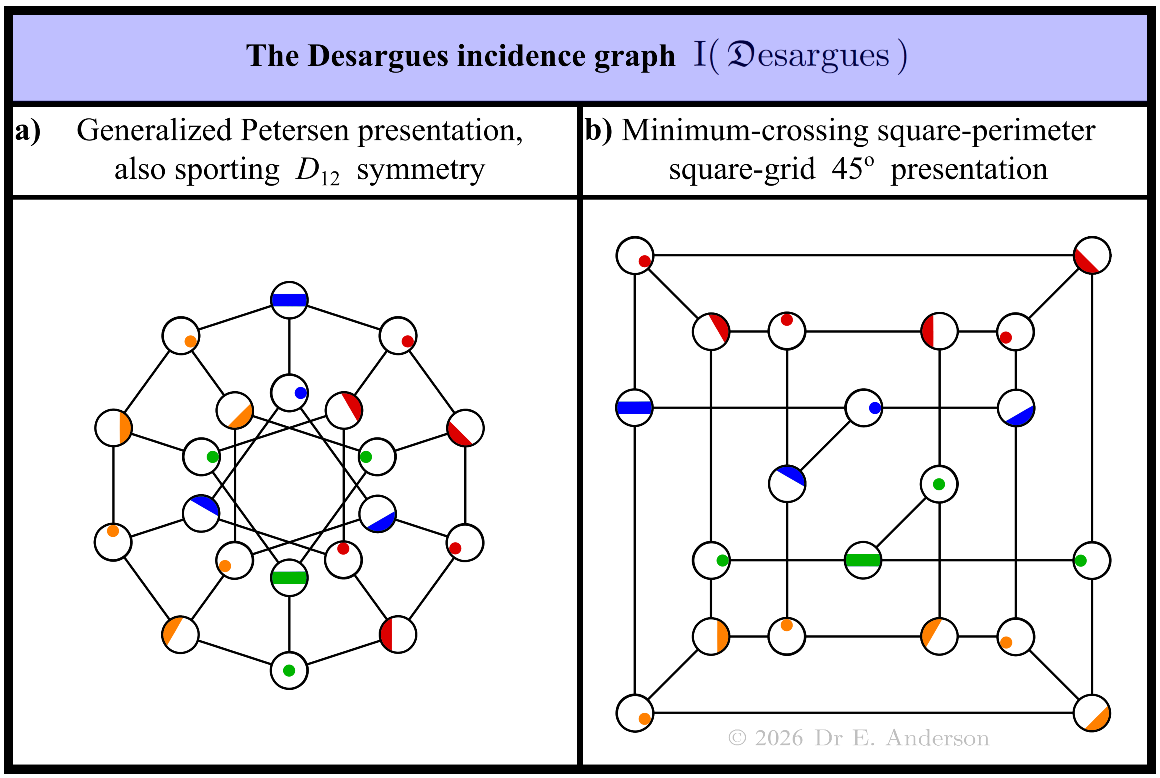

Figure 1:

This Article is ⟨3⟩: accessible to third-year undergraduates.

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1 Introducing the Desargues incidence graph

1.1 Bipartite inception

Remark 1 We follow [11]'s procedures without comment, wherever these are applicable and suffice.

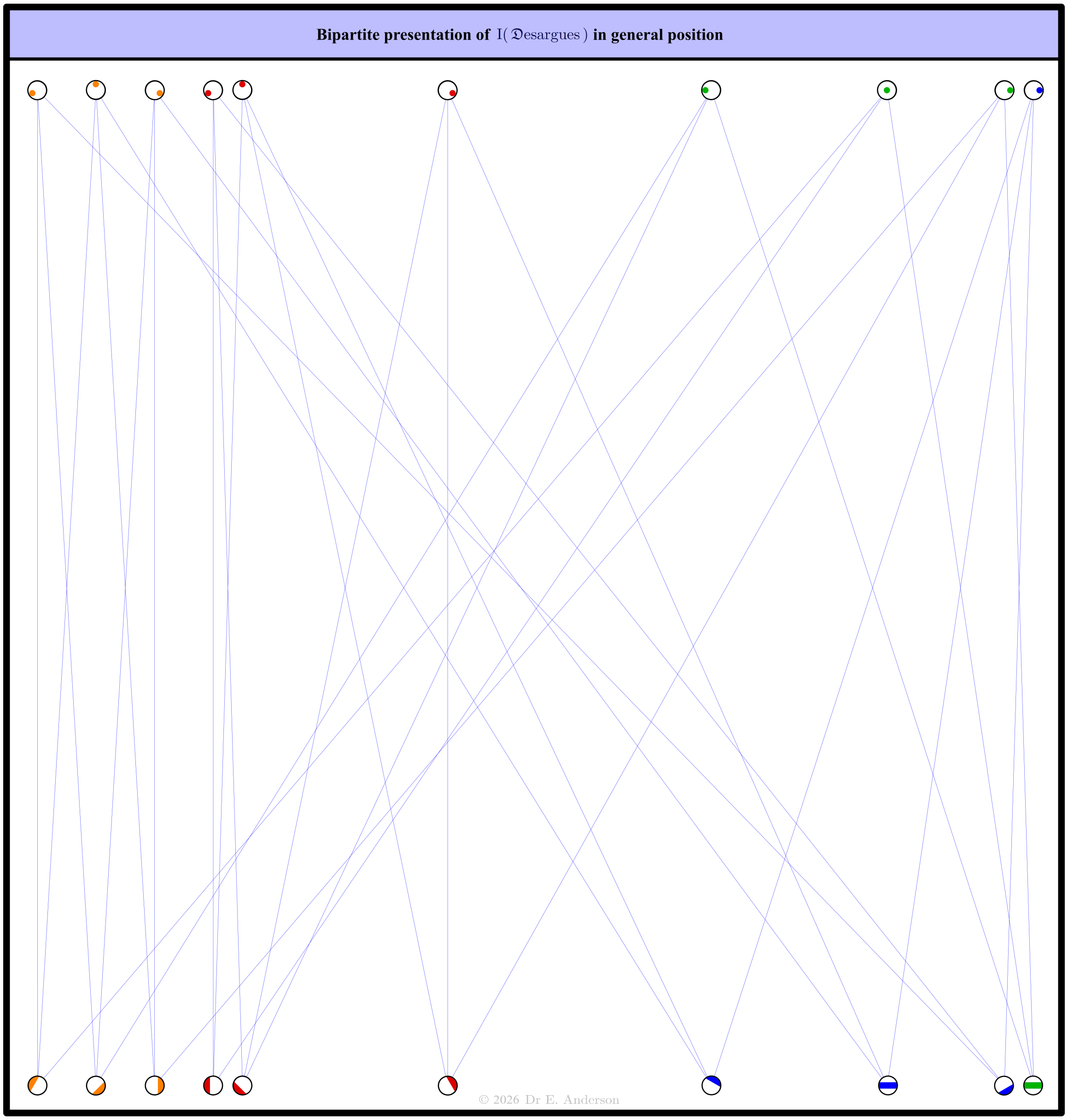

Figure 2:

Structure 1 We give the bipartite presentation of the Desargues incidence graph I(𝔇esargues) in general position in Fig 2. Using the previous Article [12] firstly to label the Desargues configuration's [1] objects with call-signs [3, 8, 10] so as to keep track of them throughout potentially lower-resolution views of our figures. And secondly to tell us which point objects lie on which line objects. With these 2 classes of object constituting the 2 parts in question, with edges between them precisely whenever such lying – incidence – occurs. We present this first graph presentation with very fine lines and high resolution in Fig 2, since it contains a little region near the centre where these features are needed to see that general position is not violated.

Open Question 1 The given presentation is upon a uniform grid of width 34 squares. What is the minimum grid width affording general position?

1.2 The generalized Petersen presentation

Structure 2 We give this in Fig 1.a); it is a well-known presentation [4], dating at least as far back as Coxeter [2]. If needs be, see [7] for a brief outline of what generalized Petersen graphs are.

Remark 1 It is also manifestly D_{12} -symmetric, of order 24 . This is rather smaller than the Desargues incidence graph's own symmetry group

$$Sym(\mathrm{I}(\mathfrak{D}\text{esargues})) \ = \ S_5 \ \times \ C_2 \ ,$$

of order

$$|\ Sym(\mathrm{I}(\mathfrak{D}\text{esargues}))\ | \ = \ 5! \ \times \ 2 \ = \ 240 \ .$$

Remark 2 Also observe that the Fano and Pappus incidence graphs are not generalized Petersen. The Desargues incidence graph is not however the smallest configuration whose incidence graph is generalized Petersen. This honour goes instead to the Möbius–Kantor incidence graph [2, 3, 4].

1.3 Some basic counts

Remark 1 Having given one presentation so as to be able to display our graph, let us make the following basic counts for future use. Its order – number of vertices – is

$$V(\mathrm{I}(\mathfrak{D}\text{esargues})) \ = \ 20 \ . \tag{1}$$

While its size – number of edges – is

$$E(\mathrm{I}(\mathfrak{D}\text{esargues})) \ = \ 30 \ . \tag{2}$$

Finally, its degree sequence is

$$\mathrm{dv}(\mathrm{I}(\mathfrak{D}\text{esargues})) \ = \ 3^{20} \ . \tag{3}$$

So it is yet another cubic graph. This 3 carries Projective significance; in further detail, the underlying Projective configuration is a 10_3 . Signifying 10 points arranged to enjoy collinearity in 3 's.

Remark 2 There is again no practical point to considering the complement of our graph, since this has far more edges than our graph itself. Now by 160 to 30 .

Remark 3 Like all generalized Petersen presentations, the Desargues incidence graph's can be viewed as an equipartition of its vertices between 2 regular-polygon shells. In the present case,

$$\frac{20}{2} \ = \ 10 \ ,$$

so we are talking about 2 regular-decagon shells; see Fig 3.a).

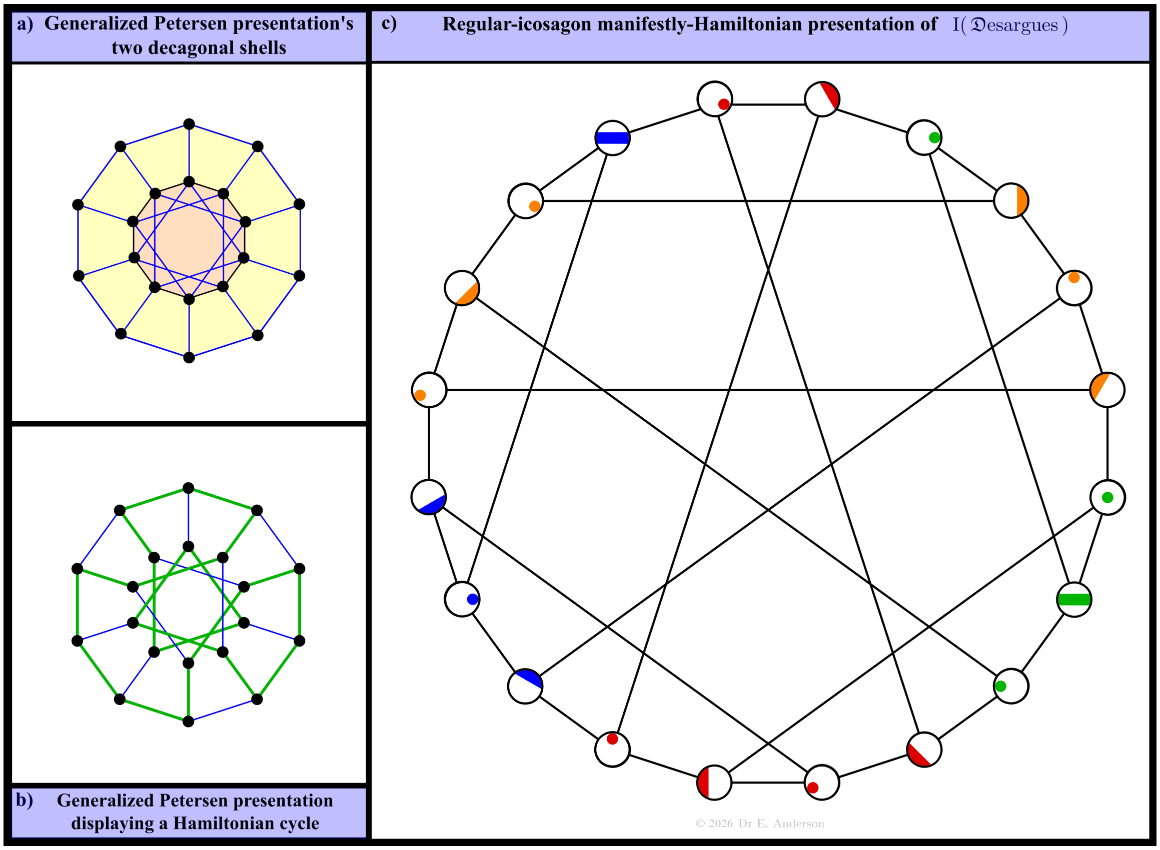

Figure 3:

2 Further properties

2.1 Metric properties

Exercise 2$^-$ Show that the girth

$$g(\mathrm{I}(\mathfrak{D}\mathrm{esargues})) = 6 \, , \tag{4}$$

And the diameter

$$d(\mathrm{I}(\mathfrak{D}\mathrm{esargues})) = 5 \, . \tag{5}$$

Why are we not explicitly noting down the circumference $c(\mathrm{I}(\mathfrak{D}\mathrm{esargues}))$ in the current Article?

2.2 Structural analysis

Remark 1 The Desargues graph is a double irreducible: class D [6].

Exercise 3 Show that a) $\mathrm{I}(\mathfrak{D}\mathrm{esargues})$ is a 8th cubeomorph of the first cubeomorph irreducible: the tetrahedron graph $\mathrm{Tet} = \mathrm{K}_4$.

b) It contains the Petersen graph as a subgraph but not the Pappus incidence graph.

c)$^+$ Does it contain the Heawood graph [9]?

d)$^+$ Finally show that it is a 6-cubeomorph irreducible, with reference to girth-6 preservation.

2.3 Traversibility properties

Remark 1 $\mathrm{I}(\mathfrak{D}\text{esargues})$ is clearly not Eulerian.

Remark 2 Consider the generalized Petersen presentation once more. In Fig 3.b), we mark upon this in emerald one of the Hamiltonian cycles. $\mathrm{I}(\mathfrak{D}\text{esargues})$ is thus Hamiltonian.

Remark 3 Subfig c) then gives the corresponding regular-icosagon manifestly-Hamiltonian presentation. This manages to capture D_5 symmetry, of order 10. Without the call-sign labelling, it is also a commonly-encountered presentation, occurring e.g. in [2, 3, 4].

2.4 Colorability

Exercise 4$^-$ Immediately write down $\mathrm{I}(\mathfrak{D}\text{esargues})$'s chromatic number. Find its edge-chromatic number. Is it uniquely colourable?

3 An optimized minimum-crossing presentation

3.1 Preamble

Exercise 5$^-$ Prove that the Desargues incidence graph is nonplanar. And furthermore that the crossing number

$$Cr(\mathrm{I}(\mathfrak{D}\text{esargues})) \geq 3 .$$

What upper bounds do the presentations exhibited so far give?

3.2 Minimum crossing presentation

Remark 1 In fact,

$$Cr(\mathrm{I}(\mathfrak{D}\text{esargues})) = 6 . \tag{6}$$

This result is covered as part of [13] and references therein.

Structure 1 Fig 1.b) has all lines vertical, horizontal or at 45°. This matches the Pappus incidence graph's new optimized presentation [10]. But in the present Desargues case, such a presentation was already well-established. E.g. Wolfram Math World [14] contains it.

Remark 2 The remainder of the current Article covers some Graph Drawing and Visualization [5, 15] finery.

3.3 Underlying shells

Structure 1 continued Fig 1.b) consists of 3 centred square shells. Each of side 2 units more than its predecessor. Including the manner in which the innermost shell can be preceded by a central point. As depicted in Fig 4.a).

3.4 Underlying grid

Structure 1 continued Subfig b) establishes that both the vertices and the crossings lie on a 6×6 square grid.

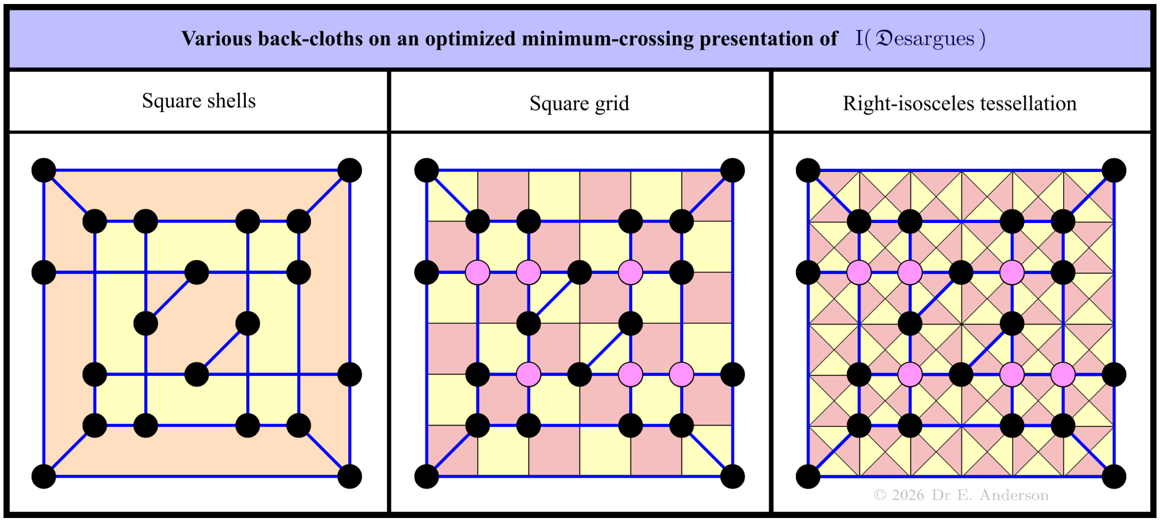

Figure 4:

3.5 Underlying right-isosceles tessellation

Structure 1 continued And Subfig c) that not only the vertices and crossings, but also all the edges, lie on a chunk of the titular tessellation.

Remark 1 These last two subsections also match our new minimum-crossing presentation for the Pappus incidence graph. The above mentions of this presentation further motivate it, while [13] provides yet further motivation for it and its sizeable potential for extensions.

4 What does our callsign tracking system reveal?

Remark 1 Consider first the generalized Petersen presentation of Fig 1.a). Here our selection of input and output groupings form 2 identical and equably-placed claws. And 2 identical and equably-placed hexagons. So that if dual indistinguishability is evoked, a $C_2 \times C_2$ subgroup of the presentation's D_12 is preserved by our colouring scheme.

Remark 2 In the minimum-crossing presentation of Fig 1.b), our choice of 4 grouping largely gets vertically sorted. With each colour and its internal edges restricted to a distinct convex piece of the square perimeter. Now if dual indistinguishability is evoked, our colouring scheme respects this presentation's sole inversive symmetry.

Remark 3 The regular-icosagon manifestly-Hamiltonian presentation is less nice in this regard. For while its claws are identical and equably-placed, its hexagons are neither. Overall reflection symmetry is maintained by this colouring, but this presentation's remaining 5-fold rotational symmetry is broken.

Remark 4 [12]'s matching interpretation picks out a Claw-of-Claws subgraph in $\mathrm{I}(\mathfrak{D}\mathrm{esargues})$. In the generalized Petersen presentation, this is realized in a manner that respects reflection symmetry. And it (broad black edges) and its (broad magenta edges) vertex complement claw turn out to be equably placed; see Fig 5.a) This second feature does not recur in any of the current Article's other presentations.

Remark 5 The 5-cycle of these pairs of Claws-of-Claws gives 5 of the unique role assignments following from choosing a point. The other 5 follow from a distinct pair of equably-placed Claws-of-Claws (Subfig b). This also renders it clear that giving any line as axis also serves to (dually) fix the role of all the other objects in the configuration. By using the 10 vertex-complement line-centred copies of the above 10 point-centred $Claws-of-Claws$.

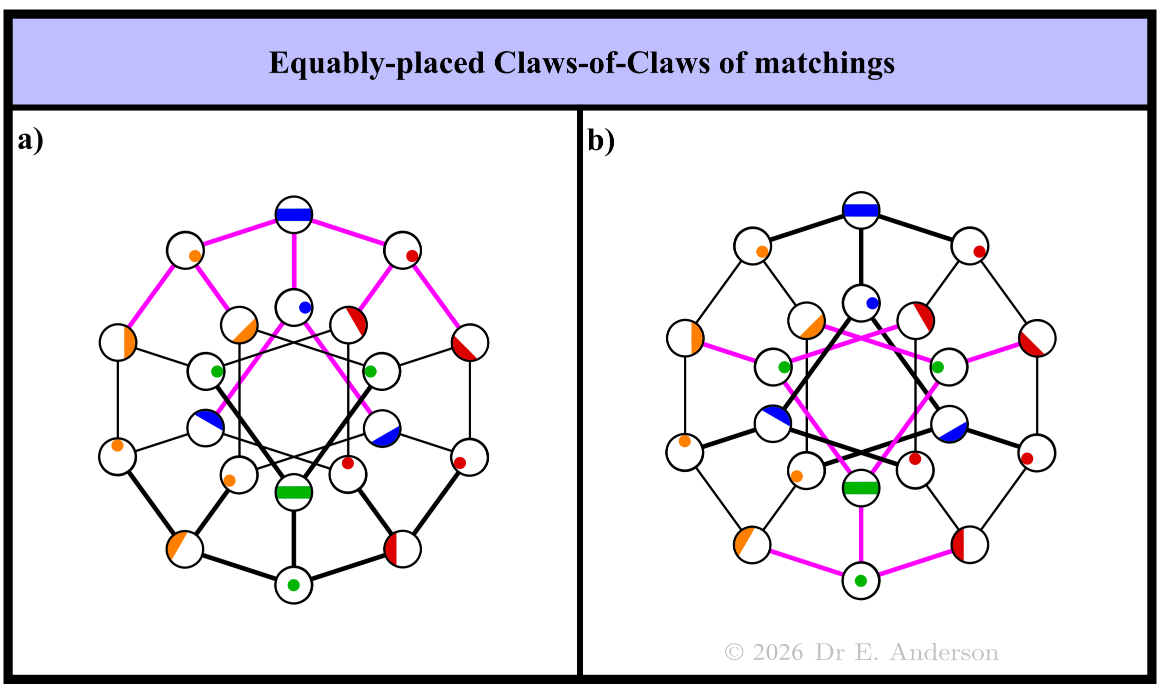

Figure 5:

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