The Fano Configuration, Plane and Graph

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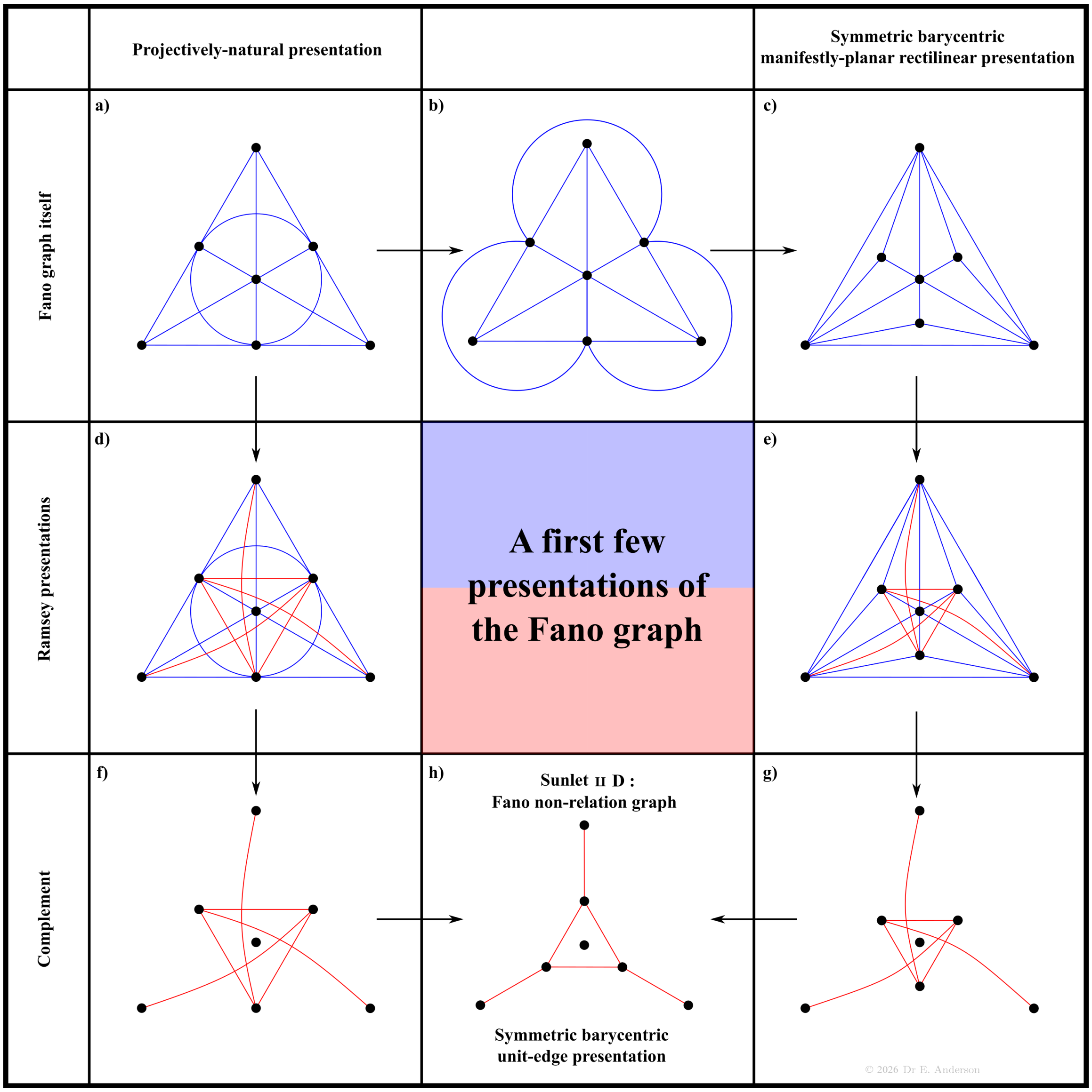

Figure 1:

The bulk of this Article is ⟨3⟩: acessible to third-year undergraduates.

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Abstract

We introduce the Fano configuration, plane and graph, displaying numerous nice presentations for the graph. Fano is 'a citizen of Kallista'. Meaning that it recurs along many conceptually-distinct routes of thought within many other areas of Mathematics. Such as the total stellations subcase of the Combinatorially and Topologically significant strong triangulations of the triangle. The total Apollonian networks of Contact Geometry. And Graph Theory's uniquely-colourable graphs, to whose theory we also contribute a poset notion.

The Fano objects serve furthermore as minimum nontrivial objects of many kinds. In particular, the Fano plane is the minimum projective plane. And its automorphism group is the smallest simple group that is not cyclic or alternating, of order 168. Many routes' second-minimum objects moreover differ, pointing us to multiple interesting structures to subsequently study. Thus exemplifying the so far typical behaviour of citizens of Kallista's multiple routes siring multiple heirs. Aside from extensions of the above-mentioned aspects, the Fano configuration has matroid, Steiner triple system, design and Hadamard-design heirs.

1 The Fano configuration

1.1 Projective planes

Subject 0 *Projective Geometry* [36, 20, 21, 13, 17, 25] is the study of *incidence*. At the level of projective planes, this is a binary relation on points-and-lines.

Structure 0 Axioms for a *Projective plane* are as follows [31, 75].

Projective Plane 0 Any 2 distinct points are incident with ('lie in') a unique line.

Projective Plane 1 Any 2 distinct lines are incident at ('intersect at') a unique point.

Projective Plane 2 ≥ 4 points are to be present such that no 3 of them are collinear.

Remark 1 0) and 1) form a *projectively-dual* pair of axioms. For planes, this is in the sense of exchanging points and lines. More generally, this aspect becomes dimension and codimension dependent. Notions of join and meet are also to be interchanged.

Naming Remark 1 The above 'lie in' amounts to being joined: by a line. While the above 'intersect' is a notion of meeting: at a point. Using 'are incident' in place of whichever of these automatically builds in the join-and-meet dual. 'Lines-and-points' can then be interpreted as a concomitant automatically-dual phrasing in the case of planes.

Remark 2 In contrast, 2) gives content to notions of plane exceeding mere notions of line. For without this, one would still be axiomatizing a Projective structure, albeit one that is just a Projective line...

1.2 Minimum Projective plane route to the Fano plane

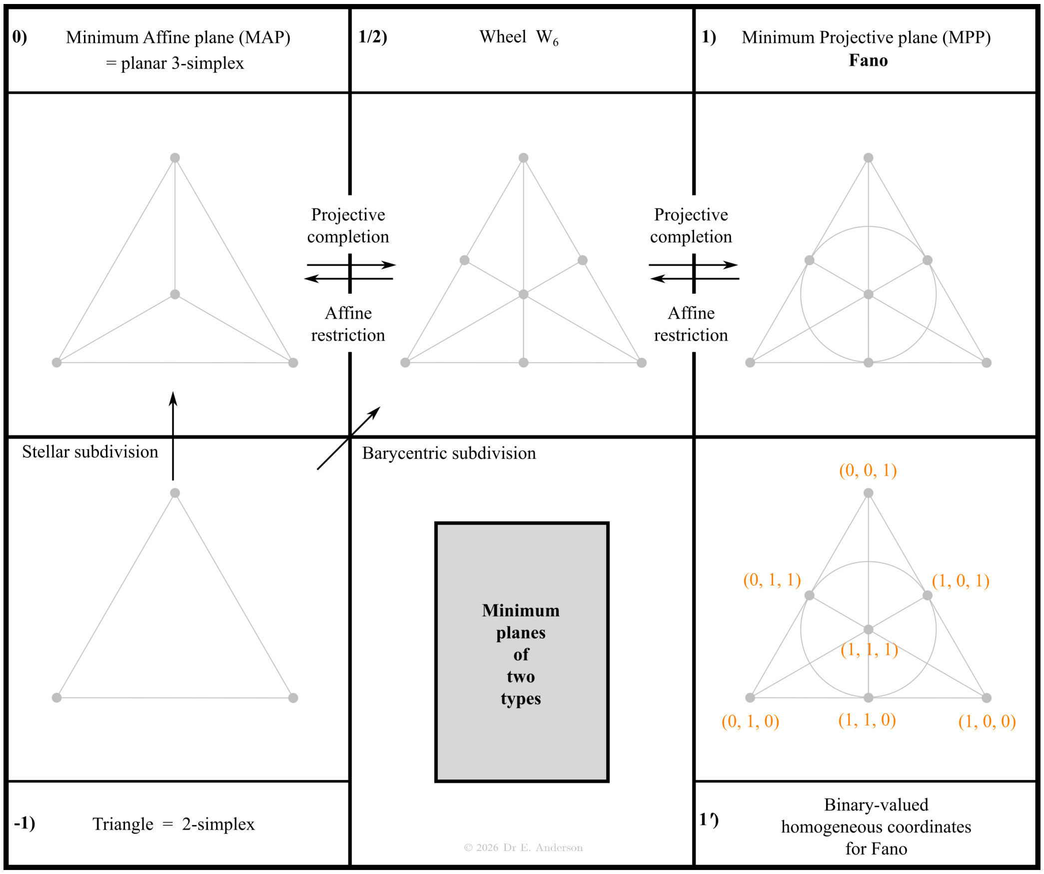

Figure 2:

Route 0 A zeroth route to the Fano plane $\mathfrak{Fano}$ [12, 16, 31, 80] (Fig 2.1) is setting out to find the minimum *Projective plane.*

Remark 3 The circle here represents a line that is no different from this Subfig's others. This points to an imperfection in the standard style of presentation for Fano . Which is brought about by the Fano plane not being embeddable in our usual real plane $\mathbb{R}^2$ ... The Projectively-natural presentation can furthermore immediately be rectilinearized while preserving the number of crossings. The preceding statement then becomes the following. For its inner triangle is a straight line just like any other in this presentation.

Remark 4 The labels in Subfig 1′) are *homogeneous coordinates*: Projective Geometry's analogue of Cartesian coordinates. Homogeneous coordinates however exceed the space in question's dimension d by 1 in number. The interconnection is that one is to form ratios out of homogeneous coordinates. And to have d independent ratios, we require $D := d + 1$ quantities [22]! So the 2-d real-projective plane has 3 independent homogeneous coordinates to the 2-d real plane having 2 independent Cartesian coordinates.

For the Fano plane, these are binary-valued. $(0, 0, 0)$ is not however among them. For no ratios can be defined between a set of numbers consisting of just zeros! Modifying the count of binary triples from

$$2^3 = 8$$

to

$$8 - 1 = 7 = |\,\text{Fano}\,| \,.$$

1.3 Completing the minimum Affine plane route to the Fano plane

Subject 1 *Affine Geometry* [18, 28, 26] is the study of *parallelism*. Historically this largely preceded Projective Geometry, from Euclid's parallel postulate to Euler's 18th century abstraction [6] of Affine Geometry. In contrast, while the first Projectively significant result was found by Pappus in the 4th century, this did not arise in a Projective context. The next were found separately by Desargues and Pascal in the 17th century. But synthesis into a coherent picture of Projective Geometry had to await the early 19th century, with work of Carnot [8], Poncelet [9] and von Staudt [10]. Projective Geometry was furthermore found to simplify many Affine results and proofs, and to complete various Affine structures.

Remark 5 Another route to the Fano plane is setting out find the minimum Affine plane

$$\mathfrak{M}\mathrm{AP}$$

of Subfig 2.0). Which corresponds to the tetrahedron graph alias complete graph K_4 . And then Projectively completing this to form a Subfig 1)'s minimum Projective plane,

$$\mathfrak{M}\mathrm{PP} \quad = \quad \mathfrak{F}\mathrm{ano}\ .$$

To make sense of this approach, we first need to present the Affine plane axioms [31, 75] ...

Affine Plane 0 and 2 These are the same as Projective Plane 0 and 2. Indeed, they build up from a notion of line to a notion of plane, without reference to what Geometrical kind of plane we mean!

Affine Plane 1 A point P and line $\mathfrak{L}$ support a unique line which contains P And is parallel to $\mathfrak{L}$.

Remark 6 This has various equivalent formulations corresponding to considerably different conceptualizations. Among these, the following due to Playfair (18th Century) [7] is somewhat of a stepping stone toward Projective Geometry.

Affine Plane 1′ Suppose that we start with a line $\mathfrak{L}$ and any point P not lying on $\mathfrak{L}$. Then there is a unique line containing P while not meeting $\mathfrak{L}$.

Remark 7 For it can be Projectively reformulated as follows.

Affine Plane 1″ Suppose that we start with a line $\mathfrak{L}$ and any point P not incident with $\mathfrak{L}$. Then P is incident with a unique line that is not incident with $\mathfrak{L}$.

Remark 8 Passage to Projective Geometry moreover involves ditching this for the dual version of axiom 0. Thus forming a more structured first pair of axioms.

Structure 1 We next need to explain what we mean by the form taken by *Projective completion* when applied to an incipient Affine plane. This amounts to adding in *points at infinity*. So as to serve as where parallel lines extend to intersect. Alongside adding in a line at infinity incident with the points at infinity [21]. Which is subsequently to be treated no differently from the others.

Proceeding in the opposite direction – *Affine restriction* – amounts to ripping out a line assigned to serve in this role. Alongside its incident points and the other line segments incident with these points.

So consider the Fano plane as presented in Subfig 2.1). Then it is the 'line depicted as a circle' that is assigned the role of the line at infinity. And is thus ripped out in passing to the smallest Affine plane.

While starting from $\mathfrak{M}$AP , 3 points at infinity are required. In our equilateral presentation, extending to these and placing new points at the intersections forms the *barycentric subdivision* of Subfig -1) to 1/2). Finally joining these up necessitates representing this line as a circle.[1]

1.4 Call-signs for the Fano presentation

Notational Remark 1 Let us denote the central point by Fig 3.a.i). The top corner-point by Subfig a.ii) and the left and right corner-points by its 3-cycles. The bottom mid-point by Subfig a.iii) and the left and right mid-points by its 3-cycles. The circle by Subfig a.iv). The bottom side by Subfig a.v) and the left and right sides by its 3-cycles. And the vertical median by Subfig a.vi), to the left and right medians by its 3-cycles.

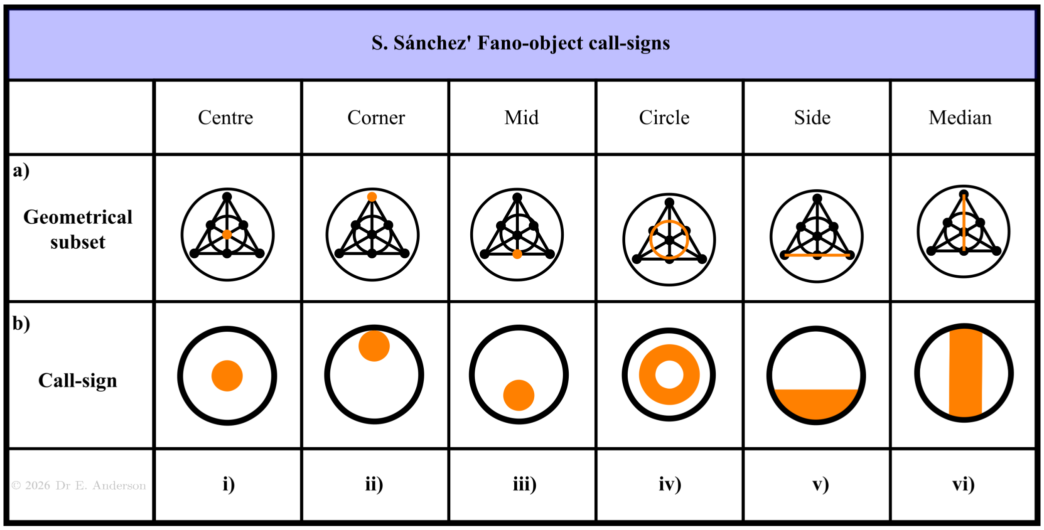

Figure 3:

Remark 1 This is motivated by frequent use in the sesuel Article's incidence graph presentations.

Notational Remark 1 Suppose that one highlights each point in the configuration in turn. Then the correct Projectively-dual depiction is to also highlight each line in turn. As opposed to each line alongside the symbols for the 3 points on it. For the Projective dual of that would be points and the 3 lines through them...[2]

Notational Remark 2 S. Sánchez further improved this notation by introducing the *Fano call-signs* in Subfig b). These both simplify the depiction and enable considerably smaller-print or lower-resolution figures.[3]

[1] Projective Geometry does not in any case distinguish between lines and circles... We might as well use 'cline' for circle-or line. Though it appears to be slightly less convenient to portmanteau 'line segment' and 'circular arc'. 'Cline sarc' is our hitherto unaired suggestion [37].

[2] This is mentioned since when the current Article was written, Wikipedia [45] was fielding the first and third sentence's dual mismatch.

[3] These arose in the context that S. Sánchez was proofreading the Author's N-body problem reviews, which contain call-signs for Flat Geometry invariants, concurrently with teaching the Author about the Fano plane. And S. Sánchez held the Fano plane to be significant enough to contribute call signs for it. Alongside setting the Author the easier challenge of devising optimal call-signs for the Pappus configuration [43]. And the harder challenge of doing so for the more heterogeneous Desargues configuration [44].

2 Introducing the Fano Graph

2.1 The projective route

Structure 1 The underlying *Fano graph* is in Subfig 1.a). By its matching the Projective Geometry configuration, let us refer to a) as the Projectively-natural presentation of the Fano graph.

Naming Remark 2 This is a *configuration graph*. Coxeter [16] uses the historical alias *Menger graph* [15], alongside chiding these for being unfaithful... This is part of why our companion Article [42] considers instead the Fano *incidence graph*. *Levi graph* [14] is now the historical alias.

Remark 1 The Fano graph is readily established to be planar. By taking the circle's arcs outside the embedded vertices' convex hull (Subfig b). Subfig c) is then a corresponding rectilinear presentation; it is always possible to find such for planar graphs by the Fáry–Stein–Wagner Theorem [33].

Remark 2 Among all the possible rectilinear presentations, this particular one is uniquely fixed by firstly manifesting the graph's full symmetry. This is $S_3 = D_3$: the symmetry group of the equilateral triangle. And so entails working within an equilateral triangle perimeter. And secondly by barycentrically placing the remaining vertices. Which is a stronger restriction as regards the middle layer of vertices than just maintaining the above symmetry group...

2.2 The stellar subdivision route

Structure 2 Indeed, the Fano graph is also the result of twice *totally stellating* a triangle. In which each input triangle subgraph $C = C_3$ (3-cycle) with Tet subgraph. Where Tet is the *tetrahedron graph*, alias 3-simplex and complete 4-graph K_4 (Fig 2.0).

Tet also happens to be the graph underlying $\mathfrak{M}$AP . Fig 2 also includes the first of the above two stellations. While the second is equivalent at the level of graphs as per Fig 2 to the following. Fig 2's Projective completion of $\mathfrak{M}$AP to $\mathfrak{M}$PP : the minimum Projective plane, i.e. Fano .

For the first stellation of a triangle, there is no difference between stellation and total stellation. But the second stellation is minimal for these to be distinct notions. For one now has 3 smaller triangles, and one could choose to stellate just 1 , 2 or all 3 . See Fig 4 for the first two cases; the second deploys cone and complement notations. While if all 3 are picked, a total stellation is being conducted.

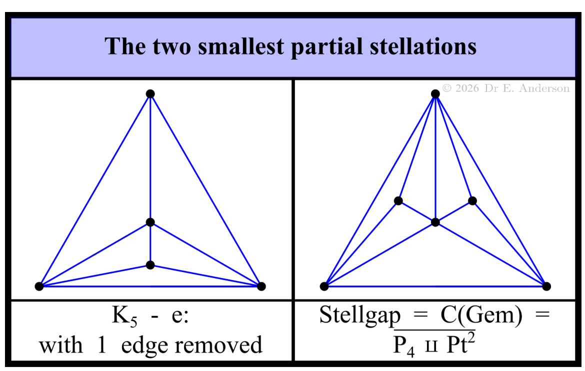

Figure 4:

Remark 3 Barycentric placings of the new vertices are natural in considering stellations. Using an incipient equilateral triangle gives the most symmetric embedding of the graph into the plane for each subsequent total stellation. All in all, let us refer to Subfig 1.c) as the unique *symmetric-barycentric* specialization among the manifestly-planar rectilinear presentations!

2.3 Ramsey presentations and complement graphs

Structure 3 *Ramsey presentations* place related and unrelated on the same footing. Consider this in the context of graphs whose edges mark related pairs of vertices and are coloured in say blue. Then unrelated pairs of vertices are not to be left unhighlighted. But are rather to be placed on the same footing as related pairs, by bringing in edges of a second colour, say red.

Remark 4 Applying this to the projective and symmetric-barycentric presentations of the Fano graph returns the corresponding Ramsey presentations in Subfigs 1.d-e).

Remark 5 One can then peel off the blue edges to reveal the complement of the Fano graph (Subfigs 1.f-g). Which can readily be straightend out to reveal the Sunlet graph alongside a loose point, D (Subfig h).

The loose point here corresponds to the Fano graph having a cone point: of degree $n := N - 1$ in a graph of size N. So that it is linked by an edge to every other vertex in the graph. In both of our presentations of the Fano graph, symmetry is maximized by placing the cone point innermost. Where it can be placed to coincide with the centre of symmetry.

Exercise 1 Show that the Fano graph's degree sequence uniquely specifies it.

2.4 The Contact Geometry route

Structure 4 We can also arrive at Tet by taking 3 touching circles and trapping a fourth between them [Subfigs a) and b) of Figs 5–7]. For sake of simplicity, we take these incipient 3 to be of the same size. We can then repeat this process of inserting touching circles into the new gaps created. Then joining up the centres of all pairs of touching circles produces the C, Tet, Fano ... sequence. See now Subfigs c)-d) of the above Figures.

Historical Remark 1 This Contact Geometry route for arriving at the 'Fano' graph in fact dates all the way back to a study of Leibniz [4]. Of a simple recursively-defined subcase of Apollonius' problem [1] of placing a circle to be in contact with multiple other circles. This has higher significance through being one of the first of what came to be known as *fractal structures* to be studied.

Naming Remark 3 In the process, a widely used alias for the stellated triangles' graphs arises: *Apollonian networks* alias *Apollionian gaskets*. Though some authors extend the meaning of one both of these from out total use to partial use as well. This corresponds to inserting touching circles into some gaps but not others.

Remark 6 Contrast also Fig 7's Contact Geometry presentation with Fig 8's barycentric stellar subdivision presentation!

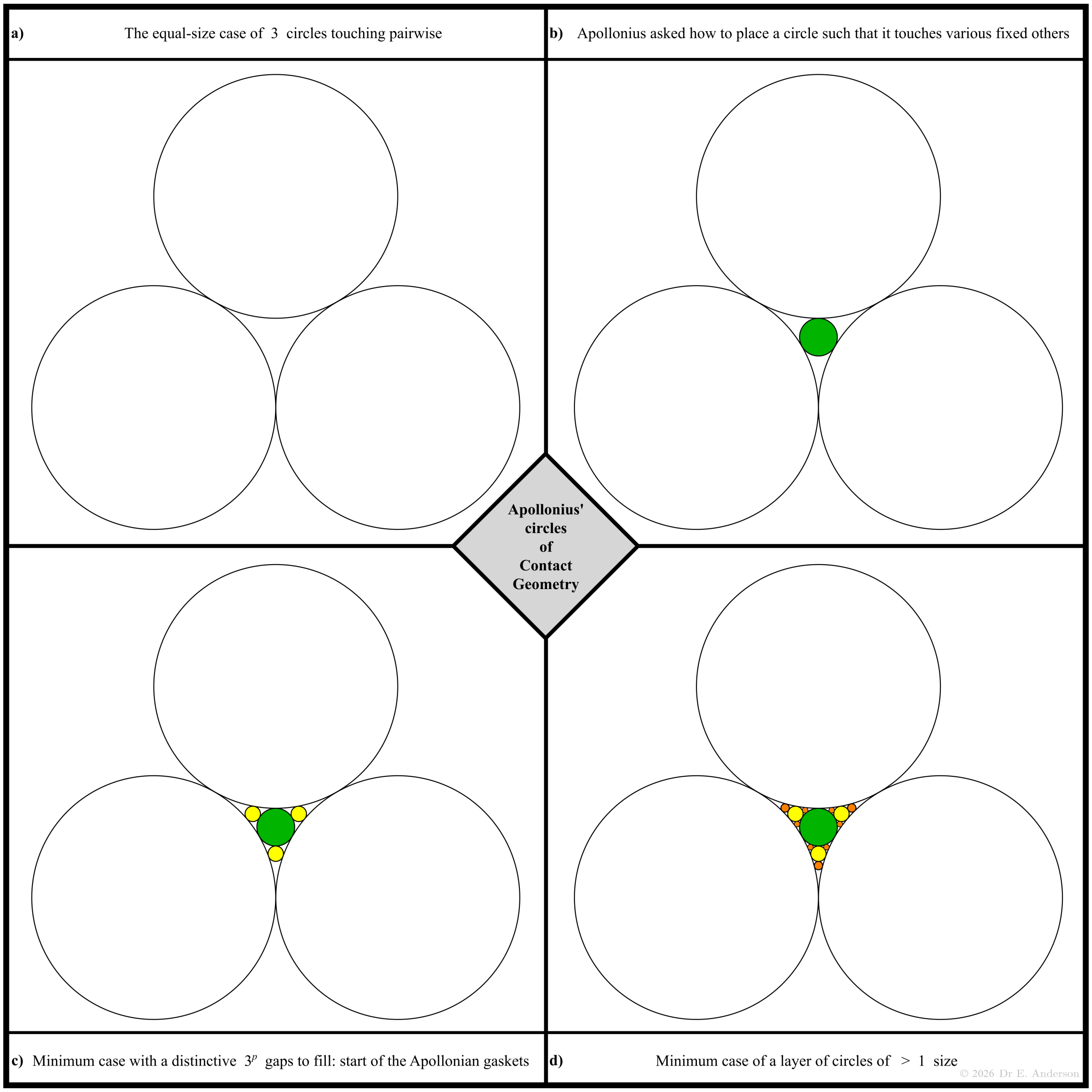
a) The equal-size case of 3 circles touching pairwise
b) Apollonius asked how to place a circle such that it touches various fixed others
Apollonius' circles of Contact Geometry
c) Minimum case with a distinctive 3^p gaps to fill: start of the Apollonian gaskets
d) Minimum case of a layer of circles of > 1 size
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Figure 5:

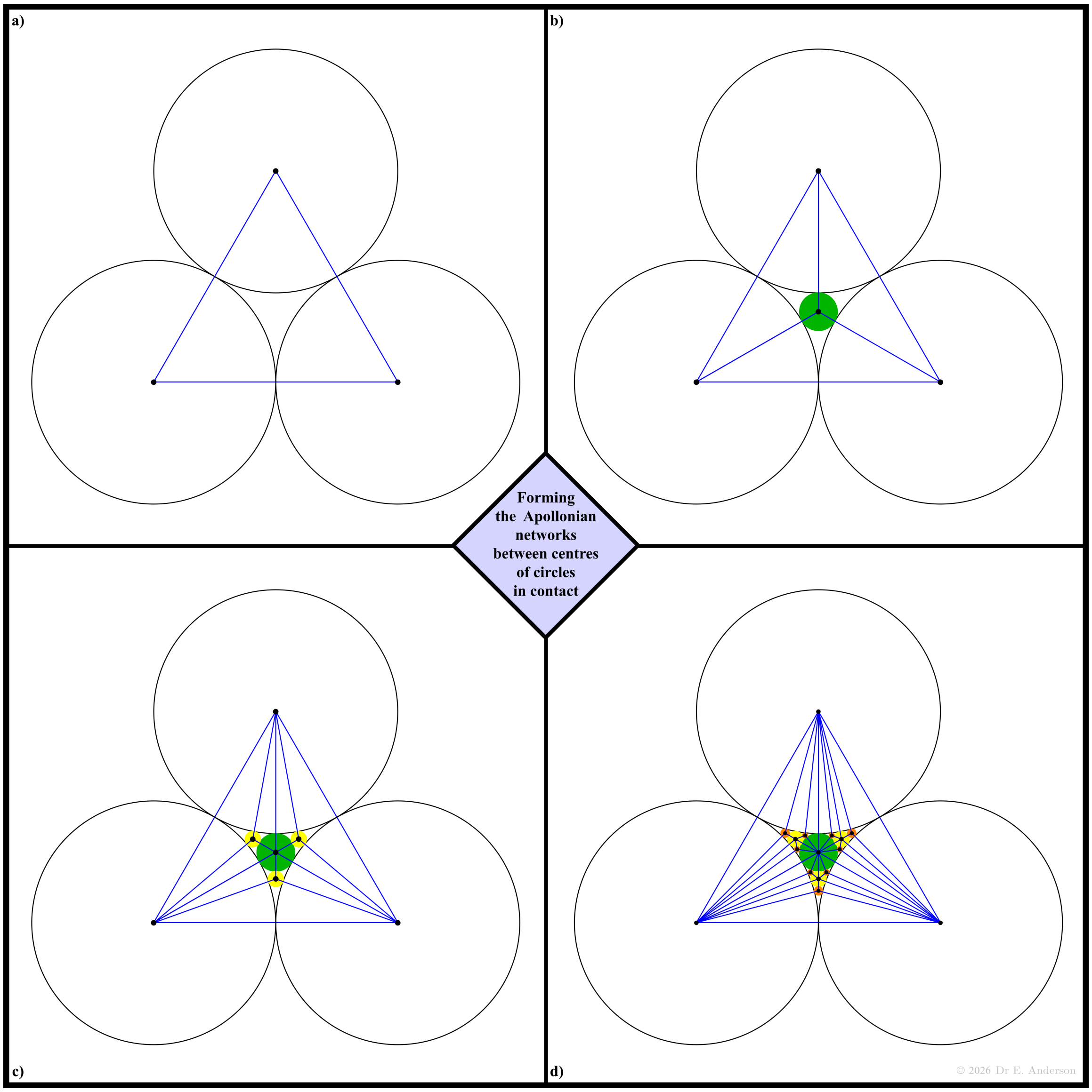

a)
b)
Forming
the Apollonian
networks
between centres
of circles
in contact
c)
d)
© 2026 Dr E. Anderson

Figure 6:

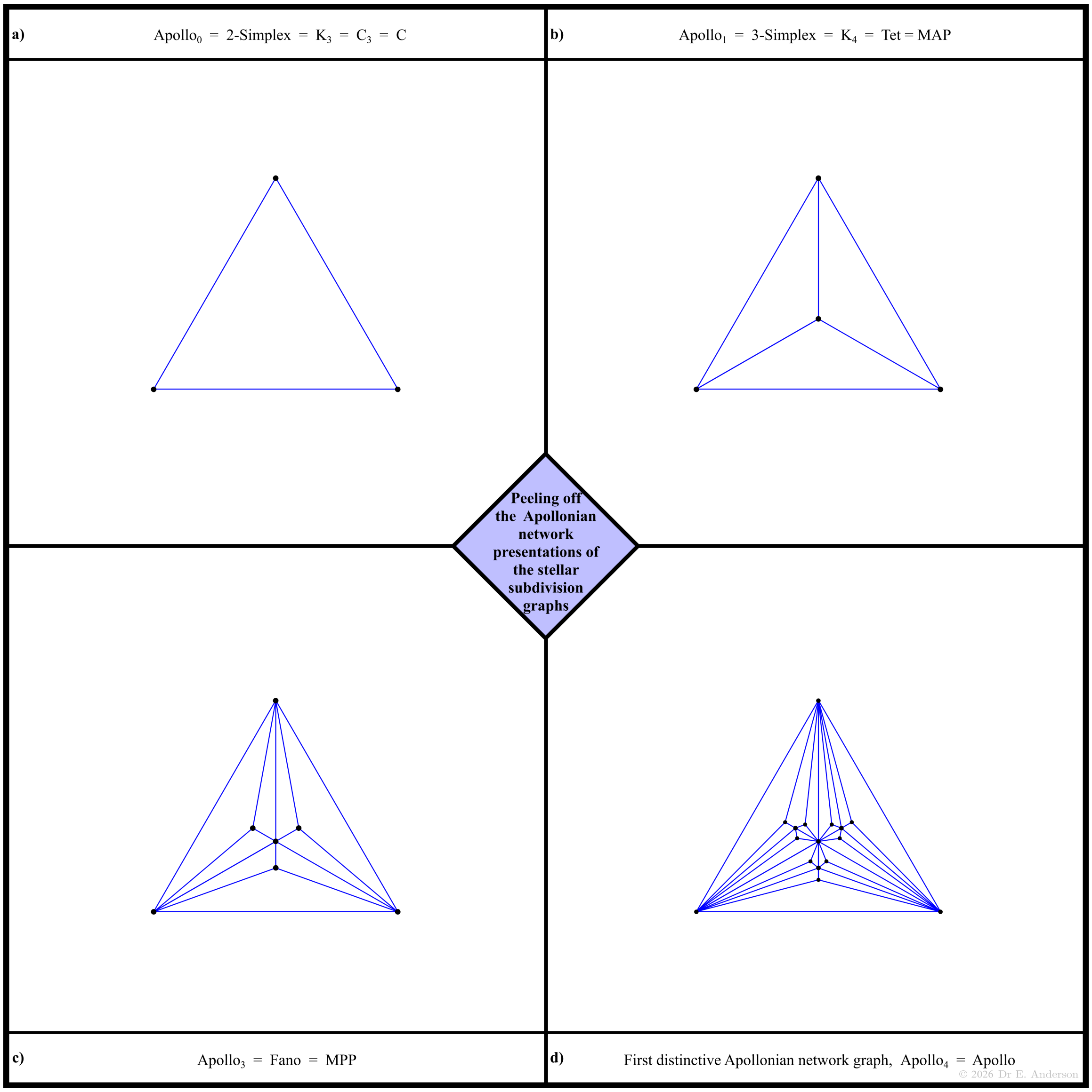

a)
Apollo_0 = 2-Simplex = K_3 = C_3 = C
b)
Apollo_1 = 3-Simplex = K_4 = Tet = MAP
Peeling off the Apollonian network presentations of the stellar subdivision graphs
c)
Apollo_3 = Fano = MPP
d)
First distinctive Apollonian network graph, Apollo_4 = Apollo
© 2026 Dr E. Anderson

Figure 7:

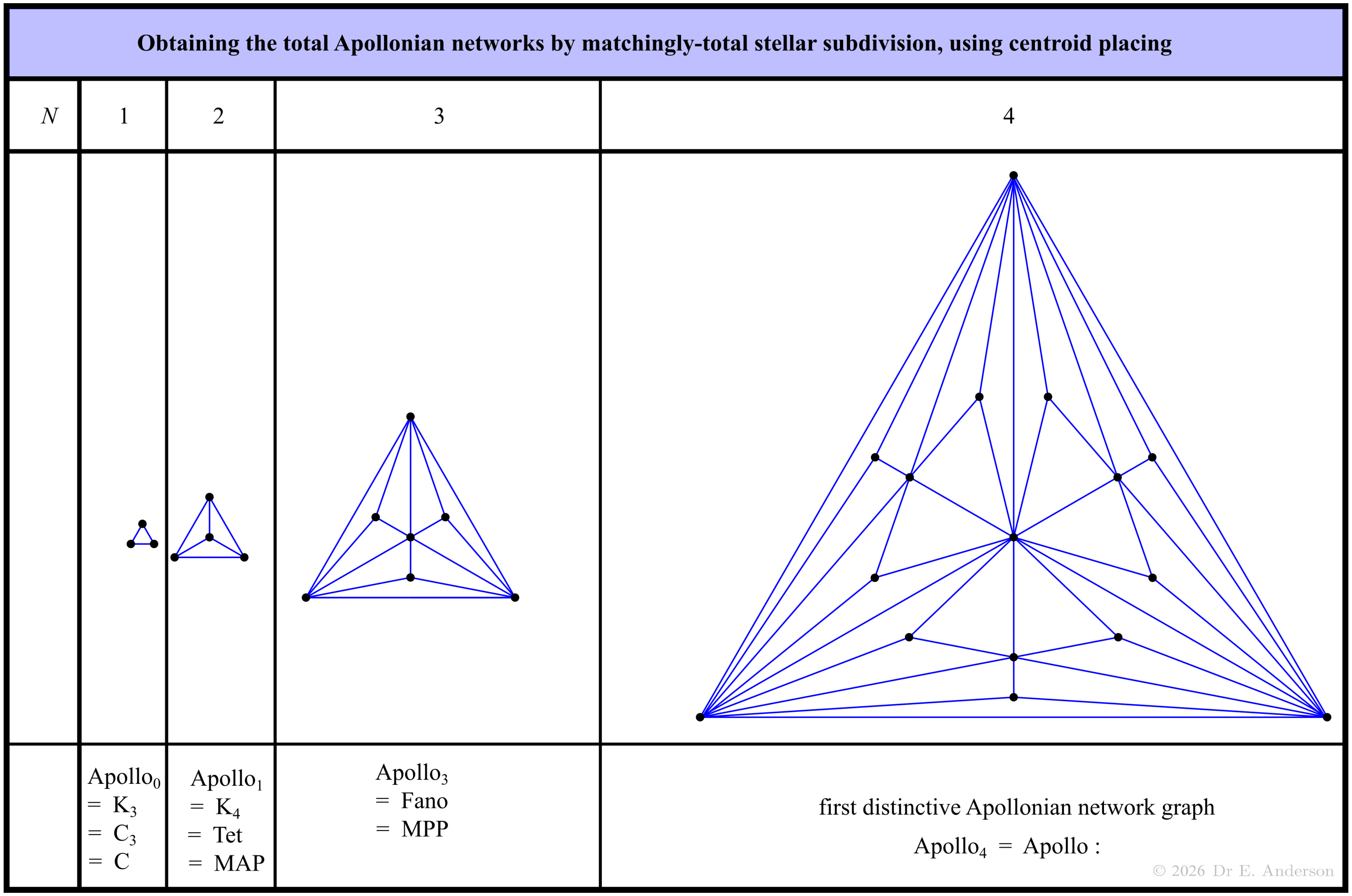

Obtaining the total Apollonian networks by matchingly-total stellar subdivision, using centroid placing
N
1
2
3
4
$Apollo_0$ = K_3 = C_3 = C
$Apollo_1$ = K_4 = Tet = MAP
$Apollo_3$ = Fano = MPP
first distinctive Apollonian network graph
$Apollo_4$ = Apollo :
© 2026 Dr E. Anderson

Figure 8:

2.5 Strong triangulation route

Structure 4 and Pointer 1 Since all faces of a triangle that has been subjected to stellation are triangles, stellations are also examples of *triangulation* [19, 65, 78]. Let us here distinguish between cutting up a polygon into triangles by edges emanating from its vertices. And cutting up a triangle specifically. In this second case, the *outer* face is a triangle too, and so *all* the faces are triangles: a *strong triangulation*. While in the first case for a $\geq$ 4-sided polygon, the outer face is not a triangle: a *weak triangulation*.

Remark 7 The Fano graph, and more generally the Apollonian graphs alias stellations of the triangle – in each case total or partial – are thus strong triangulations.

Exercise 2 Find the smallest strong triangulation that is not some stellation of the triangle.

Remark 8 This establishes that the converse of Remark 7 is false. By which strong triangulations cover more than just a reconceptualization of stellations of the triangle... So on the left tip of a trident, the smallest Affine plane and its completion the Fano plane are a terminating series. On the middle tip, stellation or the Apollonius network have these feature near the start of an infinite series. And on the right tip, both are additionally strong triangulations, which constitute *more than* just (linear) series!

Naming Remark 4 The Fano graph thus starts to fulfill the many routes condition for a Mathematical object or structure to be a citizen of Kallista [37, 38]. As we shall outline at the end of the current Article, both it and the Fano configuration are rather stronger citizens of Kallista than just this.

3 Properties of the Fano graph

3.1 Structural properties

Remark 1 The Fano graph possesses no vertices of degree 2 or 1. Thus it is respectively a homeomorph irreducible and a foliation irreducible. Which pairing we termed a *double irreducible*: class D [40].

3.2 Notions of traversability. i. Eulerian and Hamiltonian graphs

Definition 1 A graph is *Eulerian* if out of all of its edges, one can form a *circuit*. I.e. a closed loop, with no repeat uses of any edge. But with permission granted to pass multiple times through vertices if needs be.

Definition 2 A graph is *Hamiltonian* [27, 29, 33] if one can form a cycle that passes through every vertex. Unlike a circuit, a *cycle does not* grant permission to pass multiple times through any vertex.

Structure 1 These are respectively a notion of edge traversability and a notion of vertex traversability. For each of vertices and edges, graphs indeed support other distinct notions of traversability.

Remark 1 The Fano graph is immediately obviously non-Eulerian since it possesses vertices of odd degree. But Euler [5] initiated this subject with the result that Eulerian graphs must contain solely even-degree vertices.

Remark 2 The Fano graph is small enough that it is easy to espy Hamiltonian cycles therein.

3.3 ii. Planar Hamiltonian graphs

Remark 1 For a planar graph, a more systematic way of establishing Hamiltonianness is to split the graph into precisely 2 regions. Such that each contains the same number of triangles (weakly triangularizing up any larger polygons present). This is widely known as *Grinberg's theorem* [24, 29, 33].

Though in our institution, we call it the *ZIPHoN theorem* [37, 38, 49]. Standing for *zero-index planar Hamiltonian Necessity!* Where the zero index in question is the precise balance of the inner and outer triangles. Though the literature elswewhere has more confusingly proceeded by defining 'face strengths' instead of by triangulating and then just plainly counting. And have also not realized that it is a zero-index theorem in the plane. Or, indeed, an actual (non-zero) toy index theorem when conducted on surfaces [37, 49].

Remark 2 The Fano graph is already fully triangulated, outer face included! With

$$T = 5$$

triangles inside, and likewise outside. This situation – simplifying its ZIPHoN treatment – is equivalent to Sec 2's construction of the Fano graph as a strong triangulation. One choice of white-and-cyan regions with equal counts of triangles is then exhibited in Subfig 9.a) The Hamiltonian cycle is then the boundary between these 2 regions.[4]

Exercise 3 Does the Fano graph support inequivalent Hamiltonian cycles?

[Let ones related by a symmetry transformation count as the same.]

[4]For planar Hamiltonian graphs, each of the inner and outer regions must furthermore be *outerplanar* [33, 38]! This observation gives the following nice formulation [37]. Suppose that a planar graph can be split up into precisely 2 outerplanar strips. Then there must necessarily be a Hamiltonian cycle between them!

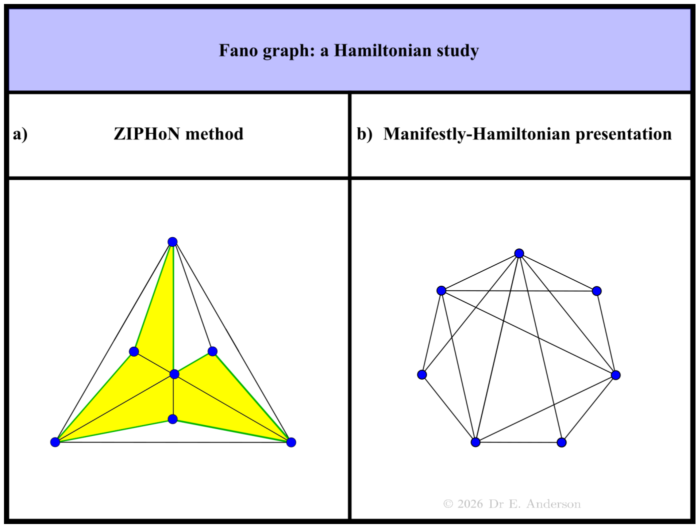


Figure 9:

3.4 iii. A manifestly-Hamiltonian presentation of the Fano graph

Remark 1 Since the Fano graph is Hamiltonian, for some purposes it is useful to form a manifestly-Hamiltonian presentation. One such is provided in Subfig b). A subsequent version of the current Article shall look into variants of this, such as minimizing the presentation's crossings. And looking into whether any symmetry elements can be exhibited within the straightjacket of a Hamiltonian-and-rectilinear presentation...

3.5 Further presentations of the Fano graph

Pointer 2 Over the next few years we will be applying the full force of the Handbook of Graph Drawing and Visualization [84] and beyond [41, 42, 46] to a collection of particularly exhalted graphs. The idea here is to give *optimized* rather than just random presentations of these graphs. And to give not only individual such presentations for each graph of interest. But also large mosaics of what they look like within the confines of a large number of competing simplicity, optimization, manifestness and aesthetic criteria.

Remark 1 Since some of our sequels use square-grid presentations, we give one of these for the Fano graph in Fig 10.a). This only manages to be single-crossing rather than manifestly planar. In Subfig b), we incline it at a $\frac{\pi}{4}$ angle so as to vertically realize its line of reflection symmetry.

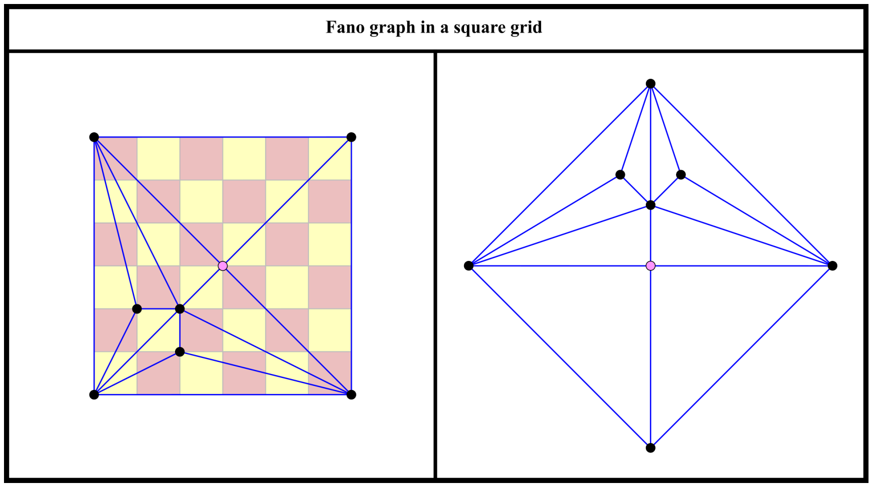


Figure 10:

Remark 2 So the Fano graph is better adapted to a chunk of equilateral triangle tessellation: no crossings. To rearrange our barycentric presentation's interior in this manner, we first fit a tetrahedron as per Fig 11.a): with 3 little triangles per side. Next we expand as tightly as general position and the tessellation jointly afford in Subfig b). This requires a patch of tessellation with 9 little triangles per side of the overall perimeter. We finally peel off the tessellation background in Subfig c).

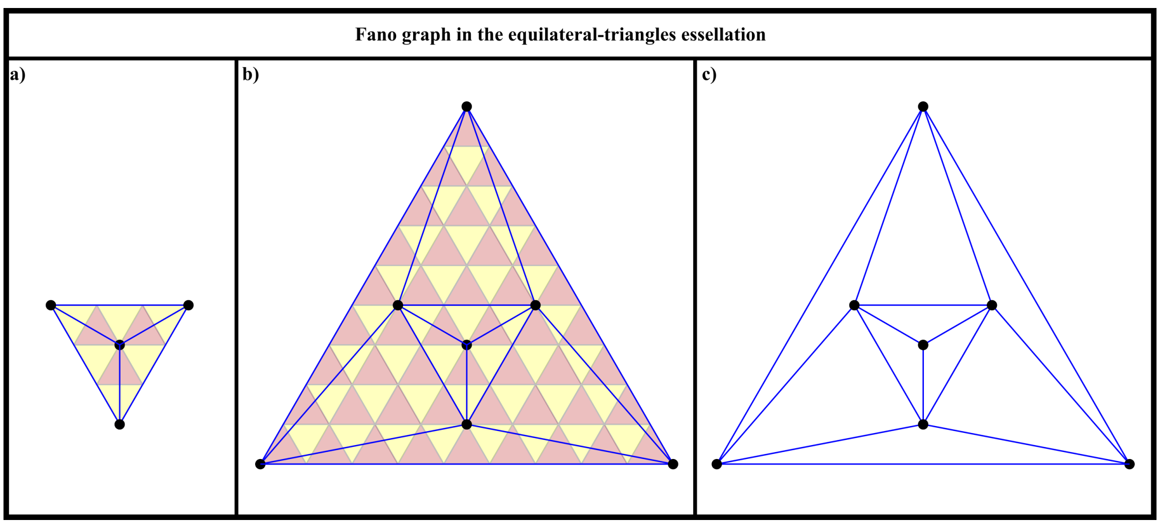


Figure 11:

Observe that this places all the vertices on points of this tessellation. But that our graph's edges are not alligned with the tessellation's. Prompting us to make the following definition [37].

Definition 1 A graph *weakly fits* a tessellation if its vertices coincide with a subset of the tessellation's. While the graph's edges do not necessarily coincide with a subset of the tessellation's.

4 Colourability properties

4.1 The basic ones

Definition 1 [27, 33, 30] A (*vertex*) *colouring* of a graph involves colouring in its vertices. According to the rule that no 2 adjacent vertices – joined by some edge – share the same colour. An *edge-colouring* of a graph involves colouring in its edges. According to the rule that no 2 adjacent edges – meeting at some vertex – share the same colour. The (*vertex-*)*chromatic number* of a graph is the minimum possible number of colours in a (vertex) colouring of it. The (*edge-chromatic number* χ' or χ^1 of a graph is the minimum possible number of colours in an (edge) colouring of it.

Remark 1 In these colourings, the colours assigned are modelled to be mutually-distinguishable and yet individually meaningless. This means that permuting the colours assigned in actually drawing a presentation of a colouring is not taken to affect that colouring. So that all colouring presentations related by such permutations are taken to be one and the same.

Remark 2 The Fano graph is 6-edge colourable: 1 less than the maximum possible for a simple 7-graph. For it is a cone graph. So its cone-point vertex is adjacent to all other 6 vertices. So each of the 6 edges emanating from it must be of a different colour. See Fig 12.a).[5]

Finally see Subfig b) for a subsequent completion of the edge-colouring scheme that re-uses 5 out of 6 colours. Which is a sharp lower bound. For the Fano graph also contains a vertex of degree 5 (in fact 3 such).

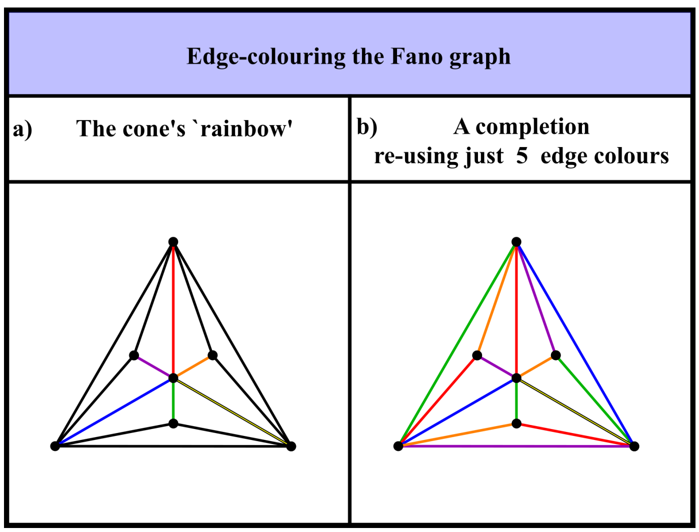


Figure 12:

Remark 3 The Fano graph is 4-colourable – the maximum possible for any planar graph by the famous 4-colour theorem [27, 33]. This readily follows from the below analysis that establishes a rather stronger property.

4.2 Unique colorability

Remark 4 *Unique colourability* is a significant enough property to be tabulated in *An Atlas of Graphs* [32].[6] Unique colourability is indeed up to the abovementioned permutation of vertex colour

[5]The complete graph K_7 illustrates how bringing in even more edges can force a seventh edge colour. No simple 7-graph can have more edges than this, so its edge number places an upper bound on all the others.

[6]Indeed, it, and the vertex and edge chromatic numbers, are the only small colour-related items to make it into these tables. The larger colour-related item tabulated there is the *chromatic polynomial* [33, 34, 35].

labels. A standard diagnostic for unique 4-colorability is as follows.

Diagnostic 1 Colouring in any triangle's vertices fixes the colours of all remaining vertices. This 4 is relevant as firstly the maximum possible vertex chromatic number for planar graphs, by the 4-colour theorem. Secondly, as the generic value of the vertex chromatic number in planar graphs. And thirdly today indeed as the vertex chromatic number of the Fano graph!

Remark 5 By symmetry, the Fano graph has 4 choices of an initial triangle to colour. As characterized by each triangle's vertices' degrees; see the top floor of Fig 13. Via the forcing moves indicated by this diagram's arrows, each of these initial probes leads all the way down to the Fano graph acquiring the same vertex colouring.

4.3 Colourability-forcing posets

Structure 2 This figure is our first public exhibition of S. Sánchez' notion of a *colourability-forcing poset*. For our Fano example, this is a fortiori a rooted tree. Which we present 'botanically', i.e. with the root at the bottom of the figure.
This poset carries furthermore a natural height function: the *colouring-in number* h_{Ci}. This counts how many vertices have been coloured in at this point in time. Corresponding to viewing colouring in as a dynamical process, with 1 further vertex being coloured in each time step.

The unique colourability property dictates that this poset has a unique bottom: the unique coloration itself! Which in the current case militates as our tree's root.

This poset's underlying tree skeleton is starlike: $\mathrm{S}(2\, 1^2\, 0)$ in excess ray length notation [38]. The root is on the innermost vertex of the longest ray.

4.4 The unique colourability route to Fano and friends

Pointer 3 In fact, unique 4-colourability and being a stellation of the triangle turn out to be equivalent criteria [64]. Giving yet another route to (the more general) Apollonian graphs. Among which the Fano graph is the first nontrivial exemplar with various stronger properties. Such as being a strong triangulation, a total stellation, and having more symmetry elements.

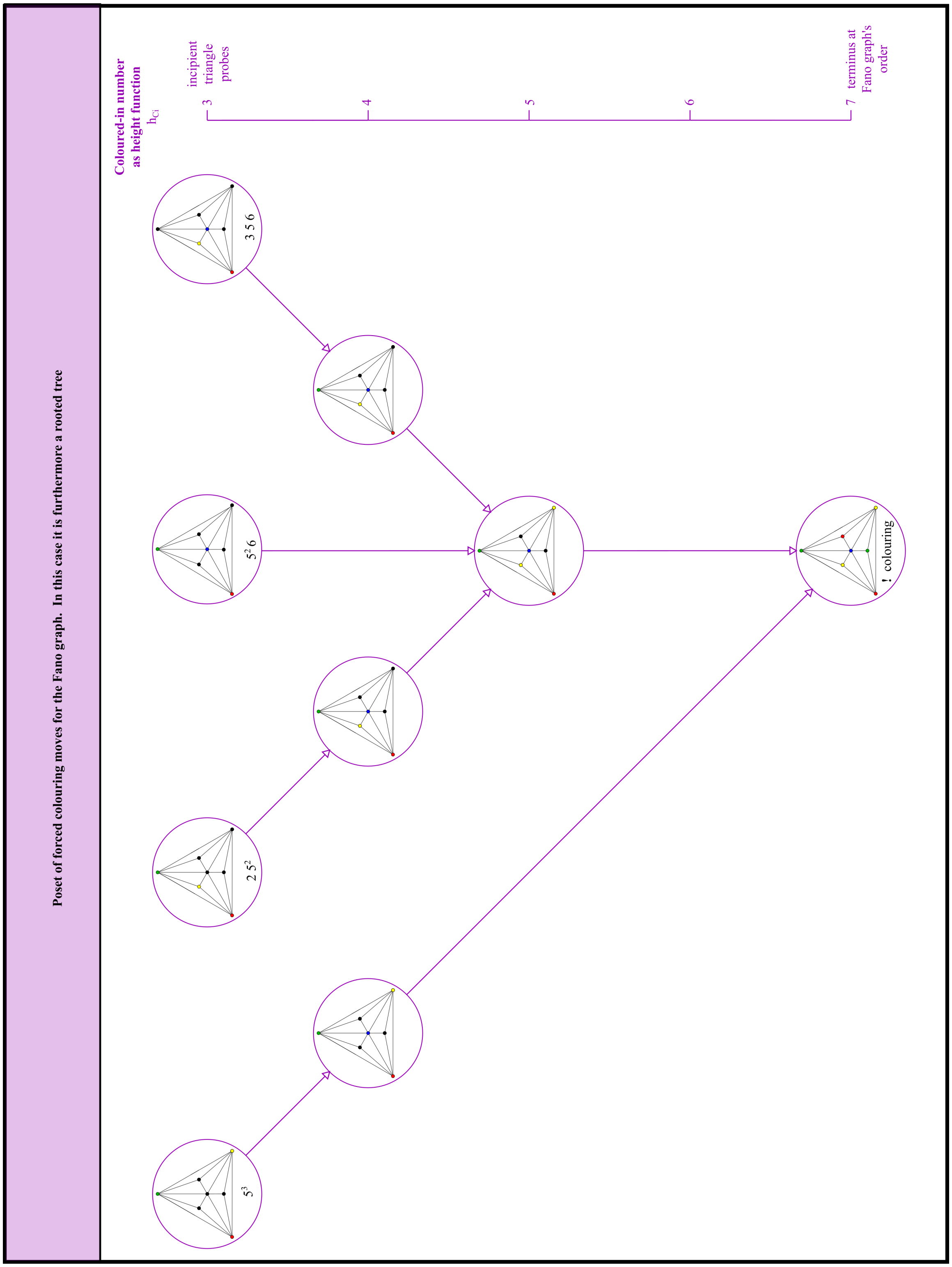

Poset of forced colouring moves for the Fano graph. In this case it is furthermore a rooted tree
Coloured-in number
as height function
h_{Ci}
incipient
triangle
probes
terminus at
Fano graph's
order
3
4
5
6
7
3 5 6
5^2 6
2 5^2
5^3
! colouring

Figure 13:

5 The Fano configuration as a citizen of Kallista

5.1 Naming update

Naming Remark 5 It remains unclear what could be used as a truer name for the Fano configuration. Firstly in a routes-unbiased manner, which the very strong property *minimum Projective plane* does not attain. Secondly, since not all of the routes leading to it may have been found yet (though our notion of *truer* name is adaptable in the face of new discoveries!) For the Fano graph, for now we [37] recommend *doubly-stellated triangle graph*. Generalizing to *n-fold-stellated triangle graph* for the stronger – totally stellated – Apollonian graphs.

Pedagogical Remark 1 The below further entrenches the Fano configuration, plane and graph as an exhalted citizen of Kallista. By which it is recommended that familiarity with this Fano example start at $\langle 2 \rangle$. Certainly by $\langle 3 \rangle$, where Projective and Graph-Theoretic notions are both widely taught, this Fano example is recommended for use as an example or long Exercise in both courses.

The below extensions are however more advanced. So we mark each with the year of study by which one might expect to encounter it in chevrons: $\langle 5 \rangle$.[7]

5.2 $\mathfrak{F}$ano's ancestry

Remark 1 Let us first display Fano's ancestors in Fig 14. We begin by forming a spine of complete graphs, alias simplices. Upon which Projective interpretations are furthermore pinned up to and including the tetrahedron, Tet . Which is also the minimum Affine plane $\mathfrak{M}$AP . We then rerun the Introduction's Projective completion, and our previous applications of stellation and of barycentric subdivision.

A first new feature marked is Projective duality. Both the triangle C and $\mathfrak{F}$ano are homodual, but Tet is not. Its dual is the *Pasch configuration* $\mathfrak{P}$asch [52], of some axiomatic Geometry fame in its own right [53]. And whose graph is the Fox (column 1 of Fig 15). This then affords a second –dual– Projective completion to the Fano plane. Now via firstly adding 3 lines and finally bringing in a 'point at infinity' for them to meet at. The new intermediary collection of points and lines, and its graph, are in turn laid out column 2.

---

[7] Take pointers 2 and 3 to be $\langle 5 \rangle$.

For all that some $\langle 3-4 \rangle$ courses on Combinatorics will mention some of them. While within the 'non-North-American' system in which students begin Ph.D.'s in $\langle 5 \rangle$, those *specializing* in Combinatorics, in Discrete Geometry or in Projective Geometry might well encounter most of them then. It is more generally recommended that textbooks, online encyclopaedias and *pedagogical* reviews *level* their examples, applications and pointers in this way. This is so as to not drown out the Reader with very mixed lists. Which for online encyclopedias are quite typically collections by some common part to their name, without any indication of extent of applicability or of around which year of study they typically belong in.

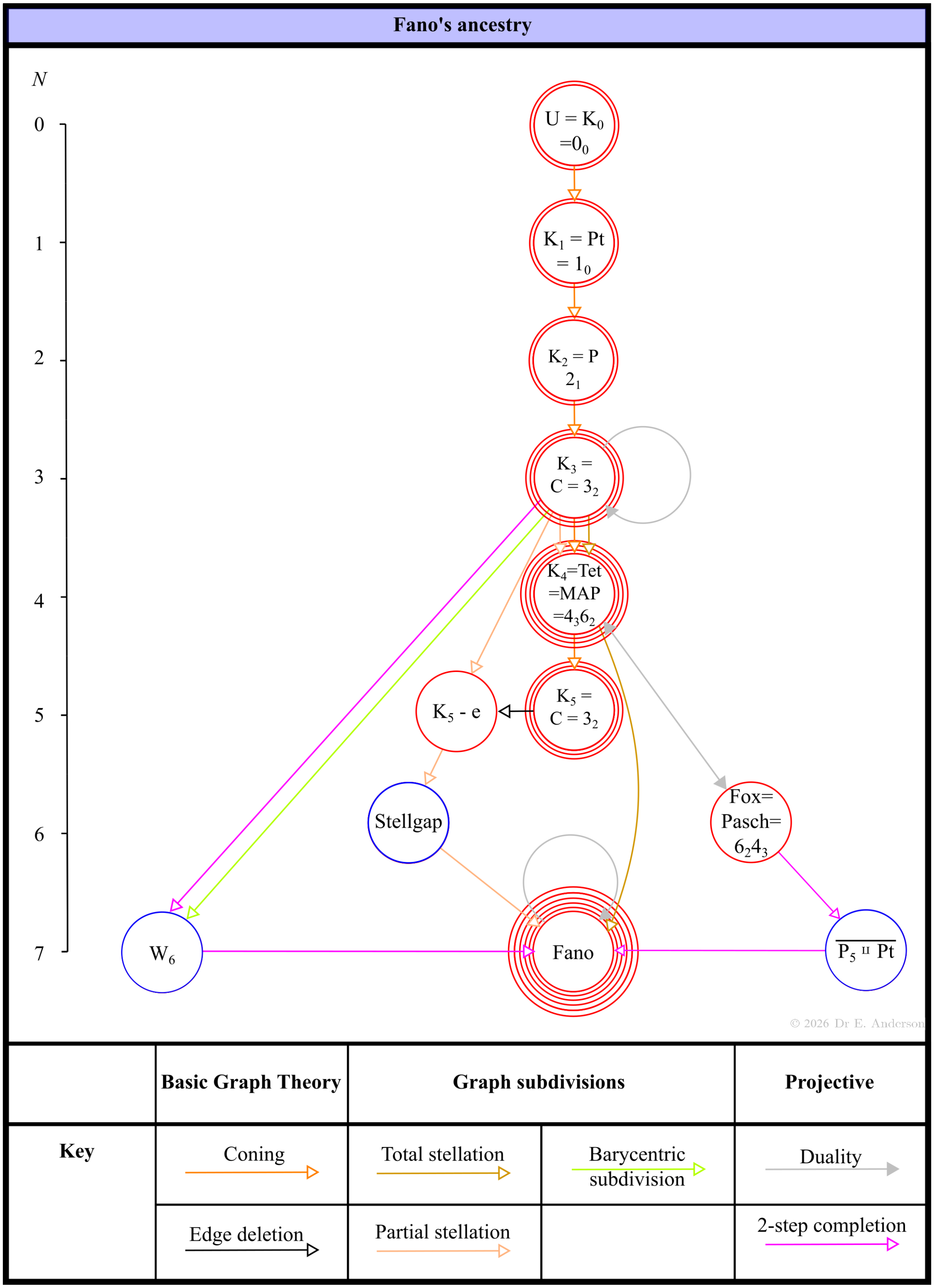
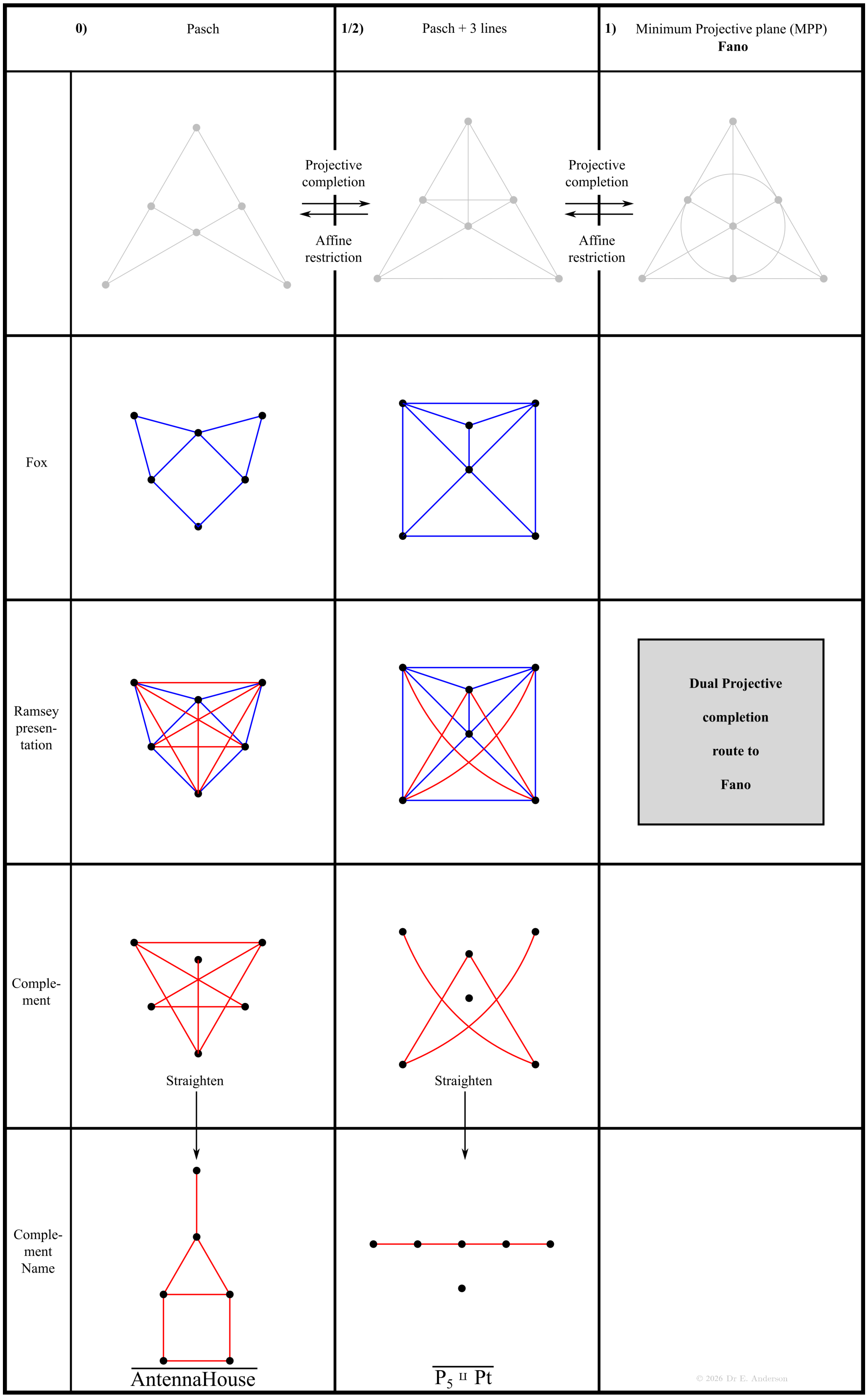

Fano's ancestry
N
0
1
2
3
4
5
6
7
U = K0 =00
K1 = Pt = 10
K2 = P 21
K3 = C = 32
K4=Tet =MAP =4362
K5 - e
K5 = C = 32
Stellgap
Fox= Pasch= 6243
W6
Fano
P5 ⊔ Pt
© 2026 Dr E. Anderson
Basic Graph Theory
Graph subdivisions
Projective
Key
Coning
Total stellation
Barycentric subdivision
Duality
Edge deletion
Partial stellation
2-step completion

Figure 14:



0)
Pasch
1/2)
Pasch + 3 lines
1)
Minimum Projective plane (MPP)
Fano
Projective completion
Affine restriction
Projective completion
Affine restriction
Fox
Ramsey presen- tation
Dual Projective
completion
route to
Fano
Comple- ment
Straighten
Straighten
Comple- ment Name
AntennaHouse
P5 ⊔ Pt
© 2026 Dr E. Anderson

Figure 15:

5.3 Projective Pointers

Pointer 4 ⟨3⟩ The Fano plane is a Projective plane [62, 75]. A necessary condition on these from finite Field Theory is that they must have dimension

$$\mathrm{d}_{\mathrm{PP}}(n) = n^2 + n + 1 .$$

This family also corresponds to completing the family of Affine planes. Whose corresponding Algebraic necessity condition is dimension

$$\mathrm{d}_{\mathrm{AP}}(n) = n^2 .$$

Let us also introduce the *Projective completion's point number*

$$c_{\mathrm{P}}(n) := \mathrm{d}_{\mathrm{PP}}(n) - \mathrm{d}_{\mathrm{AP}}(n) = n^2 + n + 1 - n^2 = n + 1 .$$

Example -1 For the minimum case of $n = 2$,

$$\mathrm{d}_{\mathrm{AP}}(2) = 2^2 = 4 .$$

This corresponds to Tet .

Example 0 While

$$\mathrm{d}_{\mathrm{PP}}(2) = 2^2 + 2 + 1 = 7 :$$

the Fano plane.

Remark 1 So Route 1 continues via $n = 3$ giving an Affine plane of dimension

$$\mathrm{d}_{\mathrm{AP}}(3) = 3^2 = 9 .$$

Which completes to a Projective plane of dimension

$$\mathrm{d}_{\mathrm{PP}}(3) = 3^2 + 3 + 1 = 13 .$$

Pointer 5 ⟨3-5⟩ But far from all Projective configurations are of this form. Route 0 continues rather via an 8-point Projective configuration: Möbius–Kantor's [16]. Though this is rather more of a counting heir than a structural heir.

And then 3 9-point Projective configurations [31, 80]. One of which corresponds to the first of Projective Geometry's 2 main structural theorems: Pappus' [2, 21, 81]. The other such configuration [44] is to be found among the 10-point Projective configurations [31, 80]: Desargues' [3, 21].

Remark 2 [83]. See Fig 16 for a point-number plot of these smallest heirs.

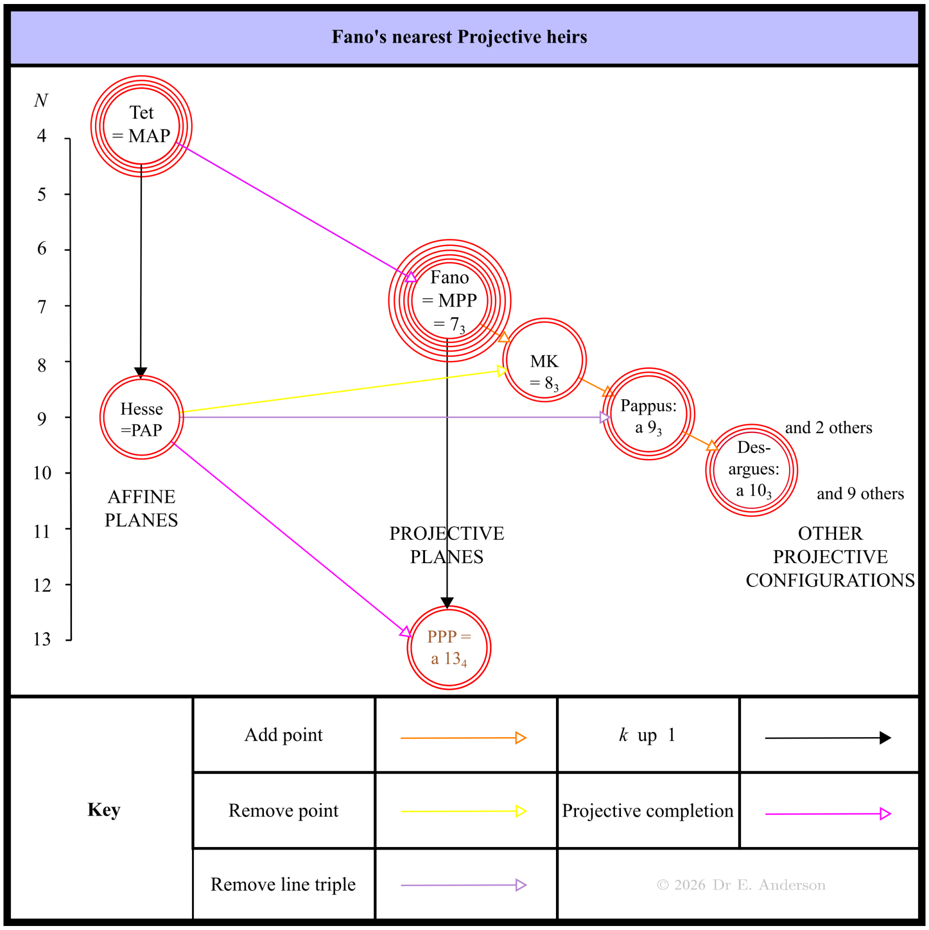


Figure 16:

Remark 3 Pointers 4 and 5 can furthermore be viewed as and part of Discrete Geometry, itself an active research field

5.4 Combinatorial Pointers. i. Design Theory

5.4.1 Steiner triple systems and block designs

Pointer 6 ⟨3⟩ The Fano plane is [31] a *Steiner triple system* [62, 71, 72].

This is motivated by longstanding Mathematical problems such as Kirkman's schoolgirl problem [51]. Whose full solution and immediate heirs' solutions took a long time to work out [54, 72].

Pointer 7 ⟨4⟩ It is more generally a *block design* [62, 60, 58, 67].

This is further motivated by Mathematical problems based on patterned squares of numbers, called such as 'magic squares' [50] and 'Latin squares' [57]. Including experimental contingency-table design, from which the name Design Theory is derived.

Definition 1 Let $v, s, p \in \mathbb{N}_0$ and $b \in \mathbb{N}$ such that $p < s < v$. Then a

$$p\text{-}(\, v,\, s,\, b\,)\text{-}\textit{block design}$$

is a v-point set $\mathfrak{X}$ equipped with a collection $\mathfrak{S}$ of its own subsets of size s : the so-called *blocks*. Such that any p points belong to precisely b blocks.

Where the above declaration of number systems that the parameters take values from should not be taken to imply that all combinations of these parameters can be realized...

Example S The Steiner triple systems are the 1-parameter subcase

$$2\text{-}(\, v,\, 3,\, 1\,)\ .$$

So the 'triple' name comes from the blocks being of size 3 . This extends to a 2-parameter family of *classical Steiner n-tuple systems*

$$p\text{-}(\, v,\, p + 1,\, 1\,)\ .$$

The modern Steiner system refers however to the 3-parameter family

$$p\text{-}(\, n,\, s,\, 1\,)\ .$$

I.e. the 'belonging to 1 block' subcase:

$$b = 1\ . \tag{1}$$

Example 0

$$\mathfrak{Fano} = 2\text{-}(\, 7,\, 3,\, 1\,)\ .$$

is a nested subcase of both. The blocks here are just points, while the subsets are the collinear triples of these.

Example 1 Viewing this as

$$2\text{-}\left(\, 2^2 + 2 + 1\ ,\ \ 2 + 1,\, 1\,\right) \ = \ 2\text{-}(\, \mathrm{d}_{\mathrm{PP}}(2)\ ,\ \ c_{\mathrm{P}}(2),\, 1\,)$$

affords generalization to the other Projective planes as block designs. I.e.

$$2\text{-}(\, \mathrm{d}_{\mathrm{PP}}(n),\, c_{\mathrm{P}}(n),\, 1\,)\ .$$

Definition 1 This restriction to $p = 2$ produces the 2-designs: another 3-parameter family.

Example 2 In fact, Affine planes are 2-designs as well:

$$2\text{-}(\, \mathrm{d}_{\mathrm{AP}}(n),\, n,\, 1\,) \ = \ 2\text{-}\left(\, n^2,\, n,\, 1\,\right)\ .$$

But nontrivial designs require [62]

$$v \geq 6 .$$

So

$$\mathfrak{Tet} = \mathfrak{M}\text{AP} = 2\text{-}\left(2^2, 2, 1\right) = 2\text{-}(4, 2, 1)$$

misses the boat. So via $\mathfrak{Fano}$, Projective planes plays a larger earlier role in the development of Design Theory.

Remark 1 While to have a Steiner triple system that is an Affine plane, $n = 3$ is forced, returning just the proxime Affine plane $\mathfrak{Hesse} = \mathfrak{p}\text{AP}$. And to have one that is a Projective plane, $n + 1 = 3$ is likewise forced, now returning just $\mathfrak{Fano}$.

Exercise 4$^-$ Rerun this analysis for the above two more general Steiner system notions.

Exercise 5 Some of you may have realized that for a complete overview a priori, one should also be parametrizing how many blocks l there are. And how many blocks r contain a given point. In the simpler case of 2-designs, show however that l, r are determined by v, s, b. Including finding explicit expressions for them.

Exercise 6 Determine that self-dual Projective plane obey

$$v = l : \tag{2}$$

the *symmetric design condition*. Alongside (1).
Deduce that this is consistent with defining the *order* of a design as

$$n := r - b .$$

5.4.2 Fano's ancestor and heirs among block designs

Example -D $\mathfrak{Fano}$ is not however minimum when viewed as a block design. The smallest nontrivial such is, rather [62],

$$2\text{-}(6, 3, 2) .$$

Remark 1 Immediate Design Theory heirs of Fano include

$$2\text{-}(7, 3, 2) :$$

up in the last parameter.

Remark 2 While

$$v = 8 \text{ forces } s = 4 .$$

[62]. So we cannot simply increase v while keeping all the other parameters fixed. Furthermore, the following identification turns out to hold [62].

$$3\text{-}(8, 4, 1) = 2\text{-}(8, 4, 3) .$$

Open Exercise 7 At this point we set working through Cameron [62]'s chapter on designs so as to understand and derive the above results for yourself. Some of what this entails is better considered after you have read the sequel Article [42]'s conclusion...

5.4.3 Hadamard designs

Pointer 8 ⟨6⟩ If one restricts to Hadamard 2-designs, however, Fano is the smallest [31].

Structure 1 *Hadamard designs* [62, 73, 61] corresponding to the following. A square matrix $\boldsymbol{H}$ of size ≥ 4 such that the modulus of every entry

$$| H_{ij} | \leq 1 .$$

This is a priori a Linear Algebra notion.

Remark 1 Among designs, Hadamard designs turn out to be restricted to another 1-parameter family, as follows.

$$2\text{-}\left(h - 1 , \frac{h}{2} - 1 , \frac{h}{4} - 1 \right) \tag{3}$$

Exercise 8$^-$ Relate h to the original design parameters. Discovering in the process which among them affords the neatest parametrization, which is indeed often found in the literature...

Open Exercise 9 Now work out this restriction's relation to the Hadamard matrices , e.g. by working through some of the last part of Cameron's chapter on designs...

Remark 2 If a Projective plane is furthermore Hadamard, then equating coefficients of b returns the following.

$$1 = \frac{h}{4} - 1 \Rightarrow h = 8 .$$

Thus once again we are talking about

$$2\text{-}\left(8 - 1 , \frac{8}{2} - 1 , \frac{8}{4} - 1 \right) = 2\text{-}(7, 3, 1) = \mathfrak{Fano} .$$

Exercise 10$^-$ What is the outcome if an Affine plane is required to be Hadamard?

Remark 2 So Projective planes are unavailable as Hadamard heirs of Fano. Many Projective configurations are not Hadamard either; one needs to go to [31]

$$2\text{-}(11, 5, 2) .$$

Exercise 11 Justify at the counting level that this is the heir

5.4.4 Jointly depicting some of Fano's design descendants

Remark 1 We provide this in Fig 17.

For all that this type of plot is now less useful. This is since the vertex-number parametrization that we have been using so far is inadequate as regards characterizing designs. For various relevant types of these are 2-or-more parameter ventures...

Also we are for now just entertaining the simple matter of parameter-increase *counting heirs*. As opposed to which of these are furthermore structurally related to $\mathfrak{Fano}$.

Finally, we include the 2-line heirs ensuing from passing from minimum to proxime planes on top of switching on design parameters. These share a pattern with the previous copy, as indicated.

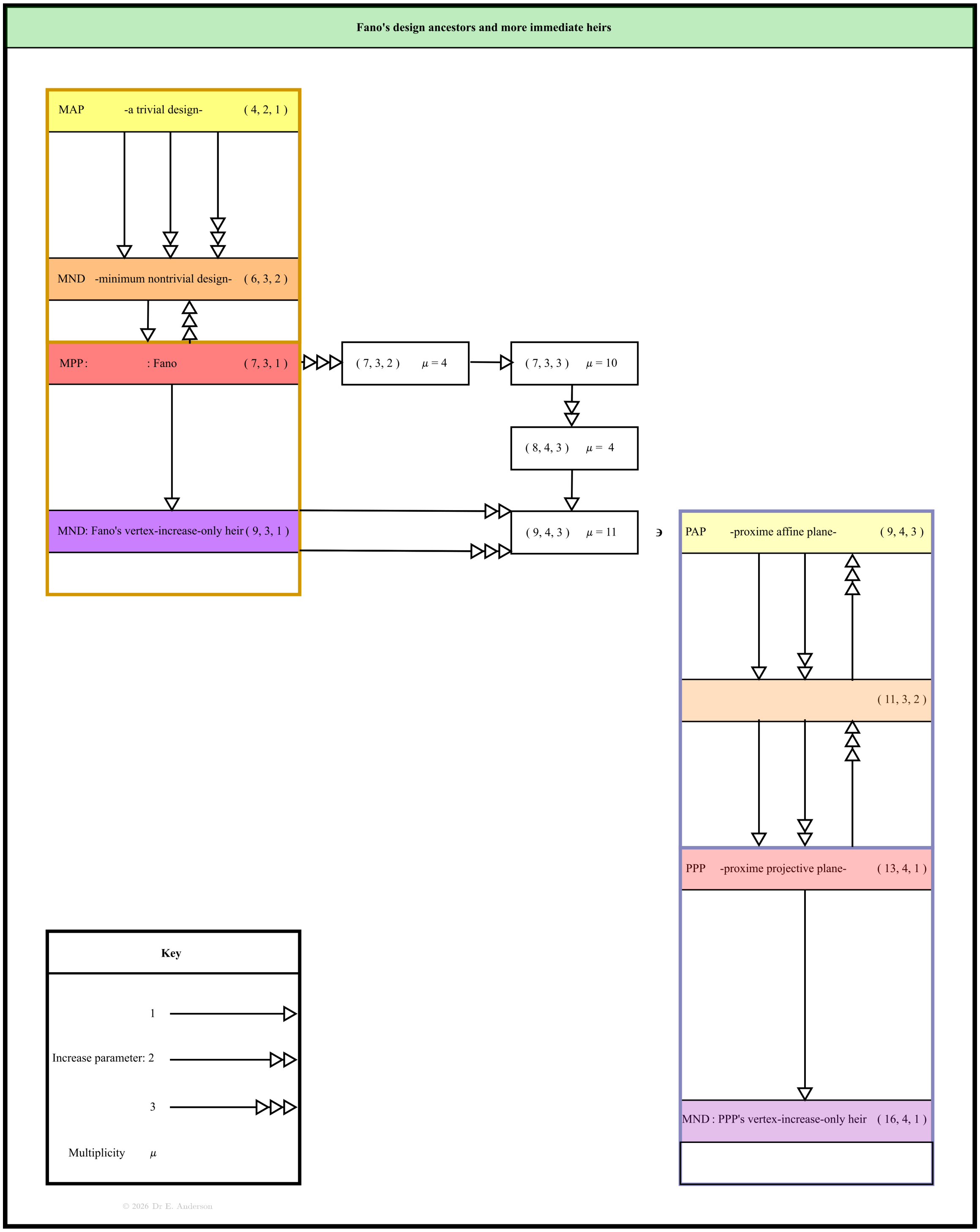


Figure 17:

End Remark 1 Design Theory is more general than the above Block Design Theory, by such as allowing variables in place of parameters. Two other names for Block Design Theory corresponding to further conceptualizations are as follows. Incidence Geometry [66, 75], bringing to the forefront the aspect of generalizing Projective Geometry. And Hypergraph Theory [59].

5.5 . ii. Matroids

5.5.1 Introduction, via the Fano matroid...

Pointer 9 $\mathfrak{Fano}$ gives rise to a *matroid* [62, 82, 79].

Structure 1 Consider $\mathfrak{X}$ and $\mathfrak{S}$ once more. Now form a further family $\mathfrak{I}$ of independent sets. These form a *matroid* if the following conditions hold.

i) $\mathfrak{I} \neq \emptyset$, here interpreted as the empty family.

ii) Hereditarity. Suppose that $J \in \mathfrak{I}$ and $I \subseteq \mathfrak{I}$. Then $I \in \mathfrak{I}$ too.

iii) Replaceability. Suppose that $I, J \in \mathfrak{I}$ and $|I| < |J|$. Then $\exists$ some element $K \in J - I$ such that $I \cup \{K\} \in \mathfrak{I}$.

This is the great Mathematician Hassler Whitney's [55] abstraction of Linear independence, with which Matroid Theory historically started.

Example 0 To form a matroid

$$\mathfrak{M}(\mathfrak{Fano})$$

in this sense from $\mathfrak{Fano}$, take $\mathfrak{I}$ to consist of the following.

$\emptyset$, now interpreted as at the usual level of the empty set.

All of the points supported.

All pairs of points.

And all non-collinear triples of points! For collinearity conditions embody a type of Linear dependence...

Exercise 12 a) Show that

$$|\,\mathfrak{I}_{\mathfrak{Fano}}\,| = 57 \ .$$

b) And that the above description of $\mathfrak{I}_{\mathfrak{Fano}}$ indeed obeys the above axiomatization for a matroid.

Remark 1 ⟨4-6⟩ There are moreover many other conceptually distinct axiomatizations [82, 79] that return the matroids. Span, basis, rank, flats, closure... In this way, *Matroid Theory is itself a citizen of Kallista*!

5.5.2 Some properties of the Fano matroid and its non-Fano matroid ancestor

Remark 1 Some of $\mathfrak{M}(\mathfrak{F}\text{ano})$'s distinctive properties are as follows.

Property 1 It is *binary*. In the sense that it is representable by [82] a matrix whose entries are just 0 and 1. Namely the 7×3 matrix of column vectors of the homogeneous coordinates in Subfig 2.1′).

Exercise 13⁻ Establish that $\mathfrak{F}$ano's 'circle-line' being Linearly independent requires

$$0 = 2 . \tag{4}$$

Remark 2 If we do not ascribe to (4), then the 'circle-line' has to be removed from $\mathfrak{I}$. Yielding the *non-Fano matroid* ancestor

$$\mathfrak{M}\left(\mathfrak{F}\text{ano}^{-}\right)$$

of the Fano matroid $\mathfrak{M}(\mathfrak{F}\text{ano})$. Its underlying configuration is the first barycentric subdivision of the triangle, i.e. the wheel graph W_6. We already depicted this in Subfig 2.1/2).

Property 2 The Fano matroid is not regular.

Definition 1 A matroid is *regular* if it can be represented over all fields.

Remark 3 So Remark 2 establishes that the Fano matroid is just represented on $\mathbb{F}_2$. It turns out that $\mathfrak{M}\left(\mathfrak{F}\text{ano}^{-}\right)$ is not regular either. It now requires $\mathbb{F}_3$, which property is termed *ternary*.

Property 3 $\mathfrak{M}\left(\mathfrak{F}\text{ano}\right)$ is a minimum binary matroid [82]. Which honour it shares with its own dual matroid...

5.5.3 $\mathfrak{F}$ano's matroid heirs

Remark 4 One heir is the next binary matroid.

Pointer 10 A more interesting heir in many ways is the minimum totally-irregular matroid: not representable over any field. Let us postpone this paragon to the Pappus configuration article [43] later on in the current Series...

5.6 Group Theory pointers

Pointer 11 The Fano configuration has a large Projectively significant-symmetry group. I.e. the Projective general linear group $PGL(2, 7)$, of order 168. This aspect is unfortunately lost by the Fano graph, whose symmetry group is just $S_3 = D_3$, of order 6 [32].

$PGL(2, 7)$ furthermore manages to be a *simple group* [63]: it contains no nontrivial normal subgroups. Simple groups are key in Group Theory's classification of the finite groups. $PGL(2, 7)$ is quite famous as marking the end of the first 'simple group desert'. For it is the smallest simple group after the basic and long-known alternating permutation group A_5, of order 60 ... On the one hand, Group Theorists have a neat way of constructing this group that is intrinsic to their subject On the other hand, more complicated groups often happen to be found via their action upon some object or structure. And for $PGL(2, 7)$, this role is played magnificently by the Fano plane!

Exercise 14$^+$ Show that

$$|\, Sym(\mathfrak{F}\text{ano})\,| = 168 .$$

By its action on $\mathfrak{F}$ano, work out this group's generators and relators. That it is indeed the stated Projective-linear group. Which in turn gives a conceptually unrelated way of making the above count.

Figure out this group's generators and relators. Demonstrate furthermore that it is simple.

Pointer 12 The classification of simple groups [77] starts with the cyclic and alternating groups. Then there are 2 copies of the Lie groups, over the finite (Galois) fields. Finally there are 26 sporadic simple groups.

The current Subsec's group is the smallest distinctly-realized Lie-type group in the easier copy: that worked out by Chevalley. It is where the distinctive A-series such start.

5.7 Afterword

Open Exercise 15^{++} Figure out both the primary object's Mathematics, and the underlying basic Graph Theory where applicable, for each of Pointers 3 to 8's first successors to the Fano object. For all that the current Encyclopedia aims to eventually cover all of these...

Pointer 13 The sequel Article [42] contemplates may further heirs via the Fano incidence graph.

End Remark 1 So far, it appears to be quite typical for exhalted citizens of Kallista to have not one but many heirs. By the many routes to each, from ab initio distinct conceptualizations that become different aspects of a citizen of Kallista, often generalizing in different ways. In this way, Fano parallels the tangle of Heron's formula, Hopf's little map [39] and triangle Mathematics being much simpler than [47] any subsequent polygon's or simplex's. .

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