Minimum-Crossing Cubic Graphs

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Abstract

Heirs of whichever of Fano and Petersen turn out to give examples of this property for almost all values (minor modifications included). Usually modulo a small amount of nonuniqueness. And with distinguished roles played by cages, Projective configuration incidence graphs, and generalized Petersen graphs GP . Whether Tutte-8 is an example for the second unknown case is singled out as a particularly interesting question. Finally use elsewhere of modern Graph Drawing and Visualization for these graphs is motivated.

This Article is ⟨3⟩: accessible to third-year undergraduates.

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1 Most beautiful among the known

Remark 1 The following path of citizen-of-Kallista [20, 24, 25, 26] cubic graphs possess the titular property [16, 21, 30]. We parametrize below by the crossing number [22] Cr and the cubic graphs' half-order adapted variable [20, 24]

$$m = \frac{V}{2} .$$

$Cr = 0$	for	Tet	=	K_4	=	Cage(3)	on	$m = 2$.	
$Cr = 1$	for	Utilities	=	$K_{3,3}$	=	Cage(4)	on	$m = 3$.	
$Cr = 2$	for	Pet	=	GP(5, 2)	=	Cage(5)	on	$m = 5$.	
$Cr = 3$	for	I($\mathfrak{F}$ano)	=	Heawood	=	Cage(6)	on	$m = 7$.	$(*)$
$Cr = 4$	for	I($\mathfrak{M}$K)	=	GP(8, 3)			on	$m = 8$.	
$Cr = 5$	for	I($\mathfrak{P}$appus)					on	$m = 9$.	
$Cr = 6$	for	I($\mathfrak{D}$esargues)	=	GP(10, 3)		= 10 .			

Remark 2 So we go through a cage [9, 18] phase. Which can be viewed as leading up to the Petersen graph [33, 13] Pet and then continuing along one of its main lines of heirs. And then pass to a phase consisting of the incidence graphs [2, 17, 26, 28, 29] of renown Projective configurations [14, 15, 25]. Which are continuously held together by the Heawood graph belonging to both: (*). For all that none of these uniquely enjoy this property upwards from Pet inclusive.

Remark 3 $Cr = 7$ is the first case not to be solved by any (exhalted) citizen of Kallista. This occurs mooreover for $m = 11$, and so $V = 22$: the first failed Moore number [26].

Remark 4 And the corresponding minimum non-Moore cage (MNMC) [6, 9, 18] then continues the table:

$$Cr = 8 \quad \text{for} \quad \text{McGee} = \text{Cage}(7) = \text{MNMC} \quad \text{on} \quad m = 12 .$$

And simple operations performed on this cage find minimum examples for $Cr = 9, 10$ as well.[1]

Remark 5 With another heir of Fano – the Coxeter graph [10] – on $m = 14$ itself serving for $Cr = 11$. While also providing by a simple operation a different example for $Cr = 9$. And a distinct

$$Cr = 8 \quad \text{is provided by} \quad \text{Nauru} = \text{GP}(12, 5) \quad \text{on} \quad m = 12 .$$

$Cr = 10, 11$ are minimum to be realized by the same value of m . From which the Reader can deduce that $Cr = 9$ occurs for $m = 13$.

[1] We celebrate by awarding the McGee cage an extra red ring in subsequent editions of [26].

2 And into the unknown

Remark 6 Beyond here, humanity currently only has bounds on m. With the highly-exhalted [1, 4, 3, 18, 26, 23, 31, 7]

$$\text{Tutte-8} = \text{Cage}(8) = \text{I}(\mathfrak{C}\text{R})$$

among the as-still-unbeaten bounding cases on $Cr = 13$ at $m \leq 15$. Where $\mathfrak{C}$R stands for the *Cremona–Richmond configuration* [14, 15, 17]: the minimum girth-4 Projective configuration.

3 Pedagogy and pointer to a hard Research Project

Remark 1 S. Sánchez called the above pattern 'the double of the voice of God' in the sense of a well-known early scene in the 1984 film "Amadeus". In that it starts with nothing much, until the 'oboe' of confirmed use of the cage aspect 'opens high above'. Until the 'clarinet' of heirs-of-Fano Projective configuration incidence graphs 'takes over its note'. Except that now it hands this back to the 'oboe' in the form of the McGee cage. Which then passes it back to an heir-of-Fano in the form of the Coxeter graph...

We do not yet know whether this more-than-doubles Mozart's Serenade N° 10 for Winds 'Gran Partita' III. Adagio... In the sense that it would if the 'oboe' and 'clarinet' then overlap for a second time, now in the form of Tutte-8. Since this is where the two main [26] joint children of Fano and Petersen – Heawood and Cremona–Richmond – themselves intersect.

Hard Research Project Either pin yet another medal on Tutte-8 or refute the above possibility...

Remark 2 Two cumulative reasons that this is hard are as follows. Firstly, finding crossing numbers is hard [22]. Secondly, converting a bound into a confirmed result at least a priori requires sweeping through every smaller cubic graph's crossing number. And this populace's size increases exponentially steeply with m. So compositionally, this places an exponentially-large number of reruns on the first hard problem...

4 Our more humble future aims

Pointer 1 To use modern Graph Drawing and Visualization [19] on the abovementioned exhalted graphs and close relations. Which we have already carried out for Tet and Utilities in [24], and shall shortly be extending to Pet [33]. And have already done for the Fano, Pappus and Desargues incidence graphs [26, 28, 29]. That already the Pappus case's minimum-crossing presentation had already not been optimized by some modern standards motivated [28], rendering it the strongest of these Articles for new research. It is then quite likely that other presentational optimizations remain to be discovered among other fairly small citizens-of-Kallista graphs. And for all that [26] served a distinct purpose, by continuing [25]'s task of starting to lay out a large-scale research program of particular significance to 'the discrete half of (easier) Mathematics': "the heirs of Fano".

And more eventually to likewise present and study the properties of the various graphs that are co-minimum in this way, and of the known bounding graphs.

Pointer 2 The current Note's problem is a 'lone wolf'. In the sense that the other commonly-encountered problems with comparable output share some common-pack properties among themselves and not with the current Note's problem. Where by 'comparable output' we mean with most smallest cases exemplified by citizens of Kallista with moderate nonuniqueness. While by 'common-pack properties' we mean advanced symmetry properties. The *cubic symmetric graphs* [23] the *cubic distance-regular graphs* [11, 31] and the *cubic distance-transitive graphs* [7] are three particularly well-known members of this pack; see also [5, 8, 12].

By their beautiful and multi-faceted nature, some citizens of Kallista manage to be on multiple such lists. Aiding in our eventual aim to cover in the abovementioned manner all citizens that make it on to at least one such list... With a cut-off at around $m = 23$, other than for problems with a proven small-finite total number of cases, and for particularly kallyxtenically strong graphs.

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