The Desargues Configuration, Theorem and Graph

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Abstract

We consider Projective Geometry's Desargues configuration [43, 54, 53], to which Desargues' Theorem applies. We firstly motivate this via its direct usefulness in Mathematics as one of the two fundamental structural theorems of Projective Geometry. In the current Article, we mostly work at the level of graphs, selecting various conceptual classes of nice presentations for the corresponding configuration graph: Projectively-natural, planar and Hamiltonian. We use Graph Drawing and Visualization [55] criteria to obtain more specific planar presentations. We secondly motivate this Desargues material via some of its particularly significant heirs. Among which the Desargues incidence graph is covered in a companion article [35] in comparable detail.

The current article is the fifth in a series on Projective configurations and their graphs. The minimum Projectively-significant configuration is Fano's [50, 54, 31]. The other fundamental theorem of Projective Geometry is Pappus' [24], corresponding to Pappus' configuration [43, 54]. This is functionally more homogeneous than Desargues' theorem. Also the corresponding Pappus configuration contains 1 point and line less than Desargues', yielding slightly smaller graphs. So we already covered this case's configuration and incidence graphs in [33, 34] respectively.

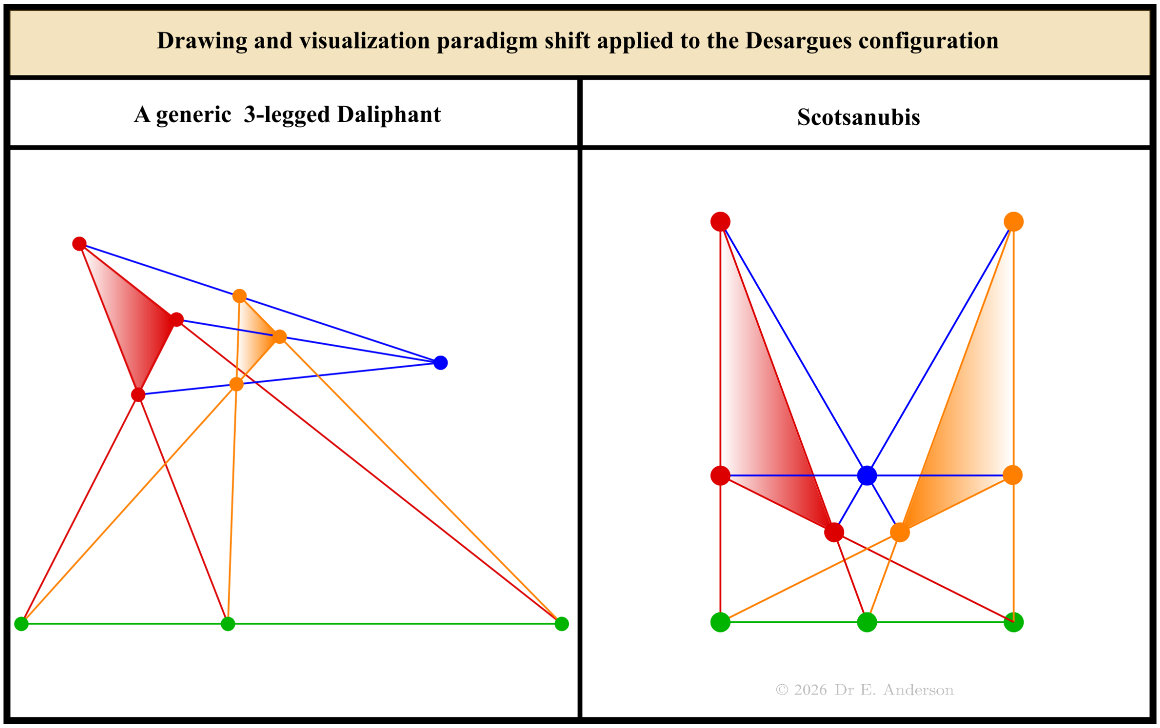

Figure 1:

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This Article is ⟨3⟩: accessible to third-year undergraduates.

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1 Desargues' Theorem in the Euclidean plane

1.1 The 2-triangle formulation

Theorem 1 [Desargues] [2, 4, 5, 8, 10, 7, 13, 16, 19, 22, 24, 25]. Work in the usual Euclidean plane. Let A, B, C be a triangle of points (orange in Fig 2). And let A′, B′, C′ be another (red). Then the following are equivalent.

i) These triangles are in perspective from a point O (blue).

ii) These triangles are in perspective from a line LNM (green).

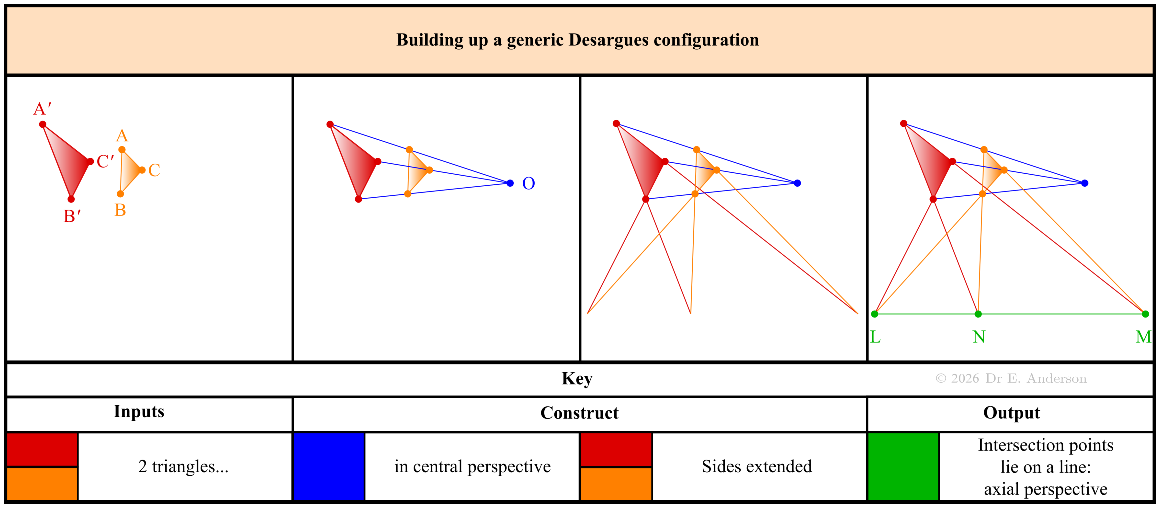


Figure 2:

1.2 Perspective

Structure 1 Consider 2 equal-sized sets of points $\mathfrak{S}_1$ and $\mathfrak{S}_2$. These are *perspective from a point* alias *in central perspective*, if the following holds. They match up into pairs, with 1 point in each pair picked from each set. With the lines joining up each pair all concurring to form a pencil emanating from a point. Which is itself termed the *centre of perspective*, O. If general position is to be attained, then the points of $\mathfrak{S}_1$, $\mathfrak{S}_2$ and O, are to all be in distinct positions.

Structure 2 $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are *perspective from a line* alias *in axial perspective*, if the following holds. Its lines meet in pairs at collinear points. The common line thus formed is itself termed the *line of perspective*, $\mathfrak{A}$. Pictures involving a horizon are casting it in this role.

2 Improving Drawing and Visualization of the Desargues configuration

2.1 The Pappus optimum as a comparative target

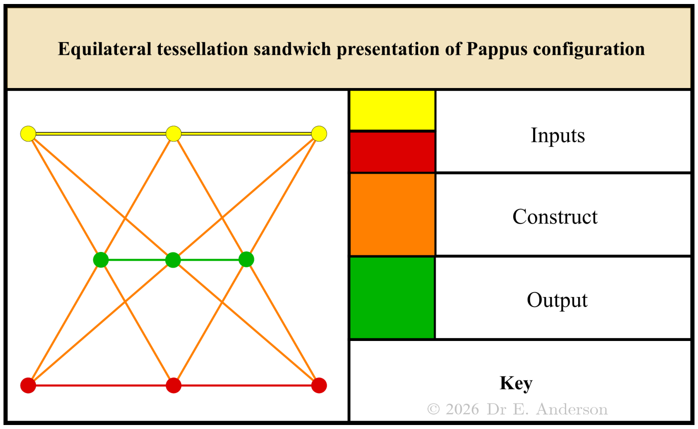


Figure 3:

Remark 1 Our detailed study [33] of the Pappus configuration

$$\mathfrak{P}\mbox{appus}$$

set a high standard for optimum presentation. We recollect this in Fig 3.

It sports a sandwich of 3 parallel lines. Whose top and bottom layer ‘input slices of bread’ – yellow and red – consist of the following. Identical-width uniform spacings of collinear triples of points. Vertically stacked. This is part of a distinguished subarena in which further parallel alignment of the ‘output cheese’ middle layer – green – is forced. Finally, the sandwich’s height is hand-picked to realize equilateral triangles, forming a chunk of tessellation.

2.2 3-legged Daliphants

Remark 1 In contrast, the generic presentation of the Desargues configuration

$$\mathfrak{D}\mbox{esargues}$$

used so far in setting up Desargues’ Theorem in Fig 2 is considerably less exhalted. The idea behind the current Section is to close down this gap...

Naming Remark 1 K. Everard called generic presentations of this kind *3-legged Daliphants*. With reference to Salvador Dalí’s distinctive insect-legged elephants. With our triangles’ sides extended supporting just the 3 legs seen in Desargues configurations $\mathfrak{D}\mbox{esargues}$ ...

Structure 1 Let us also introduce the arena of all Desargues configurations,

$$\boldsymbol{\mathfrak{D}}\mbox{esargues} \ .$$

We more generally use bold mathfrank font to distinguish arenas from configurations.

2.3 Reliant supermarket trolleys

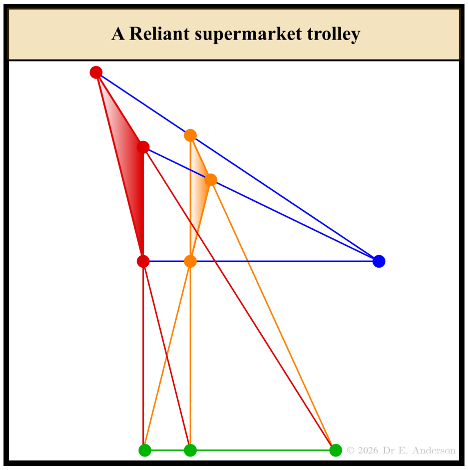


Figure 4:

Remark 1 Improving the above to a presentation that sports 2 perpendicular pairs of parallel lines is quite straightforward. See Fig 4 for a sample.

Naming Remark 2 K. Everard called this the 'Reliant supermarket trolley'. With reference to the iconic 3-wheeled British car, the Robin, manufactured by the Reliant Motor Company...

Remark 2 Back to 𝔓appus, the 2 parallel inputs furthermore force the output to be parallel as well. Thus a unique Affinely-privileged case is supported. This was celebrated in [33] by adding an Affinely-privileged bottom row to its Pappus inputs and outputs Figure 2. In contrast, 𝔇esargues' structural complexity supports multiple classes of Affinely-privileged presentations. Leaving the current Article unable to exhaustively present them all. We are instead gunning for one of the nicest presentations, for subsequent heavy lifting!

Open Exercise 1 Investigate which other combinations of parallel lines are possible in a Desargues configuration. Show that a large amount of these is capable of forcing the presentation to be arbitrarily long. Including in the case of Desarguian cranes whose 2 input triangles are similar and sport 3 pairs of parallel sides.

[This limits such a presentation's use in drawing configurations and graphs. Hence why we have opted for a non-maximum amount of Affine features in row 2.]

Remark 3 The above perpendicularity amounts to some further Metric-level privilege. Our Pappus optimum also contains an element of this. Via its vertically stacking up 2 input copies of the same width of uniform configuration.

2.4 The arena of Desarguian cranes

Remark 1 So far we have been considering pencils that realize a simple cone. By placing the 2 triangles to lie on the same side of the centre of perspectivity O.

Remark 2 And which are furthermore layered. In the sense that all points forming the red triangle are further from O than the corresponding points forming the orange triangle.

Naming Remark 3 S. Sánchez and the Author [27] called the above combination of restrictions' possible presentations the *Desarguian cranes*. For at least one of the blue pencil and the outer triangle (red) turn out to be unstable to elongation as one works out particular Desargues presentations. Conferring an anisotropic veneer to these presentations. With some aesthetic resemblances to construction-site or dock-loading cranes' configurations... Finally, if one manages to keep both of these small, then this is often at the expense of ≥ 1 of the legs becoming large. C.f. our choice of 3-legged Daliphant...

Remark 4 This elongation is furthermore undesirable in (especially printed or limited-resolution) figures. Thus setting an *optimal Desarguian crane problem* that minimizes such elongation. To which the particular Reliant supermarket trolley that the Author exhibited in the previous SSSec is an approximate solution. Subject to the further side-constraint of belonging to the stated class of Affinely-and-Metrically privileged Desarguian configurations...

At the level of arenas,

$$\mathfrak{T}\text{rolleys} \subset \mathfrak{C}\text{ranes} \subset \mathfrak{D}\text{esargues} .$$

2.5 A start on the taxonomy of pencils

Remark 1 Both Remarks 1 and 2 of the previous subsection contain restrictions which can be lifted in a number of ways. We return to most of these in Sec 9 while setting further Open Exercises and Research Projects.

Remark 2 The particular restriction that we need to lift in order to substantially outclass the Reliant supermarket trolley presentations is as follows.

Structure 1 Pencils can also realize double cones.

Remark 3 In particular, our red triangle can lie on one side of O, and our orange triangle on the other. One on each cone sheet! See Fig 5.

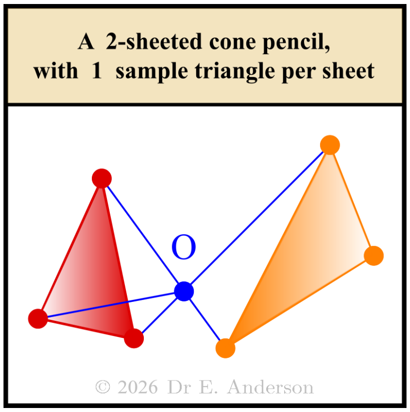


Figure 5:

2.6 The arena of Desarguian scotswolves

Naming Remark 4 S. Sánchez and the Author [27] called the ensuing configurations *Desarguian scotswolves*. An example of a such, with the output line of collinearity parallel to the input pencil's middle line, can be found on the cover of Coxeter's [10].

Structure 1 The arena of these is clearly disjoint from that of Desarguian cranes:

$$\mathfrak{S}\mbox{cotswolves} \ \cap \ \mathfrak{C}\mbox{ranes} \ = \ \emptyset \ .$$

Sec 9 shall furthermore point out why these do not collectively exhaust even the nondegenerate Desargues configurations...

2.7 The subarena of Divine Scotswolves

Remark 1 Our previous class of configurations with 2 perpendicular pairs of parallel lines supports some rather nice configurations. This is in good part since exercise 1's mishap with the 2 similar triangles in $\mathfrak{C}$ranes turns out not to extend to $\mathfrak{S}$cotswolves

Our first trick is to involve a reflection. So that 'antiparallel' elements are present. Then $\mathfrak{A}$ *is not pushed out to infinity*.

Our second trick is that placing one triangle to each side of O permits choice of a *congruent* pair of triangles. Thus supporting reflection-symmetric configurations; see Fig 6 for 3 exemplars.

Naming Remark 4 continued Within this Affinely, Metrically and symmetrically privileged class, it is somewhat easier to explain the name 'scotswolves'. $\mathfrak{D}$esargues can be decomposed into two parts. A 'Pasch canid' and the Scottish flag, each particularly visible in the current class of presentations by featuring centred and reflection-symmetrically...

The *Pasch configuration* [40, 41], alias *complete quadrilateral* [8], is the following. A planar configuration in which 4 lines meet in pairs at 6 points. It plays a foundational role in axiomatizing Euclidean Geometry [41] and is a piece of pre-Projective kinematics [39, 10]. The Author already pointed out in [31] that the underlying graph for this is the Fox . Observe furthermore that our ears contain a second triangle, 'the interior view', and it is these which constitute our input orange and red triangles.

While the Scottish flag is the D_4-symmetric 1-crossing presentation of the following. The complete graph K_4 alias tetrahedron, as covered in detail in [30, 31]. Or, more generally, its merely $C_2 \times C_2$ symmetry breakdown from a square perimater to a rectangular one. [National flags are, after all, usually rectangular.]

Aside from an overall aspect ratio

$$\alpha \ := \ \frac{\mbox{height}}{\mbox{width}} \ ,$$

the name 'scotswolves' is true enough to render memorable its other shape parameter. Namely,

$$\epsilon \ := \ \frac{\mbox{ears}}{\mbox{flag}} \ .$$

Where flag and ears are heights, of the flag and of how far above the flag the ears jut out.

Remark 2 At the level of arenas, the Divine Scotswolves live inside the following nest.

$$\mathfrak{D}\mathfrak{S}\mbox{cotswolves} \ \subset \ \mathfrak{S}\mbox{cotswolves} \ \subset \ \mathfrak{D}\mbox{esargues} \ .$$

Naming Remark 5 Various Drawing and Visualization [55, 36] finery criteria pick out specific scotswolves, including the following.

Subfig 6.a)'s, also in Subfig 1.b), is Scotsanubis , He of the taller ears. Picked to exhibit equiangularity about the centre-of-perspective cum Graph-Theoretic nexus O . To attain this feat, Scotsanubis has

$$\alpha = \frac{1 + \sqrt{3}}{2} , \quad \epsilon = \sqrt{3} .$$

Subfig c)'s is Scotsfennec , of the wider ears, so as to realize the isotropic choice

$$\alpha = 1 = \epsilon .$$

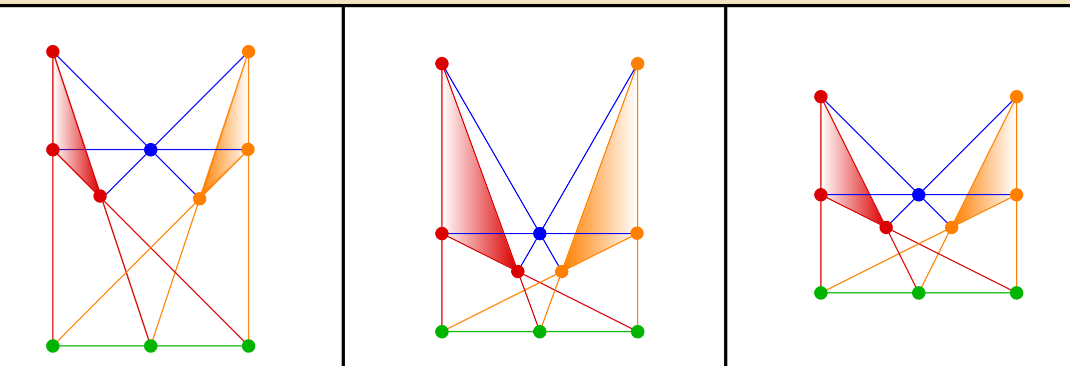
Finally Subfig c)'s is *the ordered Scotswolf*, of the shorter ears. Picked so as to contain the maximum possible amount of $\pi/4$-inclined lines, beloved of Order Theorists. The parameters here are

$$\alpha = \frac{3}{2} , \quad \epsilon = \frac{1}{2} .$$

These inclines render Ordered Scotswolf particularly suitable for building the call-signs in Sec 5. Which aligns with the good fortune of spacing out both points and faces better than Scotsfennec . As per Fig 15, itself the basis of a new line of Combinatorial Geometry problem outlined in Sec 9. Hence conferring the further alias Call-sign Scotswolf .

To approximately match Scotsanubis in divine standing, another alias for Scotsfennec is Scotsset , After another Ancient Egyptian god, who is fennec-headed. While yet another alias for Ordered Scotswolf is Scotswolfgoddess .

Some denizens of the Divine subarena of the Desarguian Scotswolves



w) Scotswolfgoddess
alias
Ordered Scotswolf,
Callsign Scotswolf

a) Scotsanubis

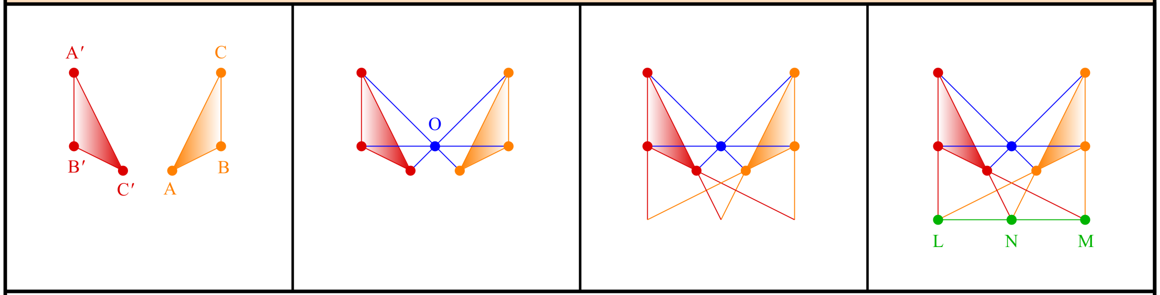
s) Scotsset
alias
Scotsfennec

Figure 6:

2.8 Desargues's Theorem in the Euclidean plane redrawn

Remark 1 Having explained our cover-figure's Drawing and Visualization paradigm shift for the Desargues configuration $\mathfrak{D}$esargues , let us now apply it to redrawing Fig 2 in the form of Fig 7. Selecting Scotsset for this role, in place of some generic 3-legged Daliphant!

Building up the Desargues configuration using Scotsset

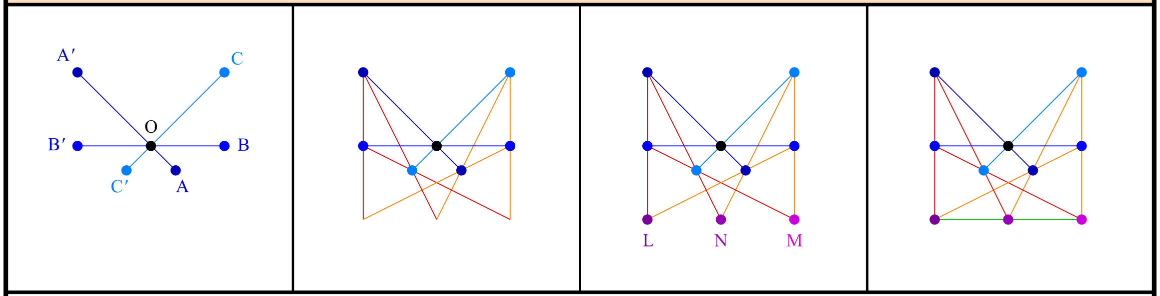


Key							
Inputs		Construct				Output	
🟥 🟧	2 triangles...	🟦	in central perspective	🟥 🟧	Sides extended	🟩	Intersection points lie on a line: axial perspective

Figure 7:

3 The matching formulation of Desargues' Theorem

Building up the Desargues configuration by matching up Scotsset



Key								
Inputs		Construct					Output	
🟦 🟦 🟦	Externally-matched pairs of points	🟥 🟥 🟥	Internally-matched pairs of points	🟧 🟧 🟧	Intersection points	🟪 🟪 🟪		
⬛	Point of coincidence						Line of collinearity	🟩

Figure 8:

Remark 1 A conceptually a priori distinct and yet a posteriori Mathematically equivalent statement is as follows. Now using not 2 $2-$triangle groupings but 3 pairs of points. Which give a matching between the previous conceptualization's triangles' vertices. This is indeed the matching arising in the above unpackaging of what central and axial perspective signify...

Theorem 1′ Let the lines AA' and cycles intersect at a single point O (all blue in Subfig b). And let AB (orange) meet $A'B'$ (red) at N (purple) and cycles. Where the colour cycle in question is pastel, middling and then intense! Then L, M, N are collinear, residing on the line that we colour in green once more.

Remark 2 We display this in Fig 8, selecting *Scotsset* again for ease of cross-comparison with the preceding Figure...

4 Proving the theorem

4.1 Split into the original theorem and its converse = Projective dual

Remark 1 Strictly speaking, Desargues' theorem is, with reference to Theorem 1, i) ⇒ ii) Its converse is however also its dual statement. By coincidence of 3 lines at a point being Projectively-dual to collinearity of 3 points on a line. Thus, within a fully Projective context, once we have a proof for the theorem itself, its converse also holds by duality.

4.2 Pre-Projective proofs

Proof 1 ⟨2⟩[1] One classical (here meaning pre-Projective) style of proof [11, 26] is to make multiple uses of Menelaus' Theorem.

Proof 2 ⟨3⟩ Another [24] is to establish the Theorem and its dual by making mixed use of Hud–Ceva's theorem alongside Menelaus'. In various more restricted senses these two theorems are also dual statements [37], and so work well together.

Proof 3 Use Linear Algebra, such as by 3 applications of the determinant formulation of Pappus' theorem [8]. For this to reside in the current section, then, we need to pick in turn a Pre-Projective proof of Pappus from the menu [24, 33].

⟨5⟩ More elaborate uses of determinants can be found in [24] under the name of 'binomial proofs'.

Proof 4 Todd [5] gives a simple vector Algebra derivation for nondegenerate configurations.

Exercise 2 Attempt proofs 1 and 3.

4.3 Desargues' theorem reappraised within Projective Geometry

Remark 1 This 17th century result harbours substantial Projective-Geometric significance. For all that recognizing this had to await the 19th and even 20th centuries; see [33] for references. In particular, it turns out to constitute one of the two main structural theorems of Projective Geometry.[2]

Proof 5 ⟨3⟩ It is straightforward to obtain Projective-pillar [23] Projective proofs intrinsically within 2-*d* [10].

Proof 6 ⟨4⟩ A larger proportion of sources, e.g. [6, 13, 19, 20] proceed however by exiting from 2-*d* to 3-*d*. This hinges on the 3-*d* situation being easier (c.f. Sec 9).

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[1] The current Encyclopedia [36] uses chevron brackets to denote "approximate year of study".

[2] See [33] for an outline of this and its interplay with the notion of incidence, or [23, 9, 10, 3, 7, 13] for more detailed accounts of these matters.

4.4 Post-Projective use of other Pillars of Geometry

Proofs 7-10 ⟨3⟩ Two Algebraic approaches using homogeneous coordinates are given in [25]. Plücker coordinates are employed instead in [24]. Finally another Algebraic approach using the Projective counterpart of Linear Algebra's dimension theorem in [22].

Proof 11 Via use of Projective Geometry's invariants: cross-ratios [24].

Remark 1 [19] give two simple proofs that employ Projective transformations.

Proof 12 ⟨3⟩ The first of these proceeds via simplifying coordinate-value choices being possible. Their particular such involves exercise 1's infinite case. Such a Projective transformation method gives furthermore a way in which Divine Scotswolves are computationally as well as presentationally significant. Though only some such meet the more stringent coordinate simplicity criteria.

Exercise 3 For now, find a Divine Scotswolf integer coordinates for all 10 of its points.

Proof 13 Their second proof rests upon a result that they prove in turn using the cross-ratio. Rendering it a mixed transformations and invariants proof.

5 Call-signs for the Desargues configuration's objects

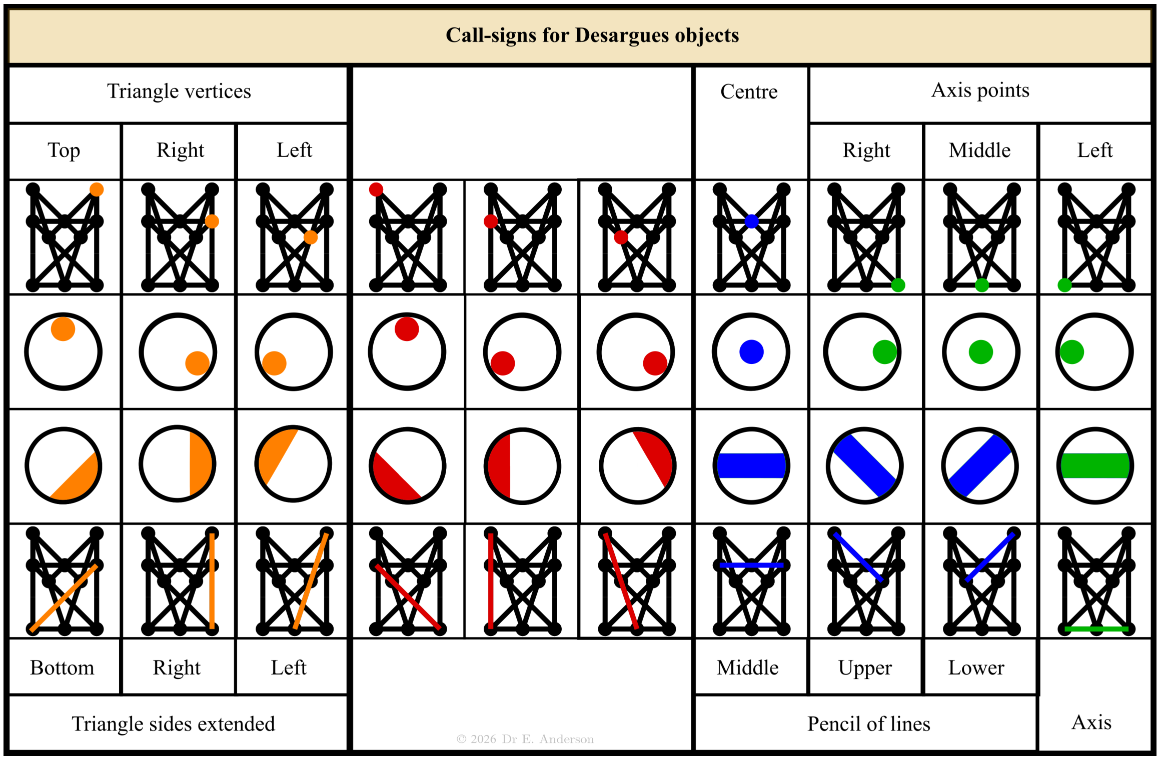


Figure 9:

Notational Remark 1 Scotswolfgoddess is most convenient as regards designing and displaying the following notation (Fig 9). Which we then apply to whichever other nondegenerate Desargues configurations. In the process, the Desargues objects highlighted in orange, red, green and blue are simplified. While all the unselected objects previously cast in black are now jettisoned. These 'call-signs' are presented inside circles, since our main use of them shall be to label the nodes of the Desargues incidence graph in the sequel Article [35].

See [31, 33, 50] for call-signs for some other Projective configurations.

6 The Desargues configuration's symmetries

Remark 1 These form the group

$$Sym(\mathfrak{D}\text{esargues}) \ = \ S_5$$

group of order

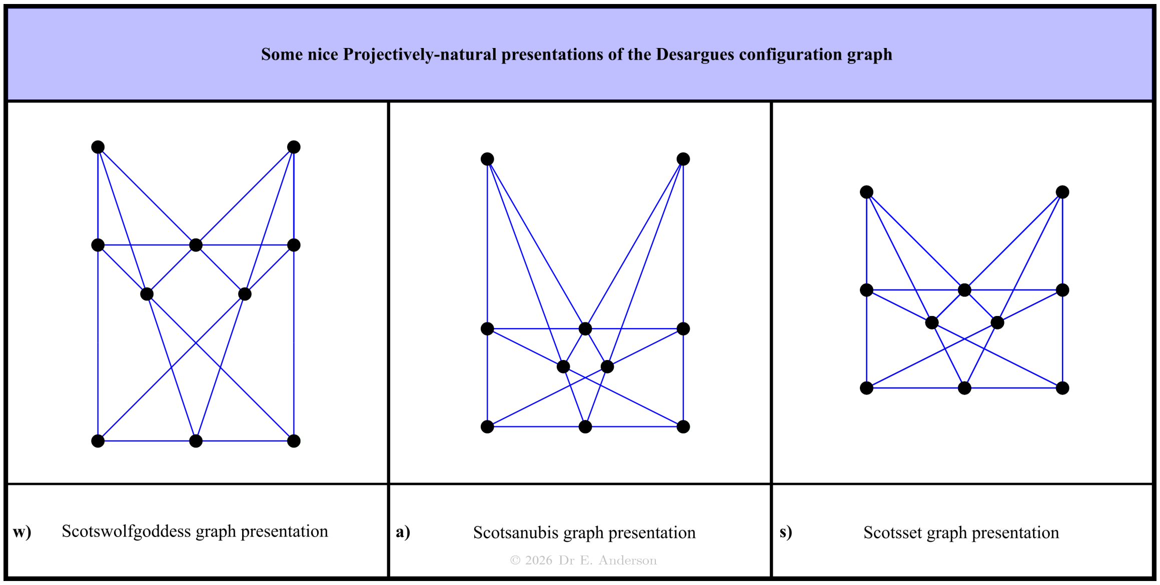
$$|\, Sym(\mathfrak{D}\text{esargues})\, | \ = \ |S_5| \ = \ 120 \ = \ 2^3 \times 3 \times 5 \ .$$

This is another situation which is more obvious to handle if one forms the 3-d version of the configuration. For this is generated by 5 planes. Which play the role of objects upon which S_5 acts naturally.

Pointer 0 So $Sym(\mathfrak{D}\text{esargues})$ is more straightforward to work out than $Sym(\mathfrak{P}\text{appus})$ [46, 45].

7 Introducing the Desargues graph

7.1 Projectively-natural presentations



w) Scotswolfgoddess graph presentation

a) Scotsanubis graph presentation

s) Scotsset graph presentation

Figure 10:

Remark 1 Let us now view Figs 4 and 6 as exhibiting *graph* presentations. For Affinely-and-Metrically privileged Desargues configurations: a Reliant supermarket trolley and three furtherly-optimized Divine Scotswolves , recast as graphs in Fig 10. Let us pick Scotsset for further representative use, since square-perimeter presentations have extra cachet in Graph Drawing and Visualization.

Remark 2 These are all *Projectively-natural* presentations of the Desargues graph $\mathfrak{D}\text{esargues}$ since they display all the requisite collinearities.

Remark 3 The last three all furthermore carry presentational C_2 reflection symmetry.

7.2 Some basic counts and properties

Remark 1 Given whichever of our four presentations of Desargues , we can count off the following.

$$V(\text{Desargues}) \;=\; N \;=\; 10\;, \tag{1}$$

$$E(\text{Desargues}) \;=\; 20\;. \tag{2}$$

So, like for the Pappus configuration graph, it is on-average quartic.

Remark 2 In more detail, the Desargues configuration graph's degree sequence is

$$\text{dv}(\text{Desargues}) \;=\; 3^4\,4^3\,5^2\,6\;. \tag{3}$$

It is consequently neither *regular* [28]: of a single degree. Nor a *cone* [28]: with

$$\geq\;[\;1\;\;\text{degree-}(V\;-\;1)\;=\;9\;]\;\;\text{vertices}\;.$$

It does however have a sole vertex that is just 3 short of this: 6 . This 'nexus' shall play a dominant role in some of our presentations below. For now, observe that Scotsanubis displays equiangularity about the nexus. While maintaining the collinearities that need to be present in any Projectively-natural presentation.

Remark 3 The following counting argument renders the complement graph $\overline{\text{Desargues}}$ less useful than Desargues itself.

$$E_{\text{max}} \;=\; \frac{V\,(V\;-\;1)}{2} \;=\; \frac{10\;\times\;9}{2}\;.$$

So

$$E\left(\overline{\text{Desargues}}\right) \;=\; E_{\text{max}} \;-\; E(\text{Desargues}) \;=\; 45\;-\;20 \;=\; 25\;>\;20\;=\;E(\text{Desargues})\;.$$

Remark 4 The current Article's graph presentations fail however to capture most of the Desargues configuration's symmetries.

7.3 The Desargues graph is planar

Remark 1 Our Reliant supermarket trolley is a 6-crossing presentation. While our Divine Scotswolves are 5-crossing presentations.

Remark 2 It is however straightforward to get rid of these crossings. Rendering the Desargues configuration graph planar.

We give a presentation with the nexus placed centrally in column 1 of Fig 11. And one with it placed internally in column 2. The second of these manages to hold on to a higher proportion of the original Projective configuration's features. Thus we investigate somewhat larger square-grid fits for it, so as to accummulate various further items of presentational finery, as indicated in columns 3 and 4.

Remark 3 So far, we have considered locating the triangle subsystems within these graph presentations. Success is more limited as regards preserving the matching collinearities, as per Fig 12. Unless one is willing to keep a single crossing. In which case we can exhibit all 3 of these collinearities rather than just 1 , as the last subfigure indicates.

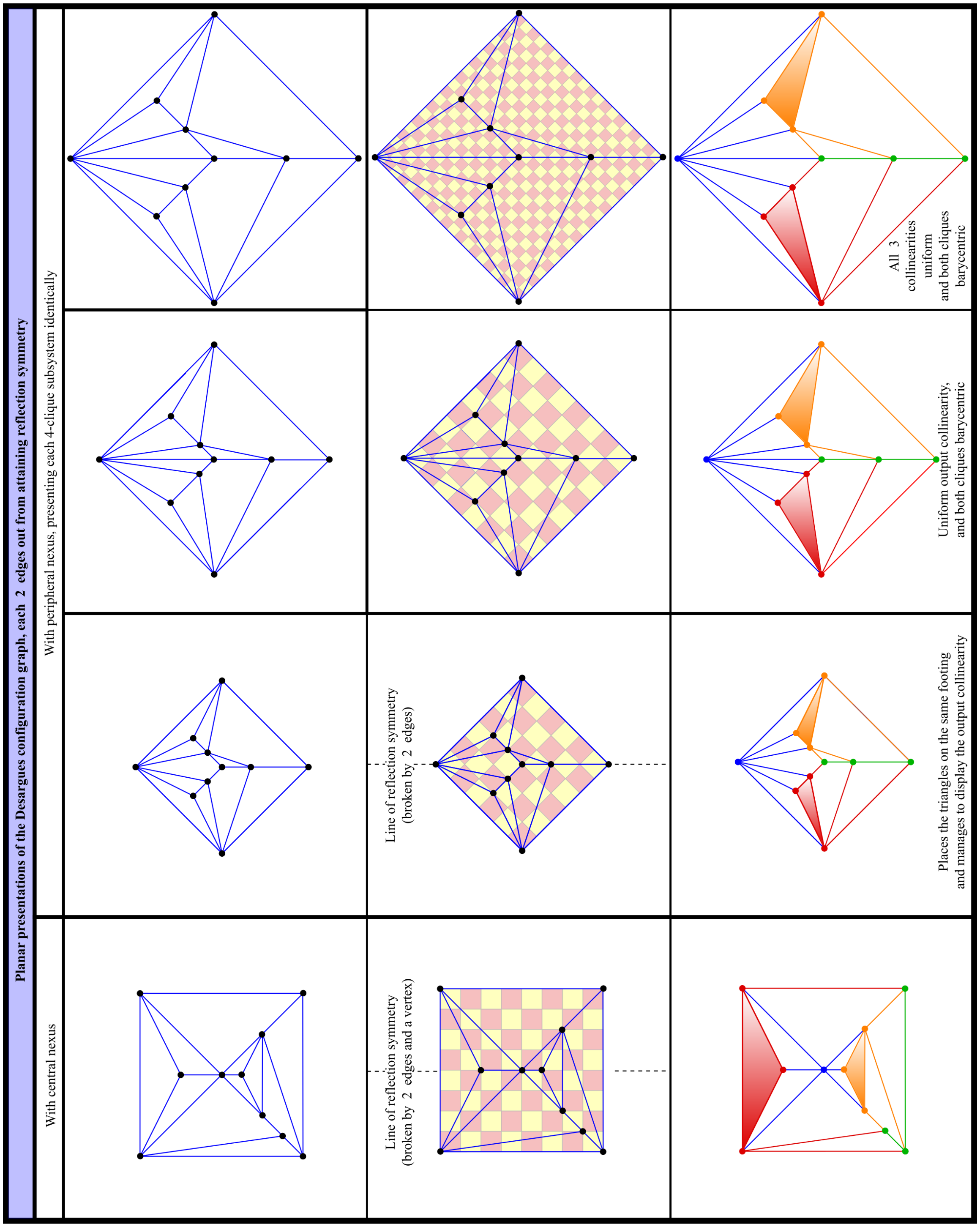


Figure 11:

7.4 Exercises

Exercise 4$^-$ Check the current subsection's graph isomorphisms.

Open Exercise 5 Which degenerate cases (extra alignmants, point coincidences...) of Desarguian configurations and graphs are possible?

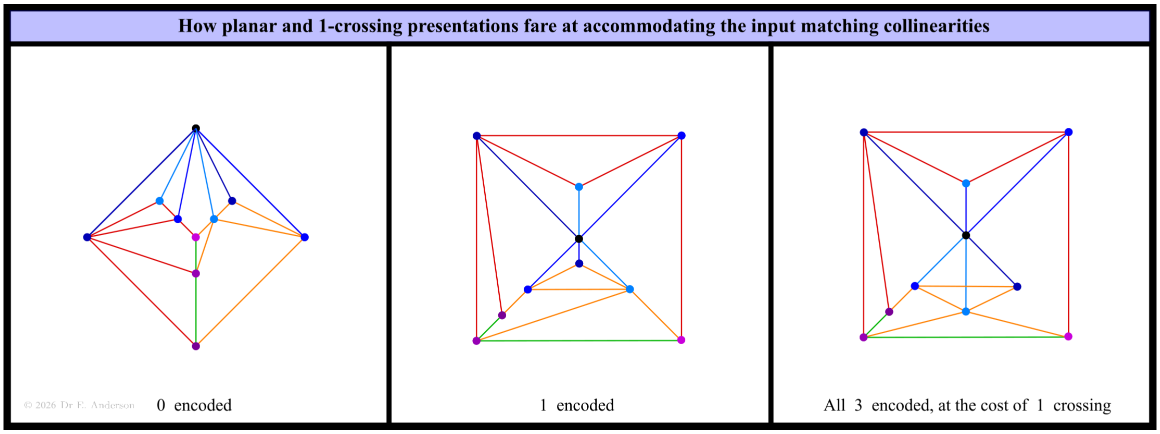


Figure 12:

8 Structural analysis

8.1 The Desargues graph is a double irreducible

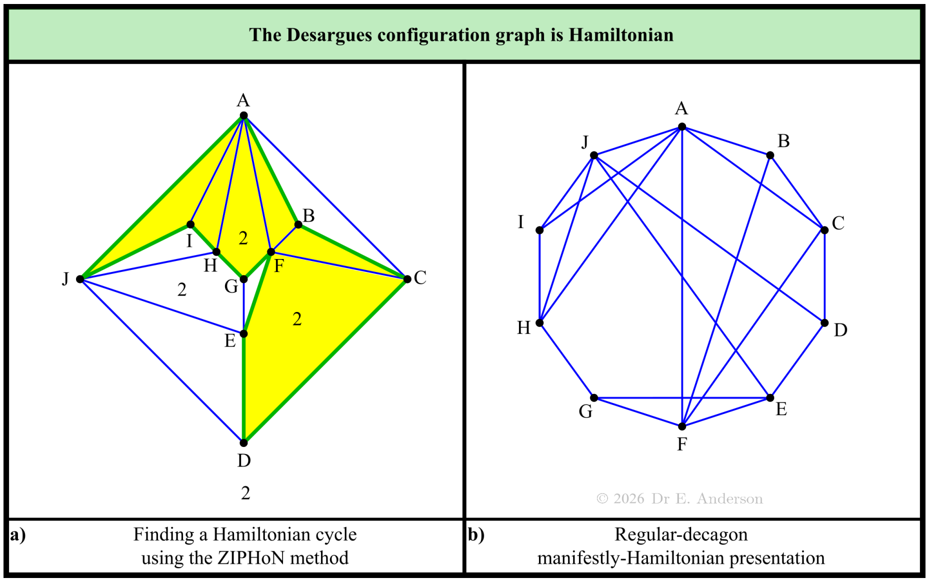
Remark 1 I(Desargues) is cubic and thus has no degree-1 or -2 vertices. Thus it is both foliation irreducible and homeomorph irreducible. Which combination we term *double irreducible*: class D [29].

8.2 Notions of traversability

8.2.1 The basics

Remark 1 The Desargues graph is immediately not *Eulerian*, since it contains odd-degree vertices.

Remark 2 The Desargues graph is straightforwardly *Hamiltonian*; see e.g. Fig 13.



a) Finding a Hamiltonian cycle using the ZIPHoN method

b) Regular-decagon manifestly-Hamiltonian presentation

Figure 13:

8.2.2 Use of the ZIPHoN method

Remark 3 With some overkill, its Hamiltonianess can be ascertained. Using the *ZIPHoN theorem.*[3] By this result, every Hamiltonian cycle splits the graph into 2 outerplanar strips. Each containing an equal amount T of triangulating triangles.

Notational Remark 1 We colour in one outerplanar strip in yellow, leaving the other in white (Fig 13.a). This includes modelling yellow and white as individually-meaningless and yet mutually-distinguishable labels. By which such strips are invariant under colour exchange. Due to this, one can w.l.o.g. never colour the outer face. We finally mark the bounding Hamiltonian cycles using thick emerald edges.

Remark 4 For Desargues ,

$$T = 8 .$$

8.2.3 Manifestly Hamiltonian presentations

Remark 5 A corresponding manifestly-Hamiltonian presentation of the Desargues graph is given in Subfig b).

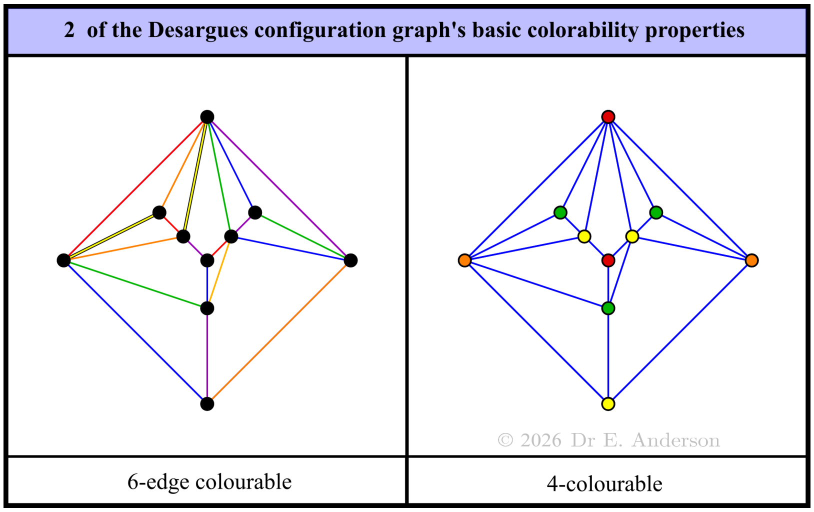
Exercise 6 Find all the distinct pairs of outerplanar strips supported that return a bounding Hamiltonian cycle. Finally draw a regular-decagon manifestly-Hamiltonian presentation for each.

8.3 Colorability

Remark 1 Since the Desargues graph has a degree-6 vertex, it is $\geq$ 6-edge colourable. In fact, it is precisely 6-edge colourable, as per Fig 14.a).

Remark 2 Since the Desargues graph contains 4-cliques, it is $\geq$ 4-colourable. In fact, it is precisely 4-colourable, as per Subfig b).

Exercise 7^- Show that the Desargues graph is not however uniquely colourable [49].



2 of the Desargues configuration graph's basic colorability properties

6-edge colourable

4-colourable

Figure 14:

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[3] Zero-index planar Hamiltonian necessity theorem [27, 28, 38]. Alias Grinberg's theorem [12, 18, 21].

9 Pointers to allied material

9.1 Further Desargues material

Pointer 1 ⟨3-4⟩ A more systematic study would start with *Desargues' law*'s simpler configurations (the second piece of Fig 7, and Fig 5). These shall be covered within our article on the simplest Projective drawings' graphs [39]. Which allows for pieces of Projective constructs, not extensive enough to possess Projective content of their own.

Pointer 2 ⟨5-6⟩ A subsequent systematic study [57] shall also cover degenerate cases, such as Desargues' little theorem.

Research Project 1 ⟨7^+⟩ To the best of our knowledge [27], the arena of (the totality of) Desargues configurations

$$\mathfrak{D}\text{esargues}$$

has so far received very little attention. Nor has the matter of which optimal Desarguian cranes and scotswolves this supports. By which it is likely that multiple further constrained optima of interest remain to be found.

This follows on in part from exercise 1's consideration of Affinely-privileged subcases of Desargues' configuration.

In part from the pencil about the point at infinity consisting of parallel lines. While layering a 3-line pencil with 2 triangles admits further general-position cases than the cranes and the scotswolves. For instance via intersecting pairs of triangles, or triangles shared between both sheets of the double cone...

And in part from exercise 5's consideration of degenerate Desargues configurations.

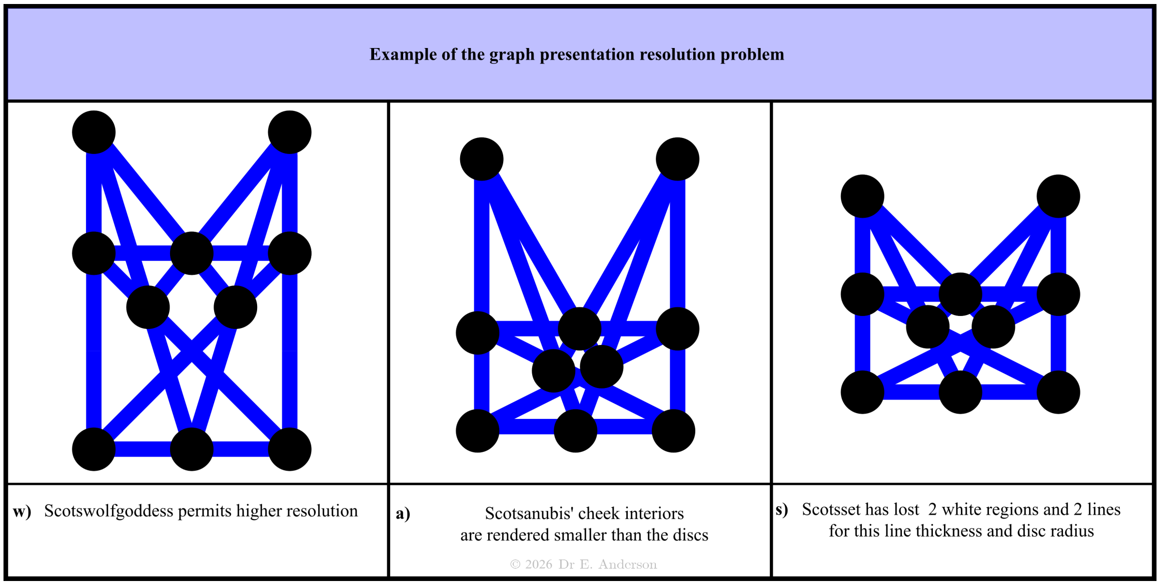
Remark 1 of Sec 6 points to the eventual onset of discrete quotients in this study.

Pointer 3 ⟨5-7⟩ It is no coincidence that the Desargues configuration $\mathfrak{D}\text{esargues}$ and the Petersen graph [56, 48] share symmetry group. On the one hand, the dodecahedron–icosahedron Platonically-dual pair merely shares symmetry group order 120 with these rather than more structurally sharing symmetry group itself. On the other hand, there is a rich tapestry of relations between all four of the current Pointer's objects, citizens of Kallista [27, 28, 37, 30, 31, 32] all.

Pointer 4 ⟨3-4⟩ Let us now follow on from Hilbert showing that Pappus' theorem is equivalent to Algebraic commutativity [41, 23]. By observing that he next showed that Desargues' theorem corresponds to the *division-ring* structure, almost as basic to Abstract Algebra! [33]'s point that Pappus follows from Desargues in finite planes can then be rooted in *Wedderburn's little theorem.* That finite division rings must be commutative...

Pointer 5 ⟨3-5⟩ Incidence is the central notion in Projective Geometry. The *incidence graph* for the Desargues configuration shall first be covered in [35]; *Levi graphs* [43] are a Graph Theory alias for incidence graphs. There and in Geometry often call the Desargues incidence graph the Desargues graph.

9.2 A line of Geometrical Combinatorics problems posed



Example of the graph presentation resolution problem

w) Scotswolfgoddess permits higher resolution

a) Scotsanubis' cheek interiors are rendered smaller than the discs

s) Scotsset has lost 2 white regions and 2 lines for this line thickness and disc radius

Figure 15:

Research Project 2 [27] Given an arena of presentations of a given graph, which can be displayed at highest resolution? I.e. which has all edges long enough, and all faces broad enough, to sustain the following? Uniform thickening of edges alongside uniform enlargement of vertices into discs. Such that no visible chunk of edge is shorter than it is wide, or than the diameter of the discs. And no visible chunk of face is smaller than any visible chunk of edge or disc.

Various variants are possible via how visible chunks of face's sizes are quantified. One possibility is by area. Another is to forbid splinters [47] – thin triangles – even if their area exceeds that of the discs and visible edge chunks.

It is surely easier to start with a fixed edge-thickness to disc-radius ratio. Such that the discs jut out sufficiently from the edges to be visible against those edges, in particular in monochromatic figures.

Remark 1 We illustrate this by displaying how Scotswolfgoddess is more satisfactory than Scotsanubis or Scotsset . At least for the particular choices of edge thickness and disc radius displayed. We leave it for another occasion to vary these parameters and to use face-area and splinter quantifiers.

9.3 Toward some of Desargues' relatives

Pointer 6 ⟨3-6⟩ The smallest Projective configuration is the *Fano plane* [31, 43, 44, 50, 52, 54]

$$\mathfrak{F}\text{ano} \ = \ 7_3 \ .$$

To explain the subsequently useful systematic notation on the right, Fano consists of 7 points that are collinear in 3's. Forming the heterodual total of 7 lines: the same count as for its points.

As a Projective configuration, this has a unique proxime – Möbius–Kantor [43, 54] –

$$\mathfrak{M}\text{K} \ = \ 8_3 \ .$$

The Pappus configuration $\mathfrak{P}$appus is non-uniquely next-smallest: a 9_3 . Sharing this honour with 2 other Projective configurations [50, 58]. Finally, the current Article's Desargues configuration $\mathfrak{D}$esargues is non-uniquely next-smallest – a 10_3 – now shared with 9 others [50, 58].

$\mathfrak{P}$appus and $\mathfrak{D}$esargues are moreover unique among the 9_3 and 10_3 as regards supporting structurally-significant fundamental theorems of Projective Geometry. A fortiori, there are *no* further such structural theorems, no matter what other larger configurations one considers...

Pointer 7 ⟨7⟩ Interestingly, abstract Projective planes are capable of being non-Desarguian [51, 52].

Finite Projective planes must contain

$$d_{\mathrm{PP}} \ := \ n^2 \ + \ n \ + \ 1 \ \text{ points where } \ n \ \in \ \mathbb{N}_2 \ .$$

$$n \ = \ 9$$

turns out to be the minimum value supporting non-Desarguian planes. This corresponds to

$$d_{\mathrm{PP}} \ = \ 91 \ \text{ points.}$$

3 such planes are supported here [44]. Firstly, the minimum heterodual set of Projective planes, named after (Marshall) Hall [42]. Secondly, the minimum homodual Projective plane not to belong to the series most obviously generalizing the Fano plane.[4] Which is usually named after (Daniel) Hughes.

Pointer 8 ⟨4-7⟩ In fact, this feature renders 2-d Projective-Geometrically the sole Goldilocks dimension. This is since for ≥ 3-d Desargues' theorem becomes obligatory, leaving no room for non-Desarguian geometries. While, at the other end, 1-d Projective Geometry is substantially structurally poorer, for instance along Group-Theoretic lines.[5] And also as regards statements about polygons, or even about multiple lines, reducing to statements about a single line... Finally, 0-d Projective Geometry is rendered trivial by there being no room for any lines at all. And thus no room for incidence, other that piling up coincidence!

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[4] Following on from [31]'s mention, this is the series churning out as its symmetry groups the Chevalley Lie-type group series A ⟨6⟩...

[5] While 1 special-Projective transformation perseveres, it now has no shears, squeezes or rotations to interact with [37]...

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