The Smallest Cubic Graphs and their Cubeomorph Poset Arena

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Abstract

We apply modern Graph Drawing and Visualization to the smallest cubic graphs, using presentations of the following kinds. Maximumly-symmetric or minimum-crossing. Regular-polygon Hamiltonian or minimum-grid. Unit-distance and its 2-distance generalization. Framed tree, Möbius ladder, generalized Petersen and diamond necklace generalizations of our graphs are pointed out.

We next consider the cubeomorph operation on these graphs. This is analogous in various ways to graphs' usual homeomorph operation. Both of these notions give rise to some type of irreducible, adding to the inventory of Topological primes.

We next apply the cubeomorph order to the arena of cubic graphs, revealing a poset structure. For ≤ 8 vertices, we combine this with the 'thread-diamond' operation. So as to form a connected poset model for cubic graphs up to this order. We end by explaining how thread-diamond requires generalization. And to some of the middling ($\leq$ 46-vertex) cubic graphs of particular interest.

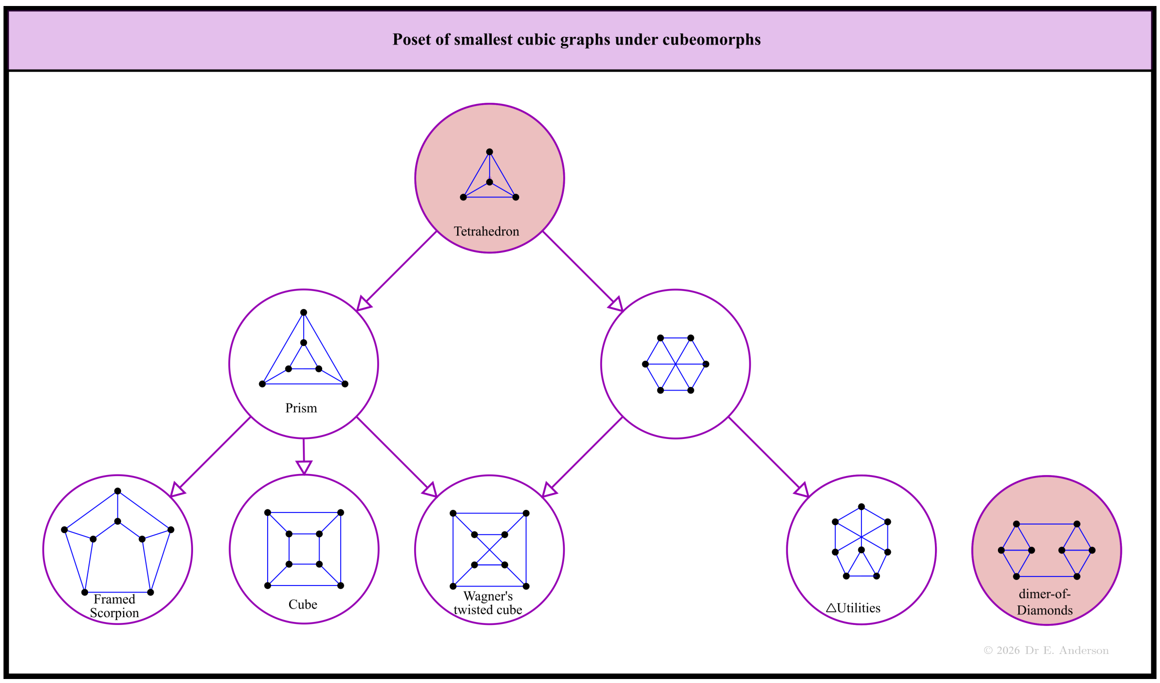

Figure 1:

This Article is ⟨4⟩: suitable for fourth-year university Students.

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1 Introduction

1.1 Simple graphs

Definition 0 Let G be a graph [33, 62]. Consisting of a set of vertices $\mathfrak{V}(\mathrm{G})$. And a set of edges $\mathfrak{E}(\mathrm{G})$ between pairs of vertices. The *order* of the graph

$$N \ := \ V \ := \ |\,\mathfrak{V}(\mathrm{G})\,| \ .$$

While the *size* of the graph

$$E \ := \ |\,\mathfrak{E}(\mathrm{G})\,| \ .$$

Let v be one of the vertices. The corresponding *degree* alias *valency* $d(v)$ is the number of edges emanating from it.

A graph is *simple* [33, 69] if the following restrictions apply. Firstly, there are no loops [148]: edges from a vertex to itself. Nor any multi-edges [148], meaning that there is at most 1 edge between a pair of distinct vertices.

Remark 1 Simple graphs' vertices can consequently have degrees take values between 0 and $n \ := \ N \ - \ 1$. For connected [125, 62] simple graphs, it is 1 to n.

Remark 2 Graph Theory's inception included the following results, which the current Article continues to make active use of...

Lemma 1 (Euler's degree-sum) [1, 62, 148]

$$2\,E \ = \ \sum_{v \,\in\, \mathfrak{V}(\mathrm{G})} d(v) \ . \tag{1}$$

Lemma 2 (Euler's formula) [2, 123, 120, 102, 106] For convex polyhedra,

$$V \ - \ E \ + \ F \ = \ 2 \ .$$

Where F is the number of faces. Appendix A.5 converts this to graph use.

1.2 Cubic graphs, and regular graphs more generally

Definition 1 The *r-regular graphs* are those all of whose vertices have degree 3. The $r \ = \ 3$-regular graphs are further alias *cubic* or *trivalent*.

Remark 1 $r \ = \ 3$ is the minimum nontrivial regularity in a vast number of ways.

Remark 2 These have long been studied [3, 12, 19, 101, 23, 31, 34, 50] and have turned out to include many delightful examples. See the Concluding Pointers Subsec 7.3 for details. To support degree-3 vertices, simple graphs require

$$N \ \geq \ 3 \ + \ 1 \ = \ 4$$

vertices. The minimum such is the complete 4-graph K_4 alias tetrahedron graph Tet ; Sec 2 serves to study this.

Corollary 1 Cubic graphs exist only on even-N vertices.

<u>Proof</u> Inserting the definition of cubic graph into Lemma 1,

$$2\,E \ = \ 3\,N \ .$$

The LHS contains a factor of 2. So some factor on the RHS must be divisible by 2. 3 is not divisible by 2. Therefore N must be. □

Remark 3 Lemma 2 and this corollary are of the 'hand-shaking lemma' conceptual type [58]. With reference to how in transactions involving hand-shakes, the number of hands shaken must be even!

Remark 4 Let us celebrate by writing

$$N \ = \ 2\,m$$

whenever cubic graphs are involved. And indeed by passing to use

$$m \ = \ \frac{N}{2}$$

as an adapted variable for the study of cubic graphs.

Remark 4 Thus cubic graphs are restricted to $m \ \geq \ 2$. So we continue our repertoire with $m \ = \ 3$ and 4 in Secs 3 and 4 respectively. As is standard, we restrict ourselves to the connected simple cubic graphs.

Remark 5 For Hamiltonian cubic graphs, m can furthermore be interpreted as the number of diagonals on whichever Hamiltonian cycle's polygon.

1.3 Arenas, as exemplified by arenas of graphs

Structure 1 Given a type of Mathematical object, the corresponding arena [127, 128, 129, 130, 133, 131, 132, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 147, 148, 149, 151, 153, 152] is the Mathematical space formed by the totality of these objects. Cumulatively on $\leq \ N$ vertices, we denote the corresponding arenas of simple graphs by

$$\mathfrak{G}\mbox{raph}[N] \ .$$

And

$$\mathfrak{C}\mbox{ubic}\mathfrak{G}\mbox{raph}[m]$$

for simple cubic graphs. We more generally reserve pieces of notation that lead with the mathfrak font to denote arenas. With round brackets for a fixed parameter, square brackets for a cumulative value, and no brackets at all for all parameter values at once!

Structure 2 Sec 5 serves to introduce the cubeomorph map [146]

$$C \ : \ \ \mathfrak{G}\mbox{raph}(N) \ \longrightarrow \ \mathfrak{G}\mbox{raph}(\,N \ + \ 2\,) \ .$$

Which is such that cubicness is preserved:

$$C \ : \ \ \mathfrak{C}\mbox{ubic}\mathfrak{G}\mbox{raph}(\,m\,) \ \longrightarrow \ \mathfrak{C}\mbox{ubic}\mathfrak{G}\mbox{raph}(\,m \ + \ 1\,) \ .$$

Structure 3 Sec 6 then uses C to place a partial order on $\mathfrak{C}\mbox{ubic}\mathfrak{G}\mbox{raph}[4]$. $m \ = \ 4$ suffices for this arena to not just be a tree. And to establish that Tet is not the only *cubeomorph irreducible*. See Sec 7.4 for a sketch of what happens for subsequent m .

Structure 4 For $m \ \leq \ 4$ vertices, we make combined use of C and the 'thread diamond' operation D . So as to form a connected poset model for cubic graphs up to this m .

Remark 1 The concluding Section includes pointers to various famous cubic graphs for somewhat larger m as further motivation. As well as how thread-diamond generalizes for subsequent m , and some general points about Combinatorial arenas.

Remark 2 For supporting material, see Appendix A for supporting material on Graph Theory. Complete, cone and bipartite graphs. Graph presentation arenas. Planar graph theorems. And simple crossing number inequalities.

2 The smallest cubic graph: Tet

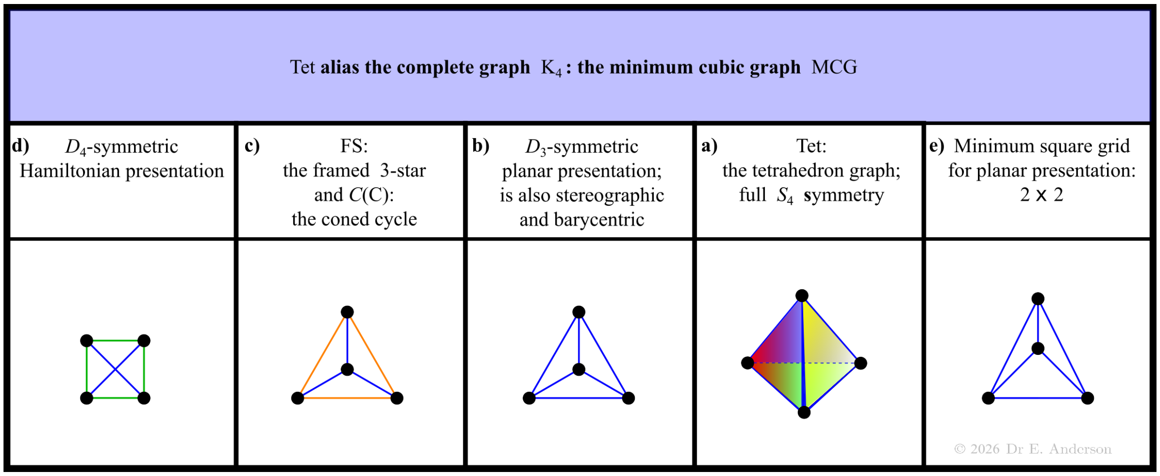


Figure 2:

Recap 1 In the Introduction, we showed that $N = 4$, and thus $m = 2$, is minimum to support any cubic graphs. There is a unique such: the complete 4-*graph* K_4 (see appendix A.1). Alias *minimum cubic graph* for our opening reason, conferring our further notation MCG . And commonly alias *tetrahedron graph* for the reason given below, whence our further notation Tet .

Remark 1 See Fig 2 for various presentations of this. Starting with Subfig a) for the 3-d regular tetrahedron presentation. This is the smallest Platonic solid [108, 102]. Of which 3-d possesses just 5 convex ones. The tetrahedron is furthermore the only one of these to enjoy self-duality.

The tetrahedron is well-known to have the symmetry group S_4 of order

$$4\,! \;=\; 24\ .$$

Which is furthermore fully captured by the tetrahedron graph. So

$$Sym(\,\mathrm{Tet}\,) \;=\; S_4\ , \quad |\,Sym(\,\mathrm{Tet}\,)\,| \;=\; 24\ .$$

Remark 2 Fig 2.b) then beats this flat onto the plane using a centred stereographic projection That retains $S_3 = D_3$ symmetry. This furthermore manages to be a planar presentation.

Remark 3 In 2-d , it is possible to retain presentational D_4 symmetry instead.

Albeit at the cost of introducing 1 crossing: c.f. the other well-known presentation in Subfig c).

And yet at the benefit of being manifestly Hamiltonian. I.e. with some cycle passing through all the vertices. Which is furthermore drawn outermost and emphasized by placing them to form a regular-polygon: highly recognizable. Which we furthermore highlight in emerald in honour of Hamilton's country: Ireland, alias the Emerald Isle...

Remark 4 Tet is furthermore the minimum *framed tree* [32]. Consisting of the claw tree Claw as framed by a triangle. This tree is alias the 3-star, conferring the further notation S_3 . Or in excess notation [148] S , standing for the first star that is not just a path. Hence our further notation FS for Tet in its aspect as a framed tree! We highlight the framing in Fireopal in Subfig d).
Alternatively, it is the cone (Appendix A.2) over this triangle, conferring the notation C(C) . Where C is excess notation for C_3 : the smallest cycle supported by the simple graphs. Or in

fact over any of its 3 other triangles. In each case with the left-out vertex serving as apex to which all other edges are attached. Which full multiplicity of apices is standard throughout the complete graphs. Now conversely the fireopal triangle is incipient. While the central vertex and the blue edges emanating from it are what the coning operation adds.

On the one hand, Tet is the unique cubic graph that manages to be a cone. On the other hand, framed trees are a recurrent feature in the study of cubic graphs. So we shall have a bit more to say about them when we encounter the second and third instances in the current Article.

2.1 Unit distance, 2-distance and grid presentations

Remark 1 Tet is the smallest graph not to admit a unit-distance presentation in 2-d . I.e. therein it cannot be drawn using a unique shared edge-length. It instead requires 2 lengths; c.f. Subfigs a) and b) for realizations. In contrast in 3-d , the regular tetrahedron indeed constitutes the unique unit-distance presentation modulo similarities. In 2-d , there are however 2-distance presentations. These include Subfig b) in the planar case, and Subfig d) with more symmetry at the cost of 1 crossing.

Lemma 3 A crude bound on minimum k-distance presentations of an N-vertex graph is as follows.

$$k \leq \left\lfloor \frac{N}{2} \right\rfloor .$$

<u>Proof</u> Pick out N-vertex subgraphs of K_N on the regular N-a-gon chassis. Count the number of diagonal lengths supported and add 1 for the side-length itself. □

Remark 2 Thus there is no merit to Tet admitting 2-distance presentations. But there is some merit to the below-mentioned instances of 6- and 8-graph 2-distance presentations.

Remark 3 As some final presentational finery, Subfig e) places the planar version on the minimum square grid. While Subfig c) can be viewed as both of the following. A barycentric presentation: the vertex not on the triangle is at its centre. And a regular chunk of equilateral-triangle tessellation [105]. Of trilinear-coordinate [111] size 4^3 .

3 $m = 3 : 2$ cubic graphs

3.1 The Prism graph

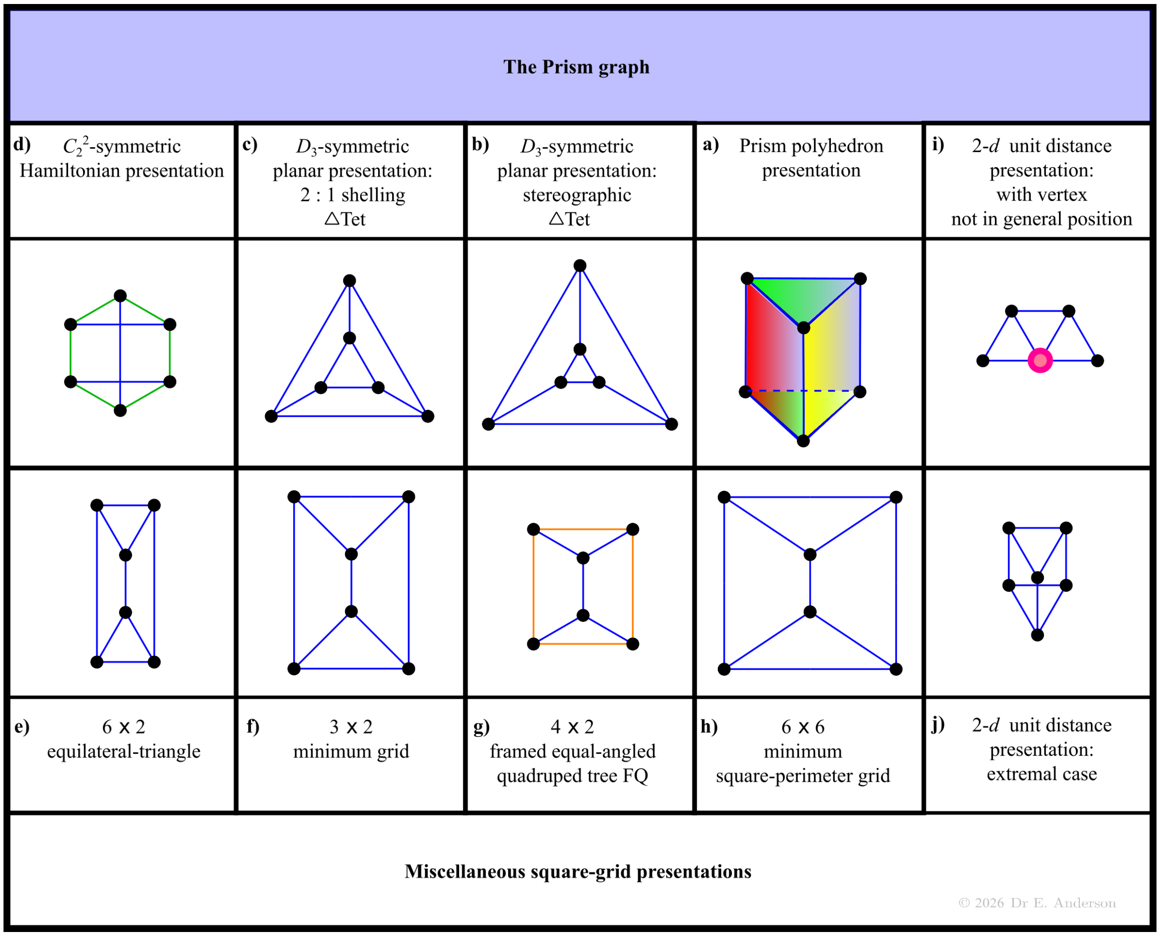


Figure 3:

Remark 1 See Fig 3 for various presentations of the Prism graph. Alias 3-prism Prism_3 in absolute rather than excess conceptualization and notation. It is the smallest such since it is built out of the smallest cycle supported by simple graphs: C . Subfig a) exhibits the 3-d presentation for this using the unit right prism polyhedron itself.

Structure 1 Polyhedra – the 3-d polytopes [100] to the polygons' 2-d polytopes – are much more numerous than the Platonic solids. Ours is convex, so Steinitz' representation theorem (Appendix A.5) furthermore applies.

Remark 2.

$$Sym(\text{Prism}) \;=\; D_3 \times C_2 \;.$$

By use of equilateral-triangle faces within a right prism. On which D_3 acts in the standard way. And C_2 by exchanging top and bottom faces. Thus

$$|\,Sym(\text{Prism})\,| \;=\; |\,D_3 \times C_2\,| \;=\; |\,D_3\,|\,|\,C_2\,| \;=\; 6 \times 2 = 12 \;.$$

Remark 3 Fig 3.b) then beats this flat onto the plane. Using a centred stereographic projection that retains D_3 symmetry. We then relatively rescale this so as to attain $2 : 1$ length ratio between shells in Subfig c).

Remark 4 Subfig d) exhibits a regular-hexagon rectilinear manifestly-Hamiltonian presentation. With Klein 4-group symmetry group

$$V = C_2 \times C_2 =: {C_2}^2 .$$

Remark 5 In Subfigs e-h), we present it instead in framed-quadruped tree form, hence our further notation FQ . Which is again ${C_2}^2$ symmetric. These presentations differ between themselves by the indicated square-gridding finery.

Pointer 1 Prism is the minimum example for one of the 3 collectively-exhaustive symmetry classes for cubic graphs contemplated in [50]. Namely the class T in which each vertex possesses precisely 2 equivalent edges. Tet is another, now for the class S in which each vertex enjoys having all edges equivalent.

3.2 Framed trees(-of-Claws)

Remark 1 Framed trees were introduced by Halin [32]. In the context of cubic graphs, we require trees-of-Claws [148] I.e.

$$\mathfrak{C}\text{ubic}\mathfrak{G}\text{raph} \cap \mathfrak{F}\text{ramed}\mathfrak{T}\text{ree} = \mathfrak{T}\text{-of-Claws} .$$

Examples 1-2 Upgrading to tree-of-Claws notation, the first 2 are

$$\text{FD} := \text{F}(\text{Claw}) = \text{Tet} ,$$

$$\text{FP} := \text{F}(\text{P-of-Claws}) = \text{Prism} .$$

Example 3 See Subsec 11 for the next such. Whose tree is the P_3-of-Claws , alias scorpion.

Pointer 2 And our first sequel the first multiplicities among them for a fixed m: specifically 5 . Comprising both the well-known P_4 versus Claw ambiguity of trees, giving the Claw-of-Claws graph as a further possibility. And the first instance of framing itself becoming nonunique! Which we denote by priming one F in Fig 31.

3.3 A start on arenas of unit-distance presentations

Remark 1 Subfig d) provides a unit-distance presentation.

Remark 2 A subspace of the 'obliquely sideways-on views' of the prism polyhedron enjoy this property.

Structure 1 Crudely, there is an $SO(2)$ rotational freedom in drawing these presentations modulo similartities. By parallelly placing 2 equal equilateral triangles 1 side-length apart. The presentations that attain this 1 side-length apart then sweep out a circle $\mathbb{S}^1$. Corresponding to the claimed freedom; see Fig 4.

However at a more refined level, this model has 2 faults.

Firstly, 6 of these presentations have vertices out of general position. Leaving one with the circle with 6 equally angularly spaced points excised. We label these B , for binary coincidence-or-collision of vertices.

Secondly, unless one regards the prism's vertices to be distinguishable, then there are 12 isometric copies of everything.

Thus for the most obvious indistinguishable-vertex model, one has just a

$$\frac{2\,\pi}{12} \;=\; \frac{\pi}{6}$$

radian arc. Which is open at one end. Corresponding to the sole surviving member of the above excisions. And closed at the other end. Now corresponding to an Arena-theoretic extremum, hence our notation E . In which our 2 input triangles are at $\frac{\pi}{6}$ radians to each other. The original $\mathbb{S}^1$ is thus marked like a clock face, with B's for even hours and E's for odd hours.

Remark 3 The particular uniform-distance presentation we exhibit in Subfig d) is moreover this extremum. The binary coincidence B is provided in Subfig j). Using fuchsia and rose for the 2 superposed vertices: not in general position.

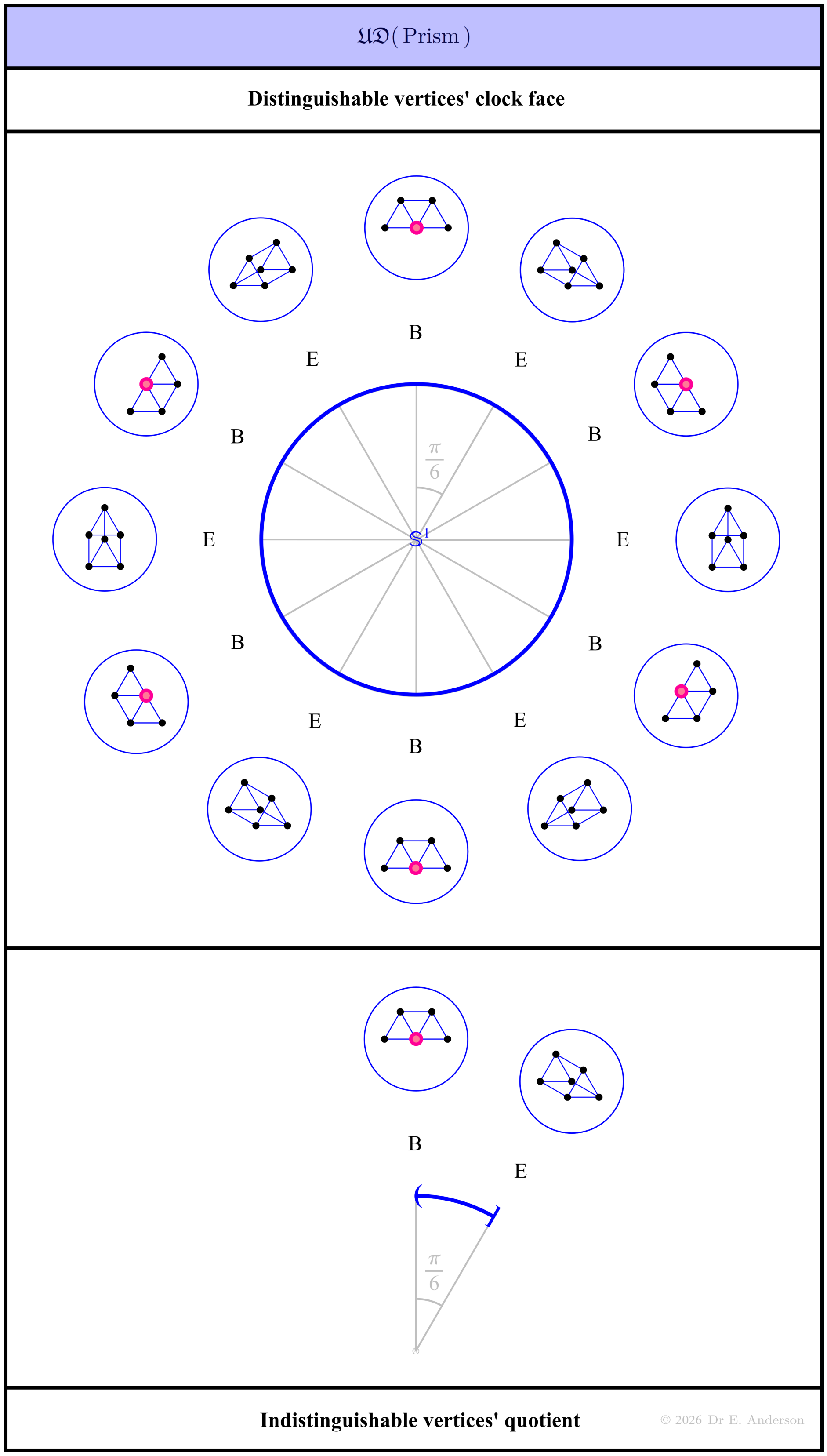

𝔘𝔇(Prism)
Distinguishable vertices' clock face
B
E
π/6
S¹
Indistinguishable vertices' quotient
© 2026 Dr E. Anderson

Figure 4:

3.4 The utilities problem

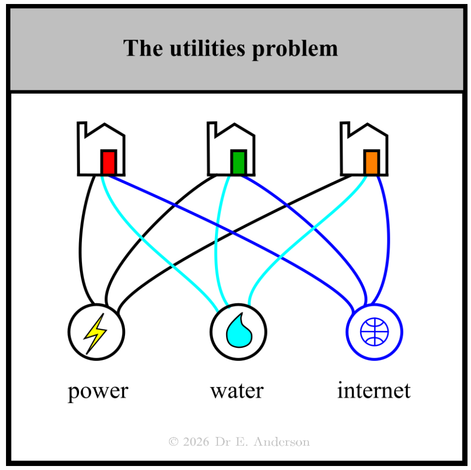


Figure 5:

Remark 1 See Fig 5 [148] for the following motivating *utilities problem*. Three new houses need to be connected to the water mains, power grid and internet. Can you find a way of doing this such that the edges (pipes, cables) do not cross over? Which has however been phrased involving gas mains instead of internet all the way back to the early 20th century [9, 47].

3.5 The Utilities graph

Remark 1 The utilities problem can be boiled down to the complete bipartite graph (Appendix A.2) $\mathrm{K}_{3,\,3}$. Presentations a) and b) in Fig 6 are of this ilk. The first places the vertices uniformly, which however results in a crossing not in general position. Which the second resolves on the next-smallest square grid. I.e. 3×1 to the uniform presentation's 2×1 .

Remark 2 Subfig c) provides the regular-hexagon rectilinear manifestly-Hamiltonian form. This presentation also does not manage to have its crossings in general position. We resolve this by bending 2 edges in Subfig d).

Motivation 1 $\mathrm{K}_{3,\,3}$ became very theoretically significant as one of the forbidden subgraphs in Kuratowski's planar-graph theorem (Appendix A.5).

Remark 3 We next throw an edge into the outer face to obtain a single-crossing presentation. See Subfigs e) and f) for versions with more specific square-grid finery. Again, just C_2 symmetry is retained. Between being able to draw such presentations and the working in Exercise A.2, one can readily establish that $\mathrm{K}_{3,\,3}$ is not only nonplanar but also specifically has crossing number 1 . By which single-crossing presentations are a fortiori crossing-minimum.

Remark 4 Utilities is not unit-distance in 2-d . This can be seen from its containing $\mathrm{K}_{3,2}$ subgraphs. Which Appendix A.3 establishes not to be unit-distance in 2-d .

Historical Remark The Utilities graph first appeared in an attempt to identify the structure of the benzene molecule [4], giving it another alias: *Thomsen graph*.

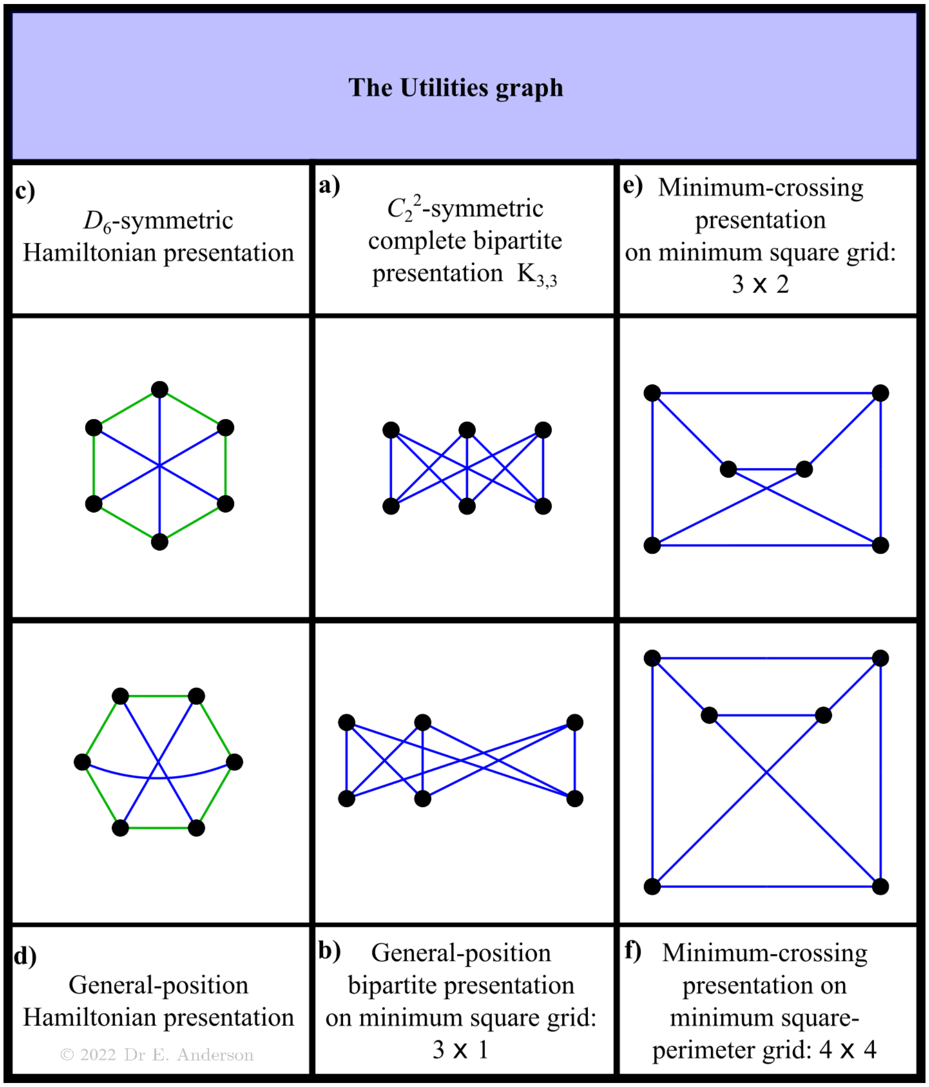


Figure 6:

Exercise 1 A graph is *maximally planar* if it is planar but adding any further edge renders it nonplanar. It is *outerplanar* if it can be embedded in the plane such that every vertex lies on the unboundedly-large external face. It is *maximally outerplanar* if it is outerplanar but adding any further edge renders it nonouterplanar. Show that there is a) just 1 cubic maximally-planar graph. And b) no cubic maximally-outerplanar graph at all.

3.6 Some guesses concerning presentations' symmetry groups

Remark 1 Hitherto the following simple guesses for graph presentation symmetry groups hold out, with the first supporting the second.

Guess 1

$$Sym\,(\,Emb_2(G)\,) \;\leq\; Sym(G) \;\text{ as groups }\;. \tag{2}$$

Guess 2

$$\mathfrak{S}\mathrm{ym}\,(\,Emb_2(G)\,) \;\subseteq\; \mathfrak{p}_{\,\leq}\,(\,Sym(G)\,) \;\text{ as arenas }\;. \tag{3}$$

Where $\mathfrak{p}_{\,\leq}$ denotes poset of subgroups.

Remark 2 While Tet already precluded the following.

Incorrect Guess 3

$$\mathfrak{S}\mathrm{ym}\,(\,Emb_2(G)\,) \;=\; \mathfrak{p}_{\,\leq}\left(\max_{Emb_2\,\in\,\mathfrak{E}\mathrm{mb}_2} Sym\,(\,G\,)\right)\;. \tag{4}$$

Remark 3 By having 2 incomparable [117] top elements D_3 and D_4 . For all that in this case it is the second that has larger group order. I.e.

$$|\,Sym\,(\,Emb_2(\,\mathrm{Tet};\,c)\,)\,| \;=\; |D_4| \;=\; 8 \;>\; 6 \;=\; |D_3| \;=\; |\,Sym\,(\,Emb_2(\,\mathrm{Tet};\,b)\,)\,|\;.$$

3.7 Utilities eliminates the other guesses

Remark 1

$$Sym(\,\mathrm{Utilities}\,) \;=\; {S_3}^2 \,\times\, C_2\;.$$

This follows from separately permuting each part's vertices, and then the exchange symmetry between the parts. Thus

$$|\,Sym(\,\mathrm{Utilities}\,)\,| \;=\; \left|\,{S_3}^2 \,\times\, C_2\right| \;=\; |S_3|^2\,|C_2|\,3!^2 \,\times\, 2 \;=\; 6^2 \,\times\, 2 \;=\; 72\;.$$

Remark 2 Presentations a) and b) retain ${C_2}^2$ and C_2 symmetry respectively.

Remark 3 For presentation c), moreover, since

$$Sym\,(\,Emb_2(\,\mathrm{Utilities};\,c)\,) \;=\; D_6 \;\not\leq\; {S_3}^2 \,\times\, C_2 \;=\; Sym(\,\mathrm{Utilities}\,)\;.$$

This sinks guesses 2 and 3. Illustrating that, rather, embedding a graph is capable of conferring group relators that the original graph did not possess.

Remark 4 Nonetheless for the current example the Combinatorial level, the following continues to hold.

$$|\,Sym\,(\,Emb_2(\,\mathrm{Utilities};\,c)\,)\,| \;=\; |\,D_6\,| \;=\; 12 \,\Big|\, 72 \;=\; |\,Sym(\,\mathrm{Utilities}\,)\,|\;.$$

I.e. at the Combinatorial level, the divisibility corresponding to Lagrange's theorem [108] perseveres in this example. For all that Lagrange's theorem itself is moot.

Pointer 3 A later version of this Periodically Updating Review Article shall cite our proof [146] that Utilities is the unique minimum simple graph to exhibit this effect. Where 'minimum' in the lexicographic sense, on V and then E . Over the presentations of a graph that attain general position. We may more eventually also provide subsequent patches of guesswork and counterexamples.

4 $m = 4 : 5$ connected cubic graphs

Aside 1 $m = 4$ is also minimum to support a non-connected such: $\mbox{Tet}^2$.

4.1 The Cube graph

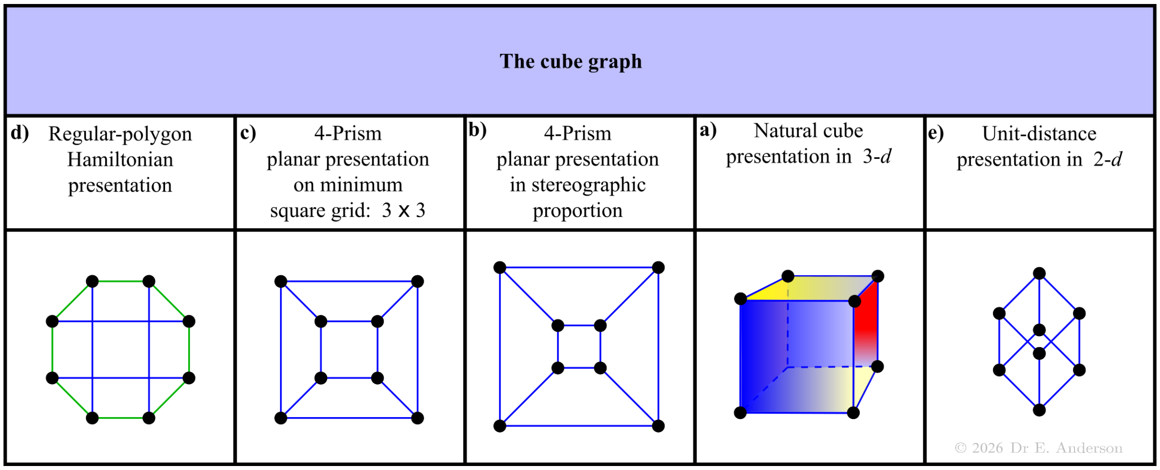


Figure 7:

Remark 1 See Fig 7 for various presentations of the cube graph. Subfig a) leads with the natural cube presentation in 3-d . This is another of the Platonic solids supported in 3-d . It is well-known to have a symmetry group of order 48 , which the cube graph itself manifests. Which is often denoted O since it is shared by the cube's dual [108] Platonic solid: the octahedron. The cube graph inherits this:

$$Sym(\mbox{Cube}) = O \; , \quad |\, Sym(\mbox{Cube})\,| = 48 \; .$$

Remark 2 Subfig b) then beats this flat onto the plane using a centred stereographic projection. It is then relatively rescaled to the minimum square grid form of Subfig c). Both of these presentations retain a D_4 symmetry group. Every right regular prism admits such presentations. The cube – in this sense alias $\mbox{Prism}_4$ is however imbued with extra symmetries and various further structural properties. That greatly outstrip the other prisms, on account of its exceptionally being a Platonic solid.

All the other prism graphs have symmetry group

$$Sym\,(\mbox{Prism}_m) = D_m \times C_2 \; .$$

So

$$|\, Sym\,(\mbox{Prism}_m)\,| = |\, D_m \times C_2\,| = |D_m|\,|C_2| = 2\,m\,2 = 4\,m = 2\,N \; .$$

The cube beats this by a factor of 3 due to having an extra generator of order 3 ; can you find it?

Exercise 2$^-$ a) Show that the remaining 3 3-d Platonic solids' graphs also manage to be planar.

b) Why are their graphs not in the current study?

Remark 3 Subfig c) relatively rescales this into minimum square-grid form, which has square perimeter as well.

Remark 4 Subfig c) has squares in $3 : 1$ ratio. Corresponding to the minimum square-grid presentation: 3×3 . And which furthermore already has square perimeter: in this case an ineffective constraint! [50] uses the $2:1$ ratio instead, corresponding to 4×4. This venerable source also contains an isometric copy of Subfig c).[1]

Remark 5 Subfig d) provides instead the regular-octagon manifestly-Hamiltonian presentation. This also retains a D_4 of presentational symmetry

Remark 6 Finally Subfig e) provides a unit-distance presentation.

Exercise 3 Describe the arena of all 2-d unit-distance presentations of $\mbox{Cube}$. Firstly allowing for vertices not being in general position and secondly excising such presentations. Also figure out how Subfig d)'s choice is the unique Arena-Theoretic extremum case.

[1]Rotated through $\frac{\pi}{4}$ radians. And using not colour but a 'join the dots' numbering to pick out the Hamiltonian cycle. being emphasized.

4.2 The Wagner (crossed cube) graph

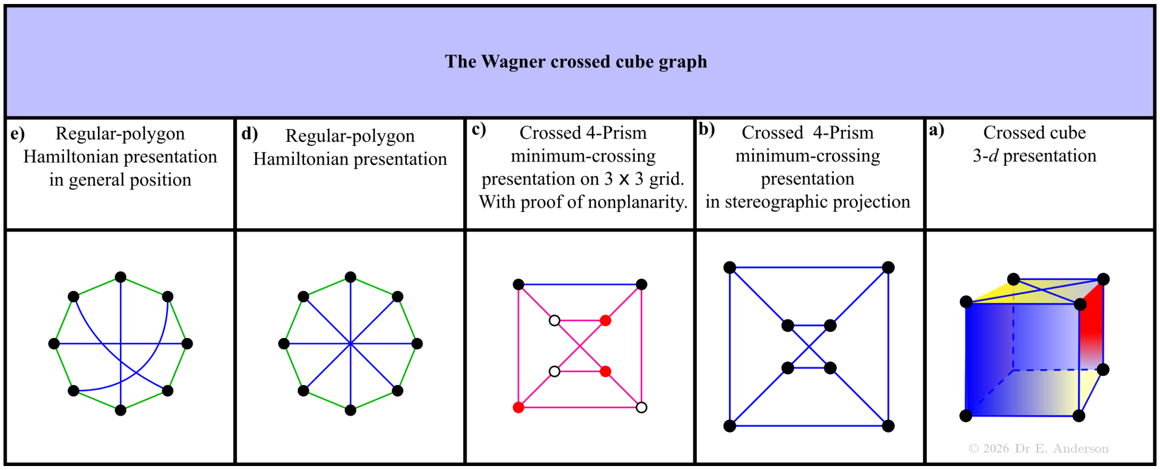


Figure 8:

Remark 1 See Fig 8 for various presentations of Wagner's crossed cube graph [14]. Subfig a)'s 3-d presentation is a manifestly crossed version of the cube. Subfig b) then beats this down into the plane using a centred stereographic projection. Subfig c) next rescales this into minimum square-grid form, which has square perimeter as well. b) and c) sport the minimum crossing number of 1 . As checked in Subfig c) by exhibiting a $\mathrm{K}_{3,\,3}$ subgraph. Both exhibit C_2^2 symmetry.

Remark 2 In Subfig d), we form a regular-octagon manifestly-Hamiltonian presentation, also featuring e.g. in [87]. This presentation – additionally exhibiting D_8 symmetry – has however a crossing not in general position. For no less than 4 edges meet at a vertexless point: the centre of symmetry. We thus additionally provide a general-position presentation in Subfig e). Using not entirely rectilinear edges, as is usually required while maintaining the regular-polygonal presentation of the Hamiltonian cycle.

Exercise 4 Show that

$$|\,Sym(\,\text{Wagner}\,)\,| \ = \ 16 \ .$$

Identify which abstract group is realized. Deos it coincide with the above presentational D_8 ?

Remark 3 Twisting the cube spoils the cube graph's own unit-distance presentations, accounting for the absence of any such from the current Figure. This follows from its containing a $\mathrm{K}_{3,\,2}$ subgraph.

Exercise 5 Calculate the length ratios in the graph presentations resulting from centred stereographic projections of the following shapes. a) The cube. b) The crossed cube. c) The prism's 3-d unit-distance presentation.

4.3 Pointer 4: the Möbius ladder graphs

Definition 1 An even *Möbius ladder* ML_m [29] is a type of cubic Hamiltonian graph. In which all opposite pairs of vertices on the Hamiltonian cycle are related by edges.

Remark 1 The above definition was originally stated for $N \geq 6$, i.e. $= 3$. They also defined odd Möbius ladders, though these are not cubic. And was The Möbius ladder's name is inspired by its similarity to the Möbius strip [106, 120, 123]. More specifically, we point out that it bears further similarities to the ruled-surface [110] and especially fibre bundle [121] models of the Möbius strip

Notational Remark 1 M_N is standard notation. We however consider ML_m to be a more distinctive notation [146].

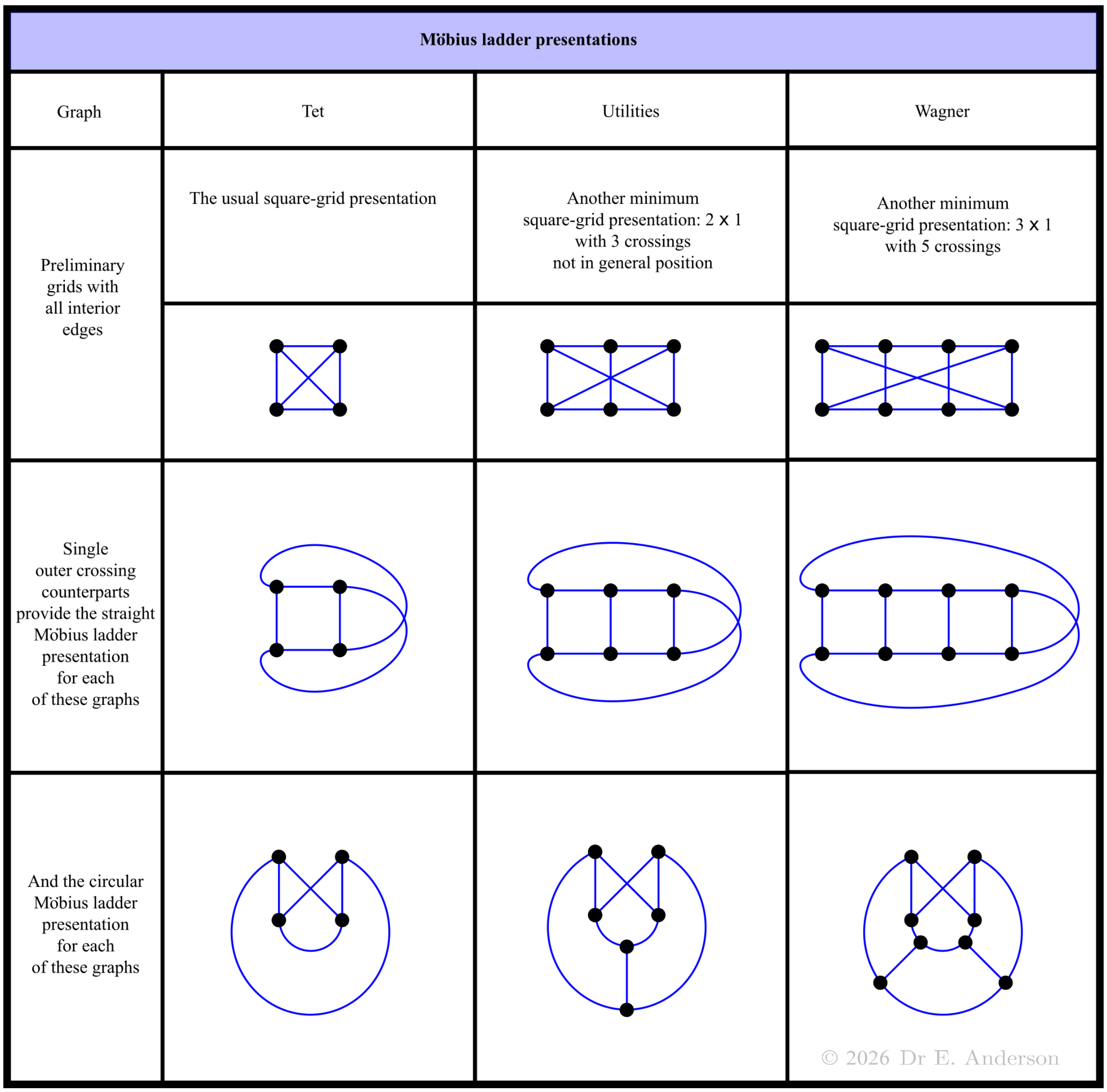


Figure 9:

♠ ◇ ♣

Remark 2 The below examples make reference to Fig 9, the first subfigure of which is a reissue of Fig 2.d). The second row follows from the first by throwing crossing edges from the interior into the exterior face. Which we called 'perimeter inversion' in [146]) by analogy with inversion in the circle in planar Inversive Geometry.

Example 1 Tet is in a sense the minimum Möbius ladder, ML_2 . It does not however have enough rungs to be nonplanar. Which gives a way in which it is not Topologically Möbius. It also has no twist-free rungs (row 3 in Fig 9: Guy and Harary's [29]'s original presentation for general $N \geq 6$).

Example 2 The next smallest – Utilities $=$ ML_3 – is clearly satisfactory in this regard.

Example 3 Wagner $=$ ML_4 is the third smallest Möbius ladder. And the minimum to more closely emulate the fibre bundle model's Differential Geometry [90], and also to have the Graph-Theoretic circulant property.

Definition 2 A cubic graph on $N = 2m$ vertices is *circulant* in this sense if it contains exactly m 4-cycles.

Remark 2 All of the above properties are subsequently persistent.

♠ ◇ ♣

Notational Remark 2 Excess notations include [146], firstly,

$$\text{Tet} = \text{ML} , \quad \text{Utilities} = \text{ML}(1) , \quad \text{Wagner} = \text{ML}(2) .$$

Using as parameter the adapted variable

$$k := m - 2 .$$

So as to centre about the naïve minimum supplied by Tet .

Secondly,

$$\text{Tet} = \text{MLT}(-1) , \quad \text{Utilities} = \text{MLT} , \quad \text{Wagner} = \text{MLT}(1) .$$

Where the parameter is now the adapted variable

$$j := m - 3 .$$

So as to centre about the minimum T-for-Topological Möbius ladder, Utilities .

Thirdly,

$$\text{Tet} = \text{MLB}(-2) , \quad \text{Utilities} = \text{MLB}(-1) , \quad \text{Wagner} = \text{MLB} .$$

Where finally the parameter is the adapted variable

$$i := m - 4 .$$

So as to centre about the minimum B-for-Bundle-Theoretic Möbius ladder, Wagner .

Exercise 6 a) Show that removing any vertex from the Wagner graph renders it planar.

b) Show that the Wagner graph is not bipartite.

c) More generally, show that for m odd, ML_m is bipartite, while for m even, it is not. And that all of the ML_m have crossing number 1 .

4.4 Great pointer 0: The Petersen graph and the generalized Petersen graphs

Remark 1 The Wagner graph's crossed square in Fig 8.a) generalizes to the totally crossed pentagon. Which is also the star pentagon [100]. Forming a prism out of a regular polygon and the corresponding star polygon gives a particularly significant graph in the case of the pentagon, as follows.

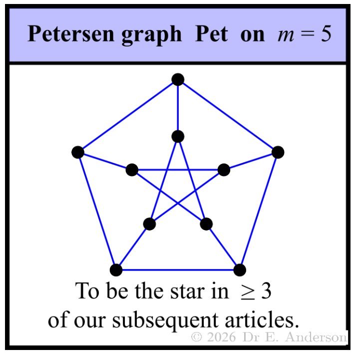


Figure 10:

Pointer 0 Fig 10's *Petersen graph* [7, 58, 62, 56] itself is quite possibly the most interesting and significant of all graphs. And certainly among small graphs studied to date. It is indeed a cubic graph, albeit on $m = 5$ and thus not in the immediate scope of the current Review. But it is significant enough to take note of structural features that smaller graphs share with it. For all that these do not share in the myriad and often deep consequences of the Petersen graph's own version of those features.

Knuth [156] described the Petersen graph as

"*a remarkable configuration that serves as a counterexample to many optimistic predictions about what might be true for graphs in general*"

A briefer description of our own [146] is that it is

"*basic Graph Theory's counterexample to almost everything*"!

Remark 2 So, on the one hand, the prisms use 2 copies of a regular polygon. While the above construct uses a regular polygon and its corresponding star. For $m = 4$, these are all the possibilities. While for $m \geq 5$ partially-crossed polygons are also possible; see [93, 96].

Structure 1 On the other hand, the star hexagon consists of 2 disjoint triangles ('star of David'). More generally, a polygon and the corresponding star polygon forms a *generalized Petersen graph*. This notion is also due to Coxeter [101], for all that its name came to be coined by Watkins [31].

On the one hand, the Wagner graph is not among these. Either by the square not being taken to support a star polygon. Or by the pair of crossed lines being allowed as a edge case. But this is not the sole content of the Wagner graph's inner shell, nor is it cubic!

On the other hand, Prism and Cube are generalized Petersen graphs.

♠ ◇ ♣

Notational Remark 1 Coxeter's notation [101] for generalized Petersen graphs is the following concatenation of Schläfli symbols [100]. Firstly for the polygon and then for the star polygon.

$$\{m\} \ + \ \{s\} \ . \tag{5}$$

Watkins' [31] subsequently standard notation is as follows. Parametrize by firstly the prism polygon's size m . And then by the the following adapted variable.

$$d \ := \ \frac{m}{s}$$

Which can be interpreted as *star-polygon density*. This can be viewed as a combinatorial count [100]. As a Topologically contentful winding number and a fortiori turning (tangent) number: winding of a path relative to its own tangent. Or as a bit of both, in terms of a covering by sides. .
Writing $\mathrm{G}(\, m, \, d\,)$, or, more distinctively,

$$\mathrm{GPG}(\, m, \, d\,) \ .$$

Example 1 In this notation, the Petersen graph itself is

$$\mathrm{Pet} \ = \ \mathrm{GPG}(\, 5, \, 2\,) \ .$$

Examples 2-3 Of direct relevance to the current Article,

$$\mathrm{Prism} \ = \ \mathrm{GPG}(\, 3, \, 1\,) \quad \text{and} \quad \mathrm{Cube} \ = \ \mathrm{GPG}(\, 4, \, 1\,) \ .$$

Remark 3 Observe the following range inequality arising from the definition of star polygon and the insistence on cubicness.

$$1 \ \leq \ d \ \leq \ \left\lfloor \frac{m \ - \ 1}{2} \right\rfloor \ .$$

Notational Remark 2 Relative variables centering about the naïve minimum example Prism are as follows. So as to use the notation

$$\mathrm{Prism}(\, j, \, c\,) \ .$$

For

$$c \ := \ d \ - \ 1 \ .$$

Given notational remark 1's parametrization, the current Prism notation is to be interpreted likewise as a family of possibly crossed prisms. Then

$$\mathrm{Cube} \ = \ \mathrm{Prism}(1) \quad \text{and} \quad \mathrm{Pet} \ = \ \mathrm{Prism}(\, 2, \, 1) \ .$$

It however makes most sense to centre about Pet , so as to use the notation

$$\mathrm{Pet}(\, h, \, b\,) \ .$$

Corresponding to the adapted variables

$$h \ := \ m \ - \ 5 \ ,$$

$$b \ := \ d \ - \ 2 \ .$$

Then

$$\mathrm{Prism} \ = \ \mathrm{Pet}(\, -2, \, -1\,) \quad \text{and} \quad \mathrm{Cube} \ = \ \mathrm{Pet}(\, -1, \, -1\,) \ .$$

4.5 The framed scorpion graph FSc

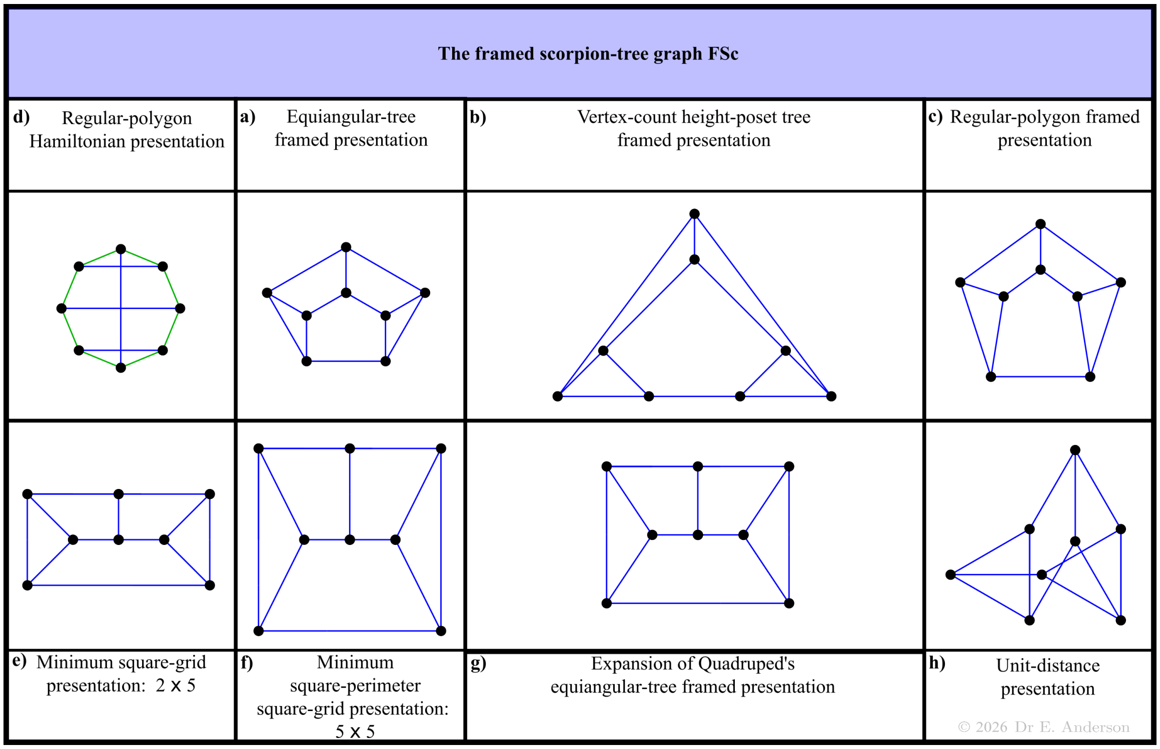


Figure 11:

Remark 1 See Fig 11 for various presentations of the Framed Scorpion tree, FSc . The Scorpion graph Sc is itself the P_3 of Claws tree. Hence the further tree-of-Claws notation FP_3 .

$$Sym(\text{FSc}) \;=\; {C_2}^2 \;, \quad \text{of order} \;\; 4 \;.$$

encoding the exchange of edges within each pincer.

Remark 2 In Subfig a), we first cast Sc in unit-distance uniform-angle presentation. In Subfig b), we exhibit instead the vertex height-count poset presentation [78]. In each of these case, we then frame this with the induced pentagon. In Subfig c), we instead first form the regular pentagon and then fit the scorpion tree to it. With the remaining 3 vertices on a second regular pentagonal shell. And a $2 : 1$ length ratio between the shells.

Remark 3 Subfig d) provides instead a regular-octagon manifestly-Hamiltonian presentation. The version we provide manages to capture all of the graph's own ${C_2}^2$ symmetry.

Remark 4 Subfigs e)-g) are square-grid versions of the framing presentation, with finery as indicated. Finally Subfig h) is a unit-distance presentation, details of which are difered to Exercise 9.b).

4.6 The △ Utilities graph

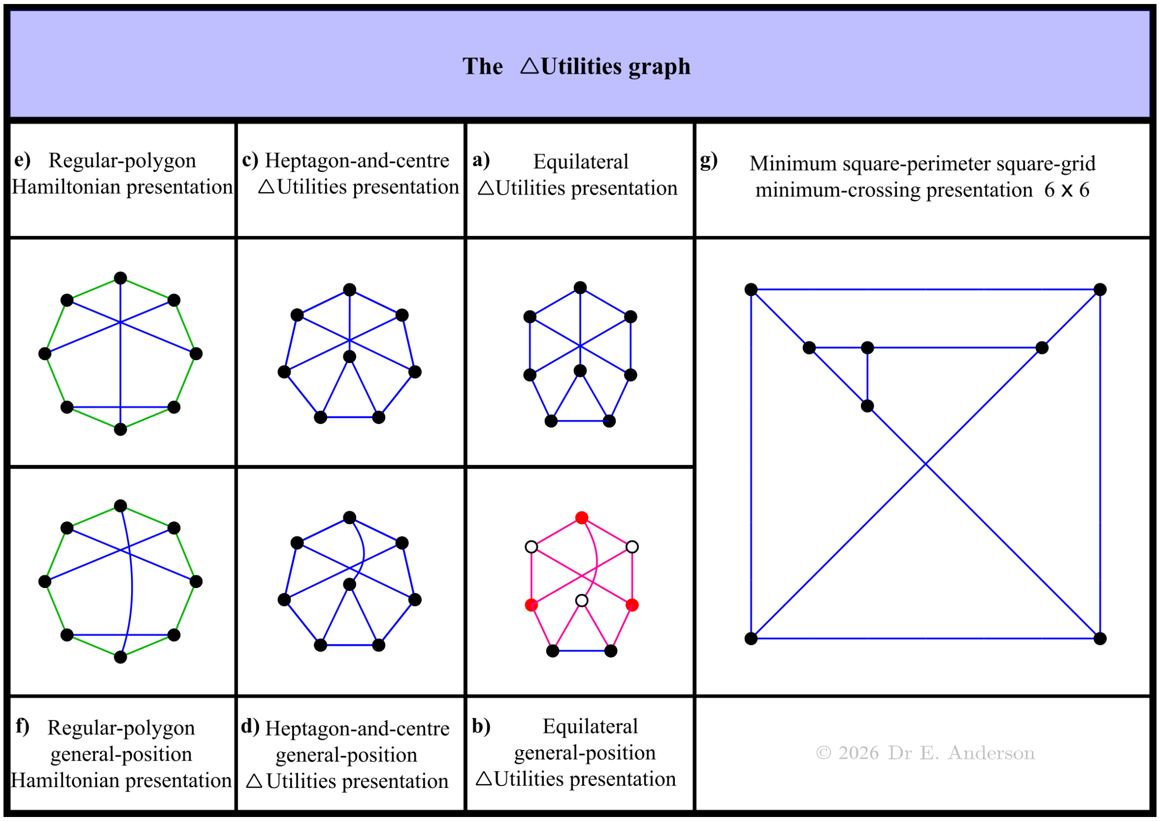


Figure 12:

Remark 1 See Fig 12 for various presentations of the △ Utilities graph. I.e. the outcome of inserting a triangle-making edge into the Utilities graph. In a similar vein,

$$\text{Tet} \;=\; \triangle\,\text{P} \;\text{ and }\; \text{Prism} \;=\; \triangle\,\text{Tet} \;.$$

Remark 2

$$Sym(\,\triangle\,\text{Utilities}\,) \;=\; S_3 \;\times\; S_2 \;=\; S_3 \;\times\; C_2 \;.$$

This is by its containing a $K_{3,\,2}$ subgraph, none of whose symmetries are broken, and no further symmetries. Consequently

$$|\; Sym(\triangle\,\text{Utilities})\;| \;=\; |\,S_3 \;\times\; C_2\,| \;=\; |S_3|\;|C_2| \;=\; 3!\,2 \;=\; 12 \;.$$

In Subfig a), we give a △ Utilities presentation. In which the inserted triangle is equilateral. And half the width of the original Hamiltonian presentation of the Utilities graph. In Subfig b), we then furthermore bend an edge to place the crossings in general position.

Remark 3 In Subfig c), we give another △ Utilities presentation. Now placing one new vertex into a regular-heptagon extension of the Utilities hexagon. And the other new vertex at its centre. Subfig d) then again bends an edge to attain general position.

Remark 4 Subfig e) exhibits a regular-octagon rectilinear manifestly-Hamiltonian presentation. Subfig f) yet again bends an edge to arrive at general position.

Remark 5 Subfig b) also takes the opportunity to explicitly establish that there is indeed a Utilities $=$ $K_{3,3}$ subgraph. By which $\triangle$Utilities is nonplanar. Subfig g) then demonstrates that the crossing number is 1. While also being more specifically the minimum square-perimeter square grid. Which comes with the extra bonus that its sole crossing lies at its centre.

Exercise 7 a)$^{-}$ Show that $\triangle$Utilities is not unit-distance. b)$^{+}$ Is it 2-distance?

4.7 The di-Di graph

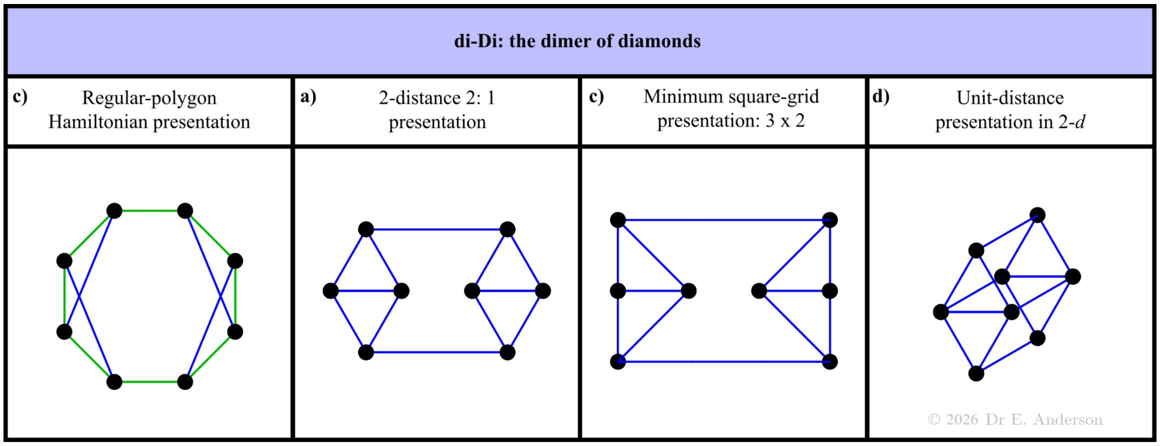


Figure 13:

Naming Remark 1 Recollect the diamond graph of Fig 14.a), which we name and denote by Di . Fig 12 then exhibits various presentations of the di-Di graph. I.e. the dimer of Diamonds [146]!

Remark 1

$$Sym(\text{ di-Di }) \ = \ {C_2}^4 \ .$$

By supporting 4 independent 2-object switches. All of which commute with each other. Consequently

$$|\, Sym(\text{ di-Di })\, | \ = \ \left|\, {C_2}^4 \,\right| \ = \ |\, C_2 \,|^4 \ = \ 2^4 \ = \ 16 \ .$$

Remark 2 In Subfig a), we give a presentationally ${C_2}^2$ symmetric 2-distance presentation. Manifesting 2 horizontally-shifted copies of the Di monomer. As joined by mutual edges in 2 : 1 ratio with the internal edges.

Remark 3 In Subfig b), we give a regular-octagon rectilinear manifestly-Hamiltonian presentation. With the same presentational symmetry group.

Remark 4 In Subfig c), we give the minimum square-grid presentation.

Remark 5 In Subfig d), we give a unit-distance presentation. Which is again an Arena-theoretic extremum that is unique modulo similarities, as you shall derive in Exercise 9.a) below.

Remark 6 Observe that Tet is also DiRing : the diamond ring alias diamond monomer.

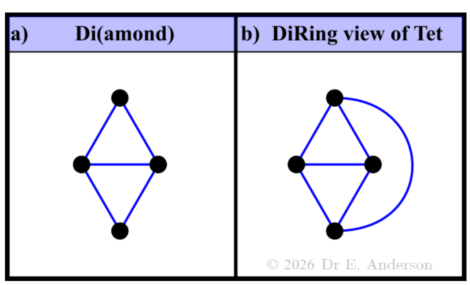


Figure 14:

4.8 Exercises

Exercise 8$^+$ [Sánchez 2019] a) For di-Di , what is its minimum planar square grid size subject to the square-perimeter constraint? *Conceptually how* is this grid size related to the unconstrained minimum planar grid size exhibited? Check your hypothesis for the current Article's other minimum grids...

b) What is the minimum convex chunk of equilateral-triangle tessellation of the plane that di-Di 's vertices fit onto? Can you fit the edges onto such a tessellation as well? What happens if you try fitting this Article's seven other cubic graphs in these ways?

Exercise 9$^+$ Rerun Exercise 3 for a) di-Di . b) FSc .

Exercise 10 [Sánchez 2019] a) Find all the cubic graphs on $m \leq 4$ as follows. By placing diagonals on the corresponding regular polygons in all possible cubic ways.

b) Show that this however has limited shelf-life as a method for finding all cubic graphs. By constructing a non-Hamiltonian cubic graph on some $m > 4$. How small can m be?

5 Cubeomorphs

5.1 Homeomorphs

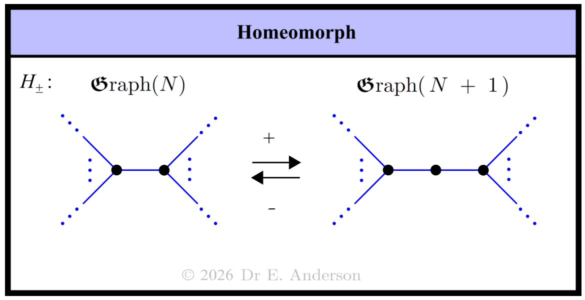


Figure 15:

Structure 1 *Homeomorphing* [63, 148] is the *expansion* of a graph by the following moves.

+1.1) Cut open an edge.

+1.2) Attach the 2 unoccupied ends thus created to a new vertex.

Or, in the reverse direction, the *reduction* of a graph by the following moves.

-1.2) Remove some degree-2 vertex from our graph.

-1.1) Join the 2 unoccupied ends thus created to each other.

Our notation here is based on these expansion and reduction operations changing the graph's vertex number by ± 1 respectively.

Working in the $-$ direction, the end product of homeomorphing as many times as possible is a *homeomorph irreducible* (*HI*) [35, 148].

Naming Remark 1 Some alternative names are as follows. Homeomorphing is alias *subdividing* or *expanding* [43] in the $+$ direction (by itself rather than in the combination 'expanding homeomorph'). And *smoothing* or (*series*) *reduction* in the $-$ direction. While [59] refer to HI trees *topological trees*.

On the one hand, the name 'homeomorph' is categorically adroit, since it alludes to Topological content. Compare *homeomorphism*, which is Topology's most fundamental map, signifying continuity-preserving map. And the homeomorph operation is indeed Topologically significant [62, 150]. On the other hand, 'smoothing' carries Differential Geometry associations. But graphs are not structurally rich enough for the current Subsec's map to have properties tightly analogous to the smoothness family of notions.

'Series' refers to many small, basic and often encountered graphs forming infinite series. In which each member arises from the previous one by homeomorphing in 1 vertex at a particular site. For instance the paths or the cycles. Or, to illustrate multiple sites, the pans with increasing cycle size, versus the pans with increasing handle size. See [148, 92] for rather longer lists of examples of this.

Remark 2 By definition, cubic graphs have solely degree-3 vertices. Thus they have no degree-2 vertices.

As a first consequence of this, cubic graphs are all HI. In fact, since they have no degree-1 vertices either, they are also foliation irreducible (FI) [148]. And thus doubly irreducible (DI) in the sense of [79].

As a second consequence, homeomorphing affords no reductions in $\mathfrak{C}\text{ube}\mathfrak{G}\text{raph}$. And all of its expansions kick one out of this arena.

However, the following more elaborate analogue is available.

5.2 Cubeomorphs

Structure 2 *Cubeomorphing* is the expansion of a graph by the following moves.

+2.1) Perform 2 expanding homeomorphs.

+2.2) Join the 2 new vertices by an edge.

Or, in the reverse direction, the *reduction* of a graph by the following moves.

-2.1) Remove an edge.

-2.2) Perform 2 homeomorph reductions to eliminate the 2 degree-2 vertices thus created.

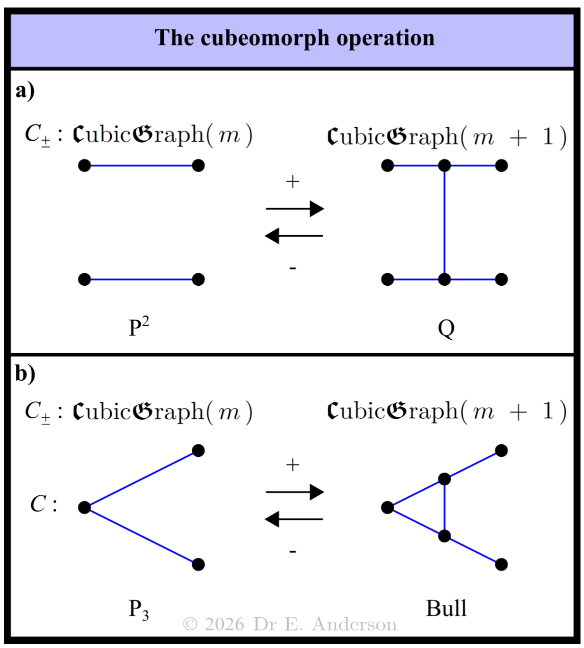


Figure 16:

Overall, these operations change the vertex count by ± 2 respectively, and thus m by ± 1 . They are furthermore designed to preserve cubicness:

$$C_\pm \ : \ \mathfrak{C}\text{ube}\mathfrak{G}\text{raph}(m) \ \rightleftharpoons \ \mathfrak{C}\text{ube}\mathfrak{G}\text{raph}(\,m\,\pm\,1)\ .$$

The Introduction's C is then a shorthand for C_+ .

Naming Remark 2 The above expansion map was used to systematically generate firstly CIs in [65] and secondly snarks in [66], The name 'cubeomorph', is however to the best of our knowledge new [146].

Alternative names are '*expansion* and *contraction of a single edge* in the theory of planar Hamiltonian graphs [146] . And *construction* and *reduction* in the literature on snarks [66]. Observing that the cubeomorph operation can furthermore be performed on any graph with enough edges and vertices gives meaning to these names in this broader context. In studying the internal life of cubic graphs, however, as the current Article does, the cubeomorph name is more appropriate.

Remark 1 Locally, cubeomorphs are of 2 types. For the 2 edges involved either have 0 or 1 vertex in common. See Subfigs 0) and 1) for these respectively.

5.3 Protected subgraphs

Remark 1 For simple graphs, cases in which homeomorphing would create a multi-edge are not to be removed. This gives a notion of a degree-2 vertex that is protected from homeomorphing.

Example 1 Any vertex in C is a such (Fig 17.a).

Remark 2 Cubic graphs exhibit the following analogue of this effect. If a vertex being removed belongs to a *Di* subgraph, then removing this vertex as part of a cubeomorph would create a multi-edge.

Example 1 One can already see this by cubeomorphing out Di's 2 degree-3 vertices (Fig 17.b). This limits the number of versions of the cubeomorph diagram to the 2 depicted in Fig 30.

Examples 2-3 Within cubic graphs themselves, Subfig.c) illustrates how Tet is a CI. And Subfig d) likewise for di-Di ; there are now 3 inequivalent edges to consider.

5.4 Motivation

Motivation 1 Observe that many graph properties are homeomorph-independent [62, 148, 79, 150].

Motivation 2 In a Topology-oriented program [146, 148, 85] with modelling diversity of notions of graph , we need a corresponding diversity of analogues of the homeomorphing operation. Aside from the current Article's cubeomorph, we have placed such on rooted graphs, digraphs [80], fixed-girth graphs, posets, and various of their specializations [91].

Motivation 3 Cubeomorphs retain less Topological content than homeomorphs. Rendering it crucial [146] to give an example of how they are not bereft of Topological content . Tet and its cubeomorphs are 3-connected. While di-Di and its cubeomorphs are merely 2-connected. Both vertex-wise and edge-wise.

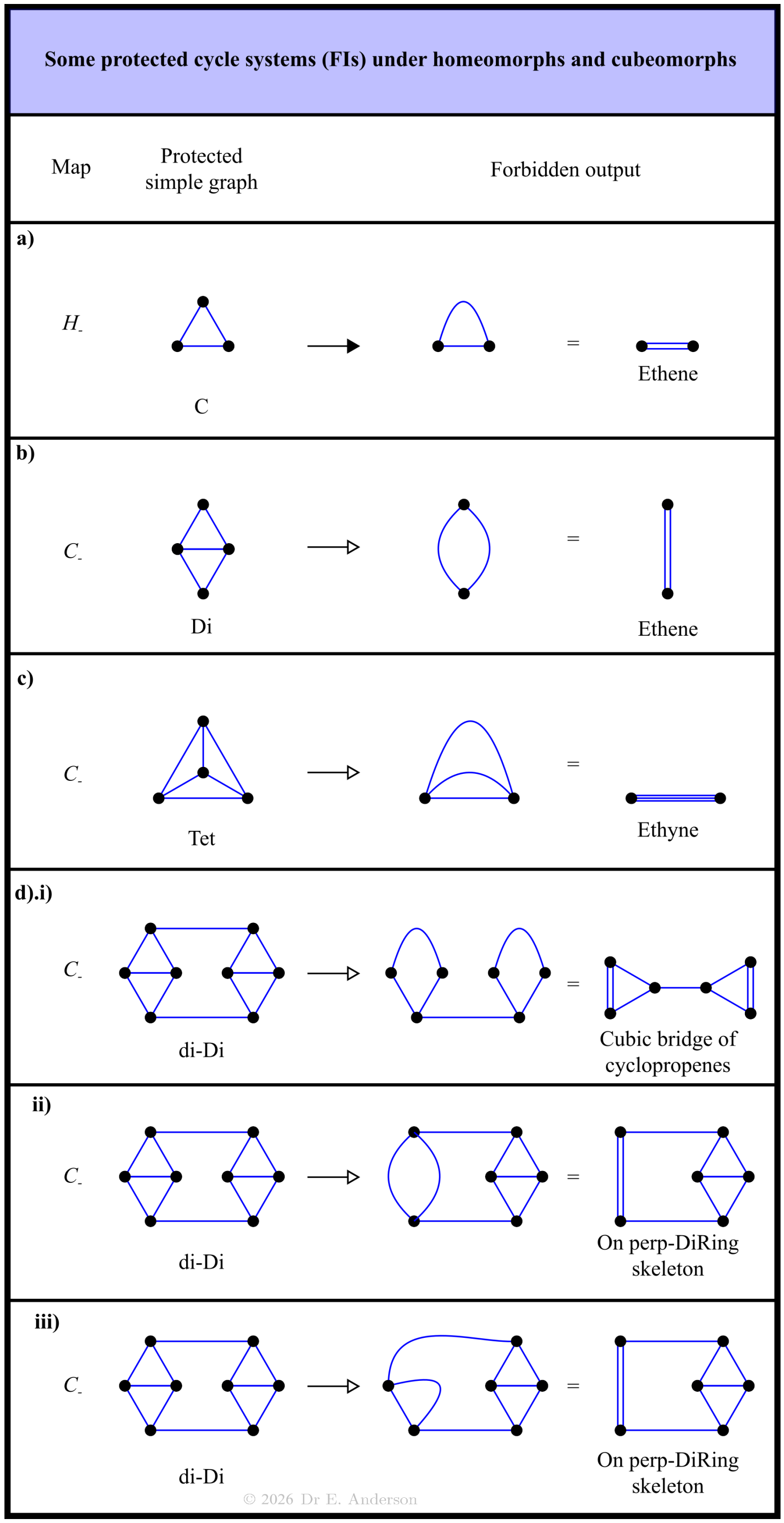

Some protected cycle systems (FIs) under homeomorphs and cubeomorphs
Map
Protected simple graph
Forbidden output
a)
H
C
=
Ethene
b)
C
Di
=
Ethene
c)
C
Tet
=
Ethyne
d).i)
C
di-Di
=
Cubic bridge of cyclopropenes
ii)
C
di-Di
=
On perp-DiRing skeleton
iii)
C
di-Di
=
On perp-DiRing skeleton
© 2026 Dr E. Anderson

Figure 17:

6 Cubeomorph order on $\mathfrak{c}\mathbf{ube}\mathfrak{G}\mathbf{raph}[m]$

6.1 $\mathfrak{c}\mathbf{ube}\mathfrak{G}\mathbf{raph}[4]$

Remark 1 See Fig 18.a) for this on the Tet chassis. Subfig b) on the Prism chassis once available. Subfig c) on the Utilities chassis once available. Fig 19.a) using the most widely recognized presentations. And Fig 19.b) with emphasis on the persistent feature of framed trees of claws.

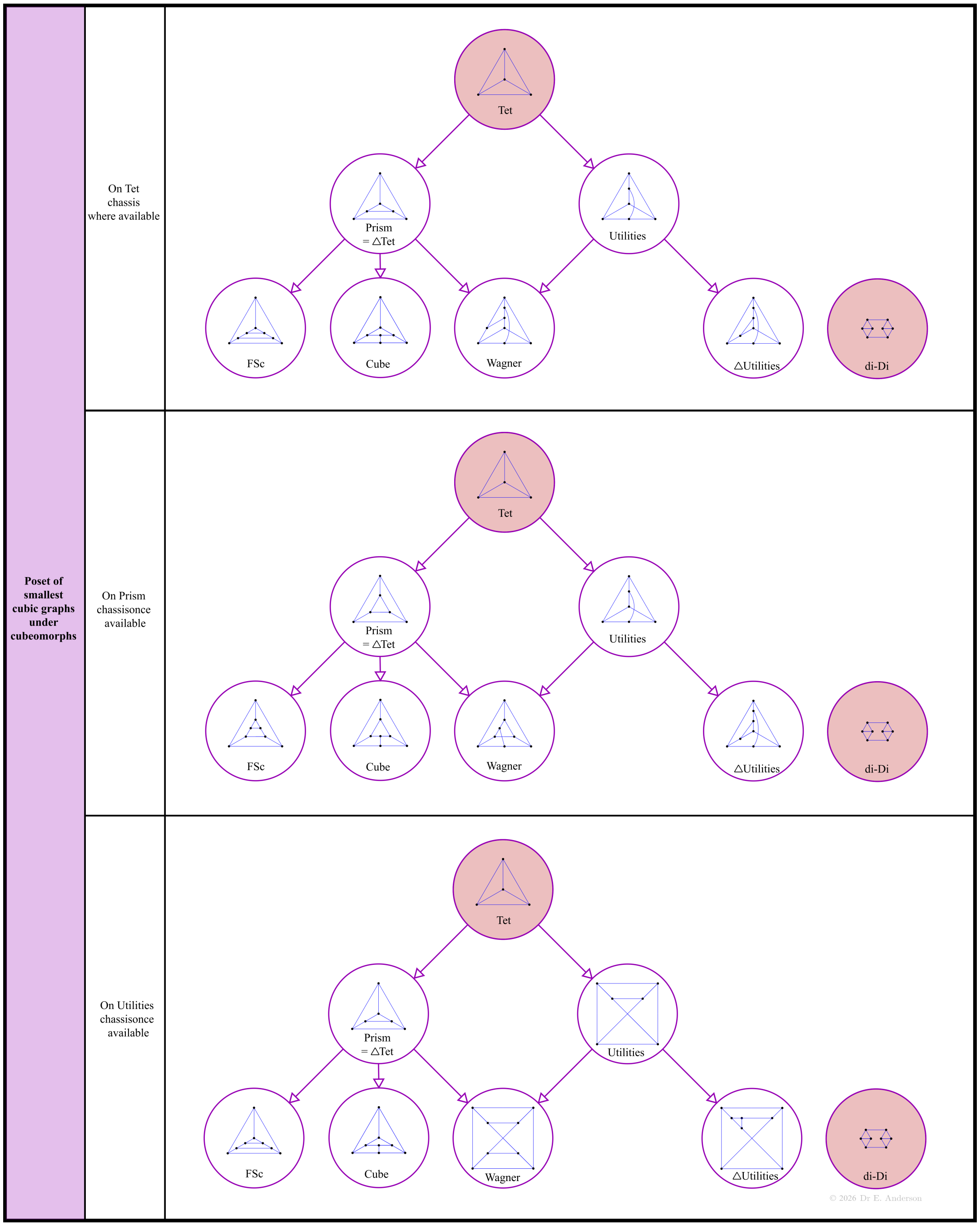


Figure 18:

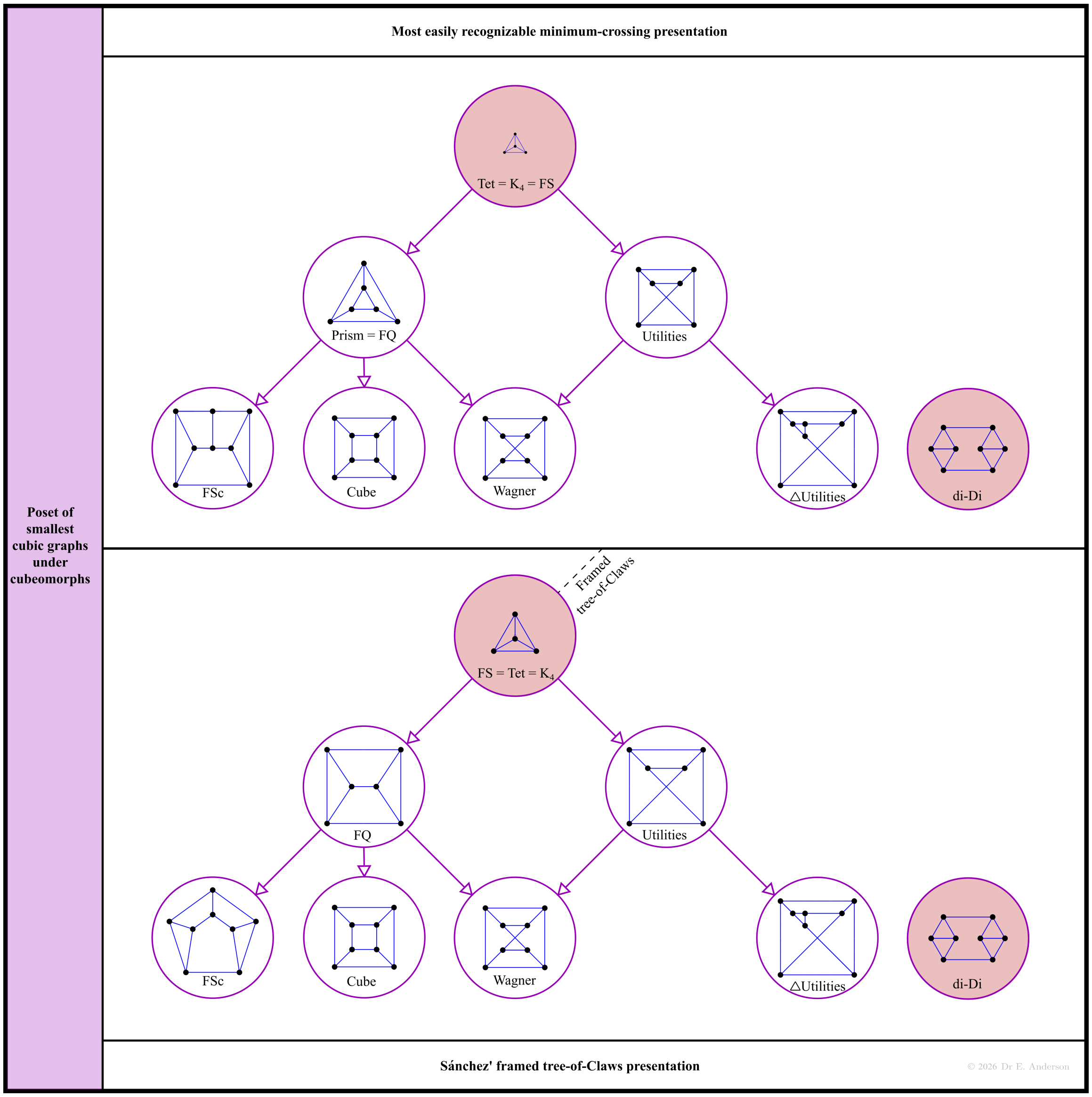


Figure 19:

Exercise 11 [Long] Show that everything within Fig 2 is isomorphic as a graph. And likewise within each of Figs 3, 6, 7, 8, 11, 13 and 12. Finally establish that each Tet chassis presentation is indeed isomorphic to the graph it is labelled as. And likewise for each Prism chassis presentation and Utilities chassis presentation.

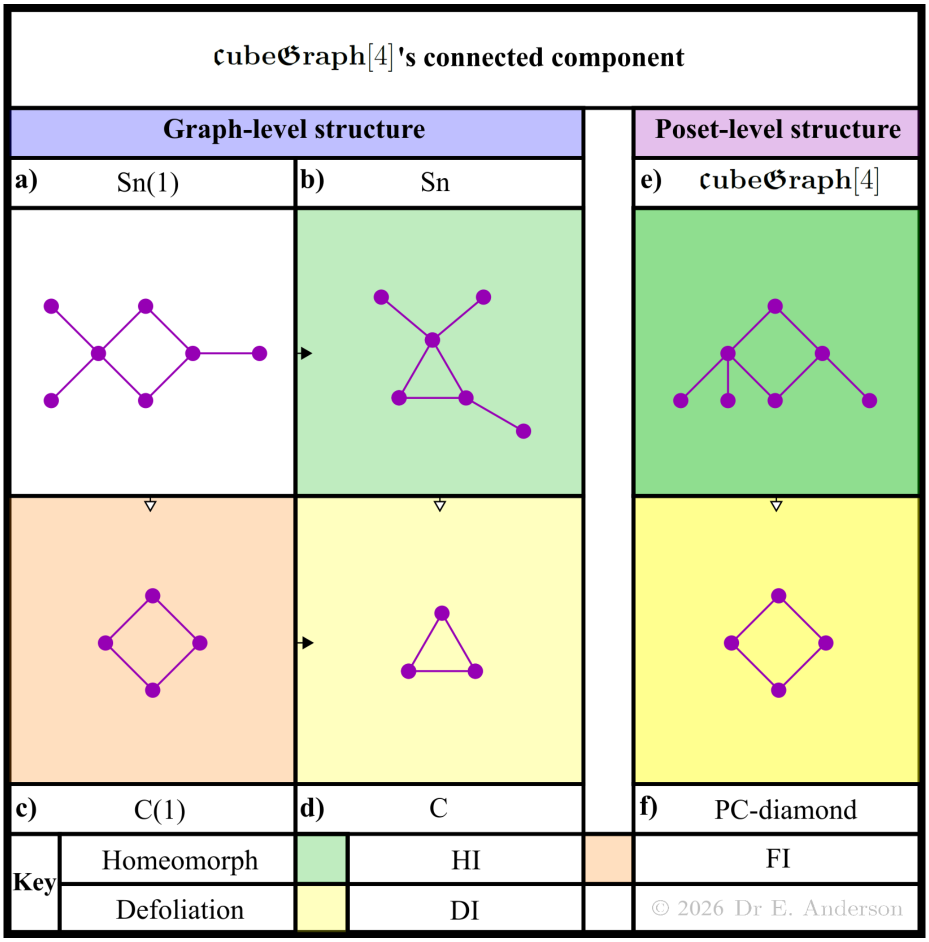


Figure 20:

6.2 Smaller cumulative arenas and some of their simplicities

$\mathfrak{CubicGraph}[2]$ is just a point.

$\mathfrak{CubicGraph}[3]$ is the bent 3-path rooted tree.

$\mathfrak{CubicGraph}[4]$ is the first nontrivial poset in the sense of being more than a tree. The underlying graph is $Sn(1)$: the first homeomorph indicated in Fig 20.b) of the Snail HI graph in Subfig a).

To obtain a nonstandard underlying graph, and exhibit breakdown of planarity and of upward planarity [114, 75], we need to consider $m = 5$ and 6 . We leave this to another occasion.

6.3 Placing an order on the CI themselves

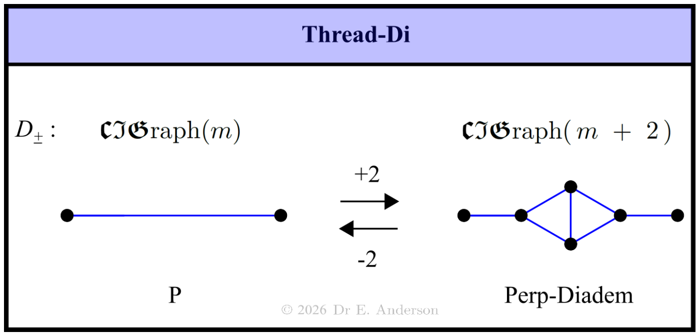


Figure 21:

Structure 1 View

$$\text{Tet} \ = \ \text{DiRing} \ = \ <1> \ :$$

the diamond necklace [64] with 1 diamond (Fig 14.b). And

$$\text{di-Di} \ = \ <2> \ :$$

the diamond necklace with 2 diamonds. In our little truncation to $m \leq 4$, placing an order on the CI is very straightforward: the *thread-diamond order*. By which our diamond necklace picks up 1 further diamond at each stage (Fig 21). With only 1 such move realized for $m \leq 4$: Tet to di-Di . Fig 22 then provides Di-centred and systematic diamond-necklace diamond-count notational upgrades for the task at hand.

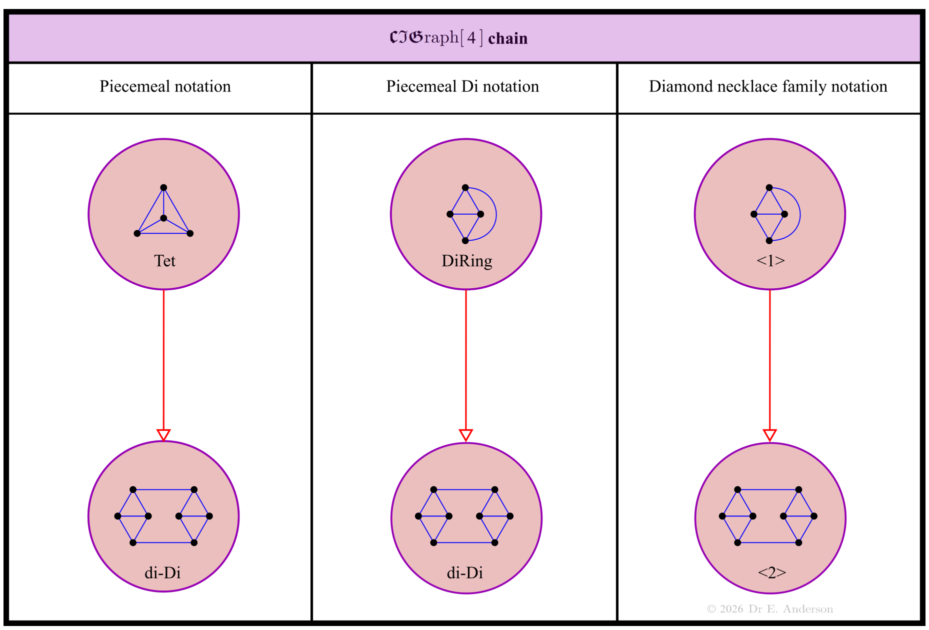


Figure 22:

Structure 2 Thus our cumulative orders on CI are the point for $m = 2$ and 3 . And then the 2-path P for $m = 4$. We exhibit this for various meaningful presentations of the CI's in Fig 22.

Structure 3 Serially continuing with this thread-Di operation gives the *diamond necklaces* that were Linear-Algebraically studied in [64] and yet not identified as CIs.

6.4 Placing a connected order on the connected cubic graphs

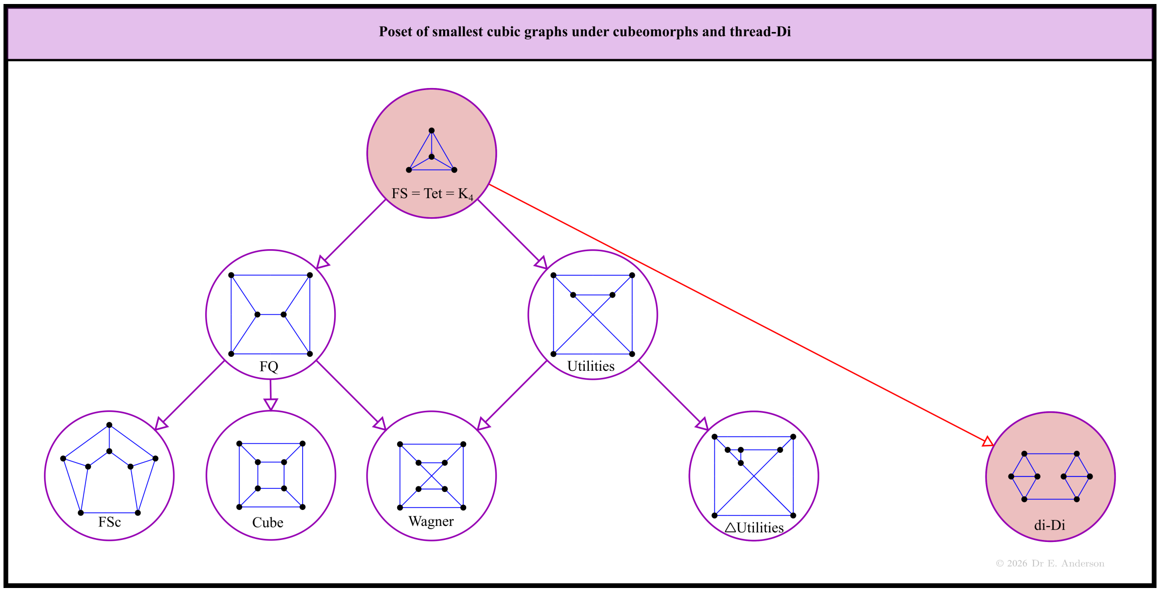


Figure 23:

Structure 1 Combined with the previous, this gives the poset in Fig 23. Whose underlying graph is now the first homeomorph top-left in Fig 24 of the top-right Sunlet + Leaf graph.

Remark 1 We consequently adjust Fig 20's DI analysis to Fig 24's.

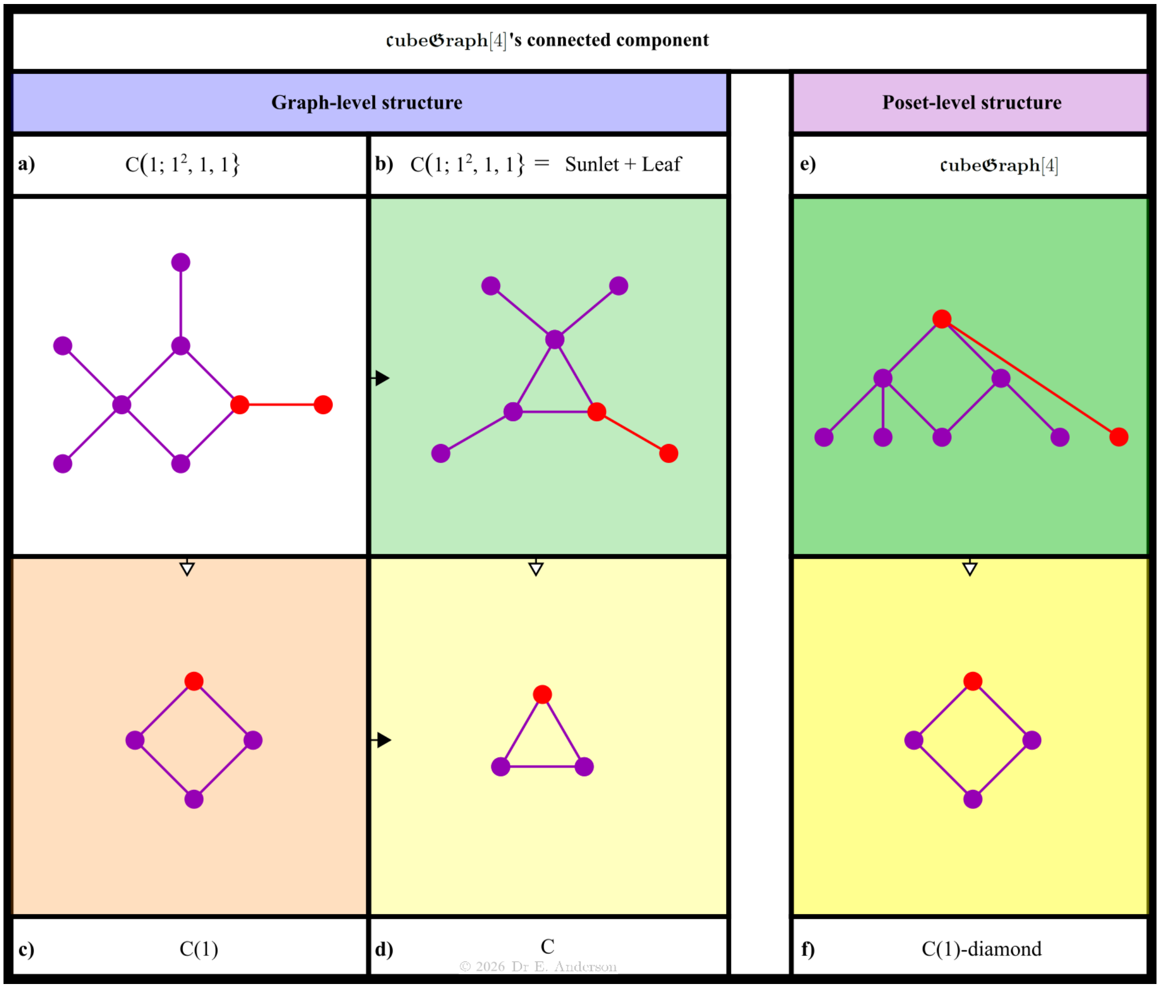


Figure 24:

7 Conclusion and motivating pointers

7.1 Classificatory key

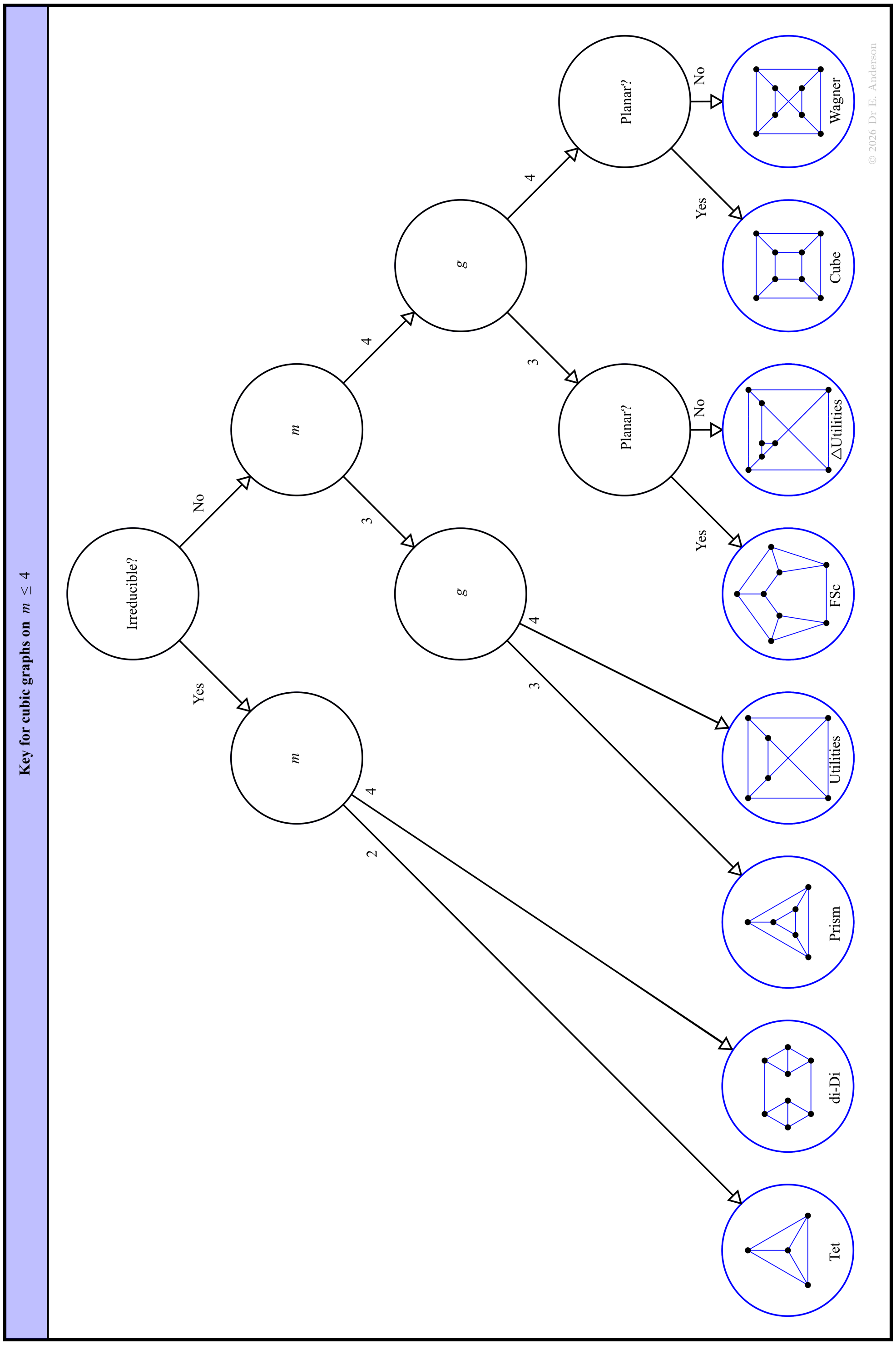


Figure 25:

Remark 1 Irreducibility, m, g , and planarity suffice to determine which of the current Article's graphs one has been dealt a presentation of.

7.2 Tabulating many aliases and notations for the 8 smallest connected cubic graphs

Compendium of basic names and notations for the smallest cubic graphs

Extremization: Graph Theory roots and Order-Theoretic bottoms			Graph Theory								Generalized Petersen graphs		
			Basic				Framed tree		A lone cone!	Δ of some smaller well-known graph	Absolute [Watkins]	Relative truer-name [Anderson and Sánchez]	
Order Theory	Symmetry	Graph Theory	Complete	Girth	Nonplanarity as quantified by crossing number	Other	Plain	of-Claws			Adapted variables built via Coxeter on Schläfli	Prism (-centred)	Petersen (-centred)
MCG: cibic graph	MCE_3G		K_4	3	0		FS	FD	C(C)	ΔP			
	MCE_2G			3	0		FQ	FP		ΔTet	GPG(3,1)	Prism	Pet(-1,-2)
MCGG: girth 4, prerequisite for supporting posets		MBCG: bipartite. MNCG: nonplanar.	$K_{3,3}$	4	1	Utilites							
				4	0						GPG(4,1)	Prism(1)	Pet(-1,-1)
				4	1	Wagner							
				3	0		FSc	FP_3					
				3	1					Δ Utilities			
				3	0								

Figure 26:

Remark 1 Let us begin this concluding venture with basic Graph Theory in blue. This includes emphasizing that nonplanarity and girth (Appendix A.6) are useful diagnostics in identifying myriad presentations of these graphs.

Compendium of basic names and notations for the smallest cubic graphs (continued, upwards!)

Geometry				Möbius ladders			Structural			
Platonic solid	Polyhedron		Absolute [graph-order]	Relative truer-name [Anderson and Sánchez]			Diamond composites		(Ir)reducibility	
	Plain	Excess		Minimum even cycle centred	Topological/ Nonplanar (-centred)	Bundle/ Circulant (-centred)	Piecemeal	Necklace	CIs	Cubeomorphs thereabouts
Tet			ML_4	ML	MLT(-1)	MLB(-2)	DiRing	<1>	CI	
	$Prism_3$	Prism								CI[1]
			ML_6	ML(1)	MLT	MLB(-1)				CI[0; 1]
Cube	$Prism_4$	Prism(1)								
	Twisted-Cube		ML_8	ML(2)	MLT(1)	MLB				
							di-Di	<2>	CI(1)	

Figure 27:

Remark 2 We evolve downwards into Extremization and Order Theory (purple). By noting significant minimum properties of some of our graphs. Which are then to serve as roots in graph models for those subarenas of cubic graphs that possess said properties. And as bottoms in order models for the same. Among these, girth $g \geq 4$ is furthermore a prerequisite [117, 148] for a graph to support any posets: the structural cornerstones of Order Thery.

Remark 3 We also evolve upwards into more advanced Graph Theory features (sky blue). And into the General Petersen graph notion (aquamarine), which is a bridge into Fig 27's Geometrical continuation of Fig 26.

Remark 4 The second figure is to be read from the bottom up. It provides basic Geometrical names and notations in green. Then Möbius ladder notions and notations in yellow: somewhat of a bridge into basic Topology.

Remark 5 It concludes with structural notions and notations in red. Built around the current Article's central notions of cubeomorph and cubeomorph-irreducible (CI). In the manner of first and second cubeomorphs of our first CI.

Remark 6 Throughout, notions, names and notations that we particularly recommend are highlighted in fireopal.

7.3 Some of the many interesting larger cubic graphs

Remark 1 Such material already started to be presented in Subsec 4.4 on the Petersen and generalized Petersen graphs. And in Subsec 4.3 on Möbius ladder graphs.

Pointer 5 ⟨ 4 ⟩. The *Frucht graph* [19, 42] is a cubic graph on $m = 6$. It shares honours as minimum *asymmetric cubic graph.* While being the only such that is both planar and a framed tree.

Pointer 6 ⟨ 6 ⟩. The *zero-symmetric cubic graphs* are a distinct notion. Corresponding to [50]'s last class – Z –, in which the edges at each vertex are all inequivalent. The minimum Z-class cubic graph requires $m = 9$ vertices and features as the cover figure on this venerable source.

Pointer 7 ⟨ 6 ⟩. The *Heawood graph* [6, 58, 101, 67] is a cubic graph on $m = 7$ vertices. Which arises in Graph Colouring as a minimum realization of the 7-colouring theorem (Appendix A.5) for graphs embedded on the torus.

Pointer 8 ⟨ 3-4 ⟩. The *Fano*, *Pappus* and *Desargues incidence graphs* [101, 67, 83, 84, 95] of Projective Geometry are cubic. The first of these so happens to rediscover [83] the Heawood graph within this a priori conceptually distinct study... The other two are on $m = 9$ and 10 respectively.

Pointer 9 ⟨ 2-3 ⟩ The *dodecahedron graph* [148] realized by the dodecahedron Platonic solid is a cubic graph on $m = 10$. Corresponding to its 20 vertices, as can be read off the nominative 20 faces of its dual [108] Platonic solid: the icosahedron.

Pointer 10 ⟨ 4-6 ⟩ Among many other properties, the Petersen and Heawood graphs are the cages of girth 5 and 6 ⟨ 4 ⟩. Where a *cage* [17, 27, 52, 68] is an extremal graph concept: the smallest r-regular graph supporting a given girth.

Exercise 12 Use the current Article's material to establish that the Petersen graph is *a such* cage.

The remaining known cubic cages ⟨ 8 ⟩ are as follows. The 7-cage is the (*Sachs*-)*McGee* graph [24] on $m = 12$ ⟨ 6-7 ⟩.. The 8-cage is *Tutte–Coxeter graph* [17, 21, 22] on $m = 15$. Which is also Projectively-significant as the *Cremona–Richmond incidence graph* [101, 67] ⟨ 6 ⟩.

⟨ 8 ⟩ 18 9-cages on $m = 29$ [57]. The *Balaban* 10- and 11-*cages* on $m = 35$ and $m = 56$ vertices respectively [36, 37]. The first shares this honour with 2 further cubic graphs [49]. And the *Benson* 12-cage [26] on 63 vertices.

Open Question Find subsequent cubic cages!

Pointer 11 The generalized Petersen graphs introduced in Subsec 4.2 are cubic graphs.

The smallest such in excess of the Petersen graph itself is the *Dürer graph* on $m = 6$ ⟨ 4 ⟩. This is named after the *Dürer solid*, which features in a famous engraving [97]. It is the abovementioned 'star of David' case.

Some famous examples are as follows. The *Möbius—Kantor graph* [67] on $m = 8$ ⟨ 6 ⟩. Which is Projectively significant. And in a sense solves *Móbius' mutually-inscribed polygon problem* in the case of quadrilaterals. The sense in question is that this is not possible in $\mathbb{R}^2$, but Kantor found it to be possible in the complex-projective plane $\mathbb{CP}^2$.

Among the previously mentioned famous cubic graphs, the dodecahaedron and Desargues-incidence graphs. are furthermore generalized Petersen.

♠ ◇ ♣

Pointer 12 The Petersen graph is also the smallest snark [40, 155, 28, 66]. I.e. connected cubic graph that cannot be 3-edge coloured. By a looser and less used definition, the next smallest is as follows. The *Tietze graph* [8] on $m = 6$. Which is a first cubeomorph – and $\triangle$ – of the Petersen graph. It is however the 2 *Blanuša snarks* [15] on $m = 9$ vertices that are next smallest by the most widely used definition. Which forbids triangles, and thus enforces girth $g \geq 4$.

Known snarks used to be very rare, but firstly Isaacs unveiled 3 infinite families of snarks [39, 41]. 2 were original to him. Comprising generalization of the Blanuša's and the *flower snark family* extension of the Tietze graph. While he credited the third family to Loupekine.

Among the flowers, the first sincere snark is on $m = 10$. The first Loupekine snark is on $m = 11$. The flower family expands on a single copy of Petersen. The Blanusa family uses 2. The first Loupekine snark uses 3 Petersen blocks, with larger members using subsequent odd numbers of these. There are moreover ambiguities in the Loupekine composition. fusing in 3 copies of this graph.

Tutte's Conjecture [28] is that all snarks contain the Petersen graph as a minor.

While Wikipedia [89] currently lists just four 'standalone snarks', these are largely only of historical interest. For instance, the smallest of these four – Isaacs' double-star snark – is now known to be [66] outnumbered on $m = 15$ vertices by over 100000 snarks not belonging to any known families.

Pointer 13 Finally the following serve as increasingly tight counterexamples to the following.

Tait's conjecture [3]. is that planar 3-connected cubic graphs are Hamiltonian.

The *Tutte graph* on $m = 23$ [16, 34] $\langle 4 \rangle$. The *Grinberg graphs* on $m = 23, 22$ and 21 [30, 44] $\langle 5 \rangle$. The *Faulkner–Younger graphs* on $m = 22$ and 21 [38] $\langle 6 \rangle$. Each 21, 22 pair here is internally related by a single cubeomorph [146]. The *Barnette–Bosák–Lederberg graph* on $m = 19$ [25] $\langle 6 \rangle$.. And its 5 close relatives as documented in [55] $\langle 7 \rangle$, with which it shares the honour of provenly-minimum counterexample.

End-Remark 1 At least over the next decade, our growing Encyclopaedia [85] shall have room to study the above Pointers' graphs in detail for $m \leq 23$. Including giving more detailed Graph Drawing and Visualization [71] presentations than previous treatments'.

7.4 CI operations in general

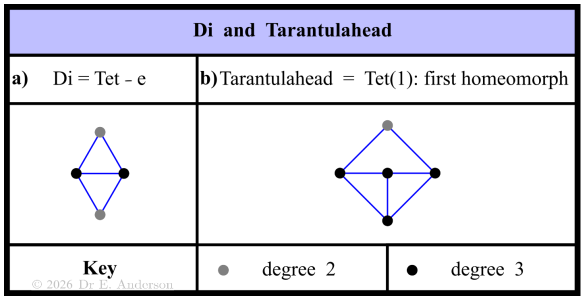


Figure 28:

Structure 1 The bigger-N complications with intrinsically generating CIs for higher N are as follows [65, 66, 146].

1) The thread-Di operation D considered so far can however be applied to form networks more general than cycles.

2) The following composite operation D' (Fig 29.a).ii) is also available.

+D.1) Homeomorph a vertex into each of 2 edges.

+D.2) Cross-weld a Di across the gap.

3) The minimum cubic bridge component [93] – Tarantulahead: Fig 28.b) – also enters play. Via the following third composite operation append-Tarantulahead T (Fig 29.b).

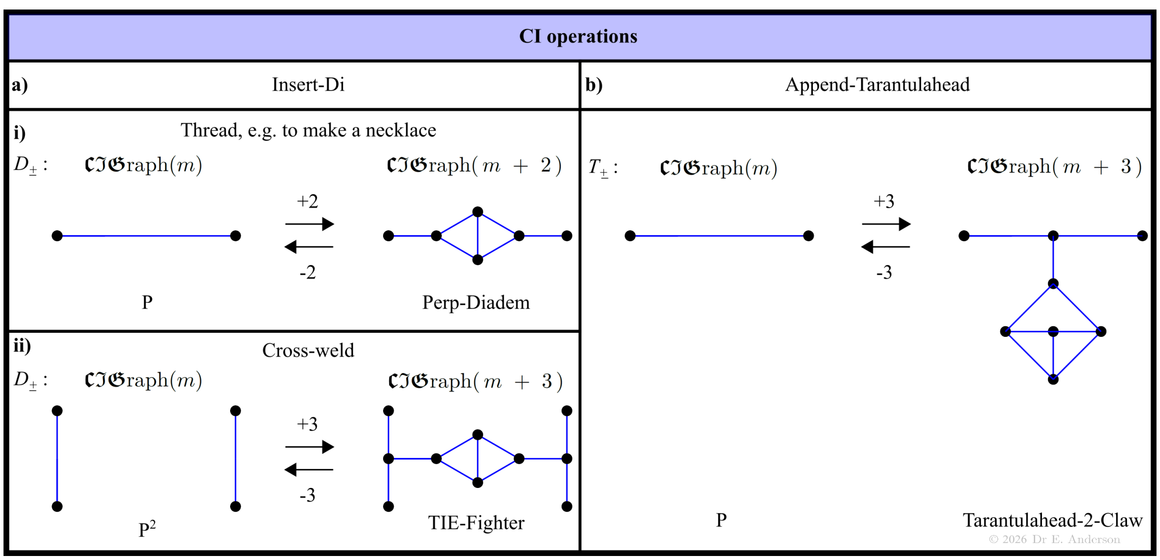


Figure 29:

+T.1) Homeomorph a vertex into an edge.

+T.2) Join this vertex to a Tarantulahead.

Naming Remark 1 On the one hand, Fig 14.a) Di-centrically renamed Tet as DiRing , which Sec 13 then interpreted as $<1>$: the single-diamond necklace. On bipa hand, the current Section Tet-centrically renames Di as Tet $- \ e$. This is a member of a well-known family, $\mathrm{K}_N - e$. I.e. the secondmost dense simple graphs on N vertices. Two advantages of this Tet-centric perspective are as follows.

Firstly, rather more conceptualizations lead to Tet rather than to Di , with row 1 of Figs 26-27 providing a fair few. These include that Tet is the smallest CI.

Secondly, in building further CIs, it is rather neat that one's incipient building blocks are Tet and two of its commonest minimum variants. Namely the unique outcome of removing an edge from Tet : Di = Tet $- \ e$. And the unique first homeomorph of Tet : Tarantulahead . Both of these uniquenesses are rooted in Tet's maximum symmetry. Which does not permit any distinction in first symmetry breakings of these types. They are means by which 2 , and 1 vertices respectively can be introduced whose degree is 1 short of attaining cubic regularity. Liberating these vertices to the application of joining operations, yielding the current Section's generative approach [65] to CIs.

Remark 1 See Fig 30 for nicer presentations of the 2 new moves' end-states.

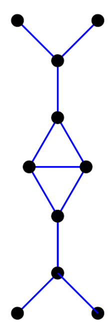
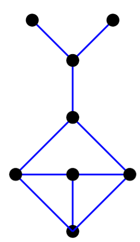
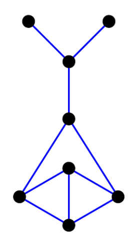
Nicer presentations of CI operation outputs		
TIE-Fighter = Claw-Di-Claw	Tarantulahead-2-Claw	
		
Unit-distance, uniform-angle presentation	Uniform-angle presentation	Diamond presentation

Figure 30:

Pointer 14 This provides further motivation for our subsequent study [93, 96] of $\mathfrak{C}$ubic$\mathfrak{G}$raph[5] and $\mathfrak{C}$ubic$\mathfrak{G}$raph[6] . And of the arena of CI's themselves

$$\mathfrak{C}\mathfrak{I}\mathfrak{G}\text{raph}[m]$$

somewhat beyond these values of N . For at least for small N , CIs do not quickly grow in number [65]. Nor does the acquisition of increasingly general Order-Theoretic features for the corresponding arena. At least over the next decade, our growing Encyclopaedia [85] shall also have room [94] to study CIs and their arenas on $m \ \leq \ 12$.

Pointer 15 Let us also leave the systematic notations for cubic graphs that Coxeter and Frucht set up with various coauthors [25, 50] to the above sequels.

7.5 Tet as a citizen of Kallista

Remark 1 A *citizen of Kallista* [146, 148, 152, 85] is a Mathematical object arising from multiple a priori conceptually unrelated lines of thought. These consequently tend to have many names. And the lines of thought are rather prone to diverging as regards what next-most object each picks out. I.e. citizens of Kallista tend to have multiple heirs that split up the citizen's many aspects and properties between them.

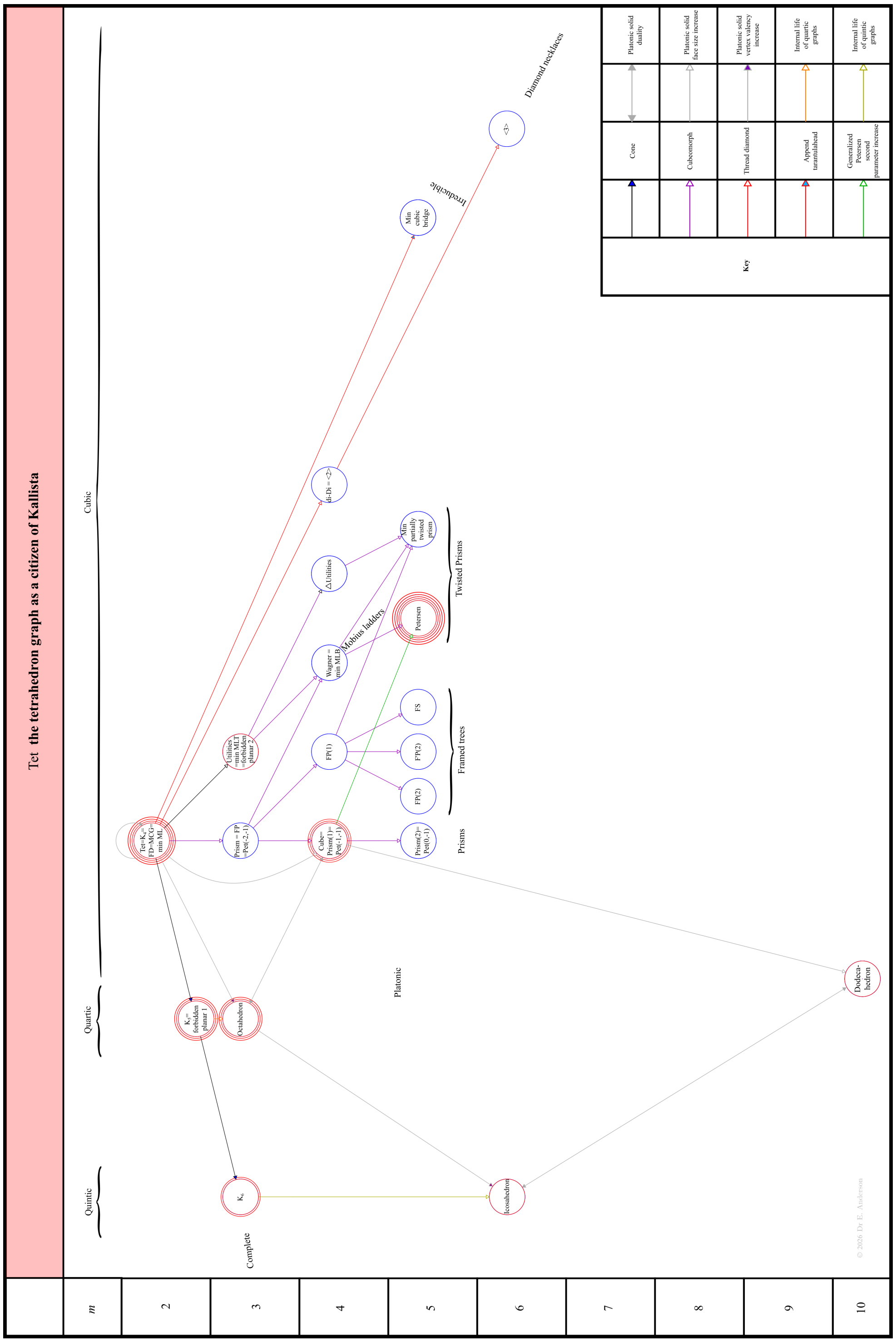


Figure 31:

Remark 2 Tet is a citizen of Kallista, as evidenced by the many notions, and consequently names, entering its row in Figs 26 and 27. To celebrate, the Author drew out Fig 31 and presented it to S. Sánchez. Twenty minutes of vigorous penstrokes later, S. Sánchez replied with 32.

One salient difference stems from isolated out the Platonic graphs to one side, on account of their containing quartic and quintic exemplars. The Author's version put the cube outermost among the cubic graphs on account of its also being Platonic. But S. Sánchez chose instead to put the framed trees on the outside, picking out the Platonic pattern in ivory instead. Three benefits from this change in layout are as follows.

Preliminarily the 2-parameter prisms are given continuous representation. And the trees-of-claws are cast as 'the second course's cutlery', with framed such as 'fork' to Tarantula-leaved such as 'knife'.

Most of all, this permits centred presentation of the Petersen graph Pet : an even more exhalted citizen of Kallista! With its precursor 'virus head' picked out symmetrically. From which a vast number of 'tail fibre' hairs emerge (in sequel figures!) to 'infect almost all of Graph Theory' with marvellous consequences! Symmetrically is here modulo firstly using the adapted variable m as height function. And secondly isolating an impressive number of series on labelled fixed-gradient runnels. The first is a 'clan hallmark' – Order Theorists! – while the second is a distinctive personal signature. All in all, when an eventual greater heir to lesser sires emerges, it should be placed to dominate each sire's Kallista diagram.

See Fig 33 for the 'Petersen virus' diagram. To Fig 34 for the 'Platonic home with satellite dish' looped arrow-graph arena of 3-d convex Platonic solids that the Author's own diagram was designed to isolate.

Finally S. Sánchez' 'first-course cutlery' is the super-cubic Platonic 'fork' to the diamond necklace 'knife'. Within Fig 32, this amounts to splitting out 1- from 2-connected CIs. Alongside identification of top and bottom elements for fixed m . Albeit only available $0 \pmod 2$ and $2 \pmod 3$ respectively.

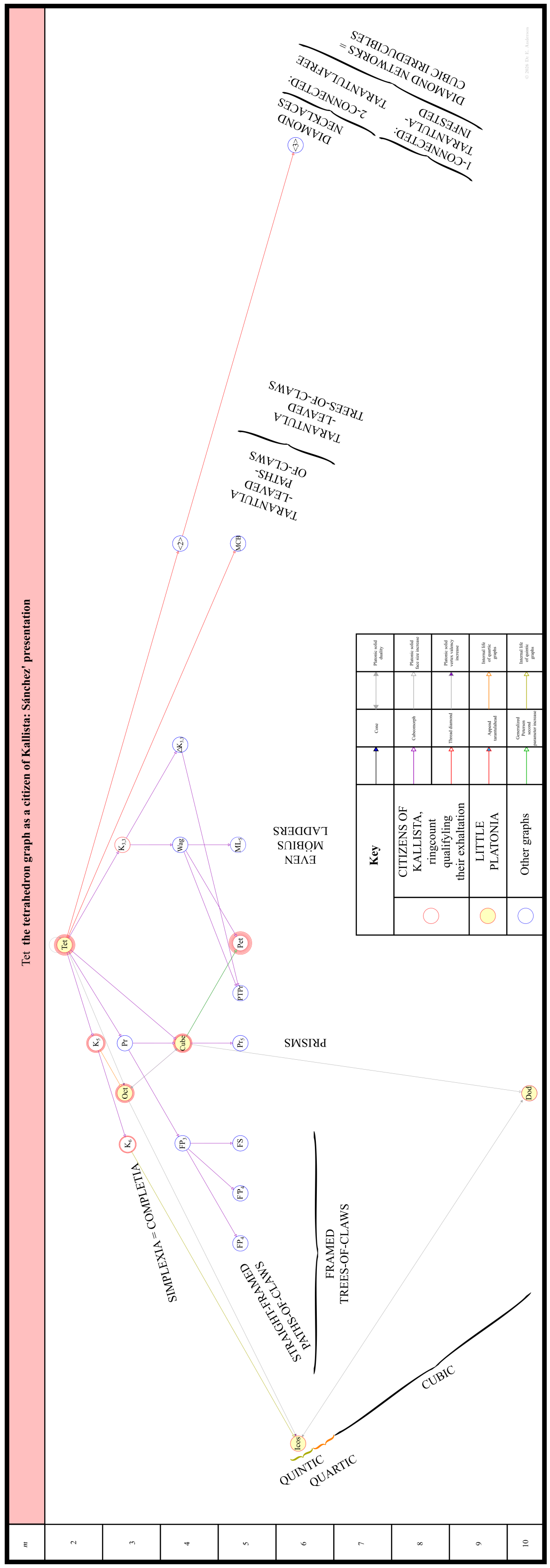

Tet the tetrahedron graph as a citizen of Kallista: Sánchez' presentation
m
2
3
4
5
6
7
8
9
10
SIMPLEXIA = COMPLETIA
STRAIGHT-FRAMED PATHS-OF-CLAWS
FRAMED TREES-OF-CLAWS
PRISMS
EVEN MÖBIUS LADDERS
TARANTULA-LEAVED PATHS-OF-CLAWS
TARANTULA-LEAVED TREES-OF-CLAWS
DIAMOND NECKLACES
2-CONNECTED: TARANTULAFREE
1-CONNECTED: TARANTULA-INFESTED
DIAMOND NETWORKS = CUBIC IRREDUCIBLES
CUBIC
QUARTIC
QUINTIC
Key
CITIZENS OF KALLISTA, ringcount qualifying their exhaltation
LITTLE PLATONIA
Other graphs

Figure 32:

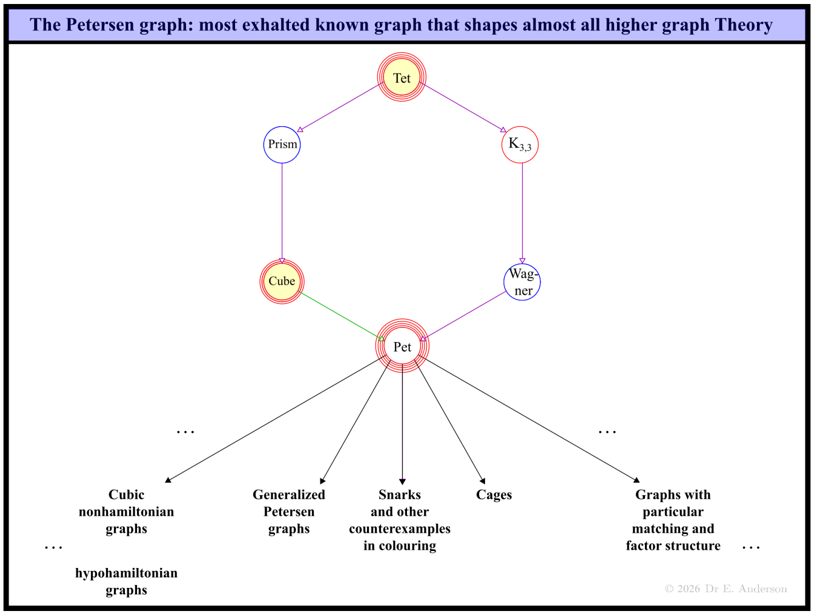


Figure 33:

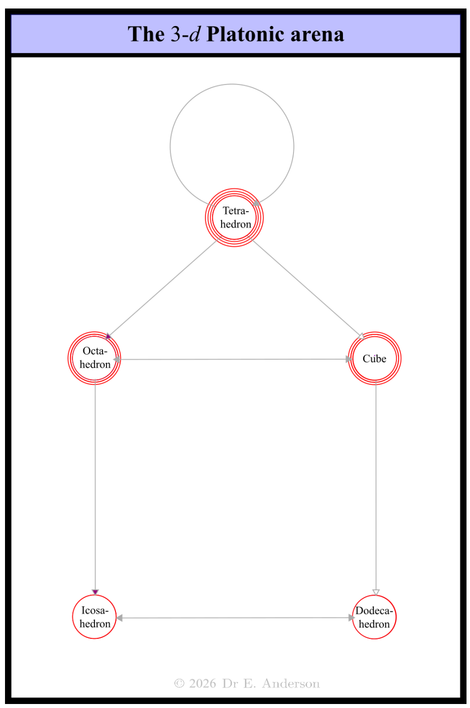


Figure 34:

Pointer 16 There are further Platonic solids in dimension ≥ 4. This is part of why the $K_N = n$)-simplex for $N := N - 1$ carry more red rings in our figures. Coxeter has however already magnificently studied these [100, 102]. So we elect to leave them out of our figures, where they would dominate the left flank with much depth in vertex-number.

[**Motivation 1** Instead, we have now set the stage in which to perform three small tasks and a large one. The three small tasks are to sweep out the middle of our figure to completion for $m = 5$ and then 6 [93, 96]. With deeper fire down the rightmost wedge's CIs, systematically down to

$m = 12$ and with a select few constructions beyond. The large task is of course to study the Petersen graph and its many heirs. This gives a concrete focus as to which cubic graphs to consider in the swiftly expanding repertoire from $m = 7$ to 17. See the 'Petersen virus' Figure for some of the main known topics in this regard!

Motivation 2 It is a distinct citizen of Kallista – the *Tutte fragment* [16] – that bounds the above-mentioned even larger and yet just planar examples on $m = 23$ working backwards down to $m = 19$.

Motivation 3 Other notable examples of citizens of Kallista include *Hopf's little map* [122, 126]. The *Catalan numbers* [157, 158]. The *power sets* [129, 148]; the cube graph Cube is tied to one of these, accounting for some of its red ringing... And the *Fano configuration* and its associated graphs [101, 82, 83], with for instance projective, matroid, design and simple-group heirs.

7.6 More general Arena-Theoretic pointers

Remark 1 Topology has two 'easy ends' which are substantially different from each other, with each offering a largely distinct collection of insights. The 'discrete end', covering for instance Combinatorial objects such as graphs and orders. And the 'continuum' end, with such as the real line $\mathbb{R}$ – seat of Analysis – flat space and manifolds.

Remark 2 Next consider the Arenization problem. Each subject has many object types and thus engenders many arenas. And each of these has very good chances of being a gateway into an a priori 'unrelated' field of study.

Pointers 16-17 Some accurate fixed-point results concerning arenas are as follows.

Gromov [104, 109] established that the arena of metric spaces is itself a metric space: carrying the Gromov–Hausdorff metric.

S. Sánchez established that the arena of posets is itself a poset (reported in [148]).

These provide two small 'lamp-posts'. I.e. here instances in which arenization does not kick one out of one's a priori subject matter if that so happens to be the easy continuum end's metric spaces. Or the easy discrete end's posets. One then has a rather better chance of 'finding one's keys' if one is a single-subject player...

Pointer 18 An approximately 'trapped-subject' observation is as follows [148, 85]. Combinatorics' arenas can be passably modelled using Graph Theory. And can be rather more signifcantly modelled by Order Theory. This is 'trapping' since within the conventional perspective, Graph Theory and Order Theory are themselves branches of Combinatorics... Compare Topology and Dynamical Systems' 'trapping set' [124] and GR's 'trapped surface' [103].

Pointer 19 Among other matters, we have however here established ([146], in 2018) that Graph Drawing and Visualization's [71] own arenas are not necessarily confined to the Combinatorial field of study.

A Various supporting pieces of Graph Theory

A.1 Complete graphs

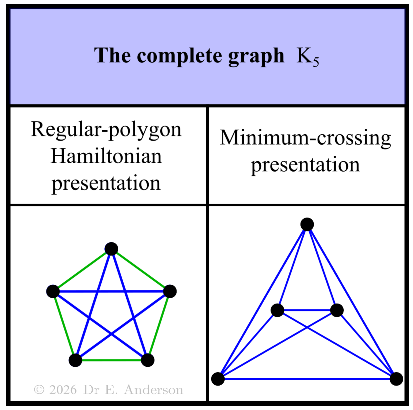


Figure 35:

Definition 1 A simple graph is *complete* if edges join every one of its vertices to every other. It is standard to denote the N-vertex such by K_N .

Example 1

$$K_1 = D_1 = D = Pt .$$

Where D denotes discrete graph and Pt its graph-free conceptualization as just a point.

Example 2

$$K_2 = P_2 = P .$$

Where P stands for path.

Example 3

$$K_3 = C_3 = C .$$

Where C stands for cycle. The suffix-less versions so far constitute an excess notation [148] and are suggested [81] for use in the HI context.

Example 4 K_4 is the smallest cubic graph, and thus the subject of Sec 2.

Example 5 K_5 is depicted in Fig 35. This is the smallest quartic graph: of regularity $r = 4$.

A.2 Cone graphs

Definition 1 The *cone* C(G) *over a graph* G is a graph consisting of the following. G alongside one extra vertex o incident to all of G's vertices. Let us also call o the *cone point* alias *apex* of C(G) .

Remark 1 In a complete graph, every vertex serves as a cone point.

A.3 Bipartite graphs

Definition 1 A simple graph is *bipartite* if its vertices can be split into 2 sets. Neither of which possesses any self alias internal edges. It is *complete bipartite* if all of the following are realized. The mutual alias cross edges between the 2 vertex sets.

Notational Remark 1 The general complete bipartite graph is denoted by $K_{p,q}$. For parts of size p and q .

Definition 2 If $p = q$, the complete bipartite graph is *homogeneous*. While if $p \neq q$, it is *heterogeneous*.

♠ ◇ ♣

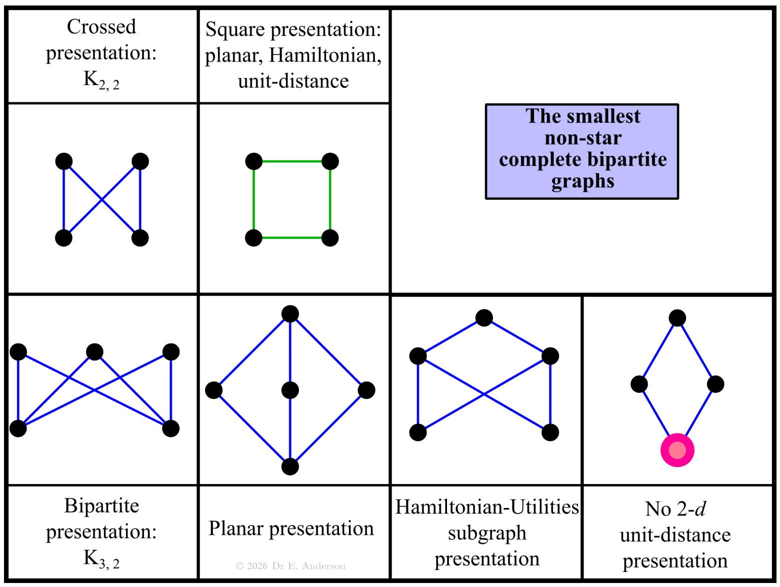


Figure 36:

Example 0

$$K_{p,1} = S_p :$$

the p-pointed star graphs.

Example 1

$$K_{2,2} = C_4 = C(1) .$$

I.e. the 4-cycle graph as viewed in crossed presentation (row 1 of Fig 36). For which C(1) is an excess notation. Based on 4 being 1 larger than 3 : the minimum cycle size supported by simple graphs. This is the smallest homogeneous complete bipartite graph that is not just a star. Or indeed just a path, since

$$K_{1,1} = S_1 = P_2 = P .$$

Example 2 Row 2 of Subfig 36's $K_{3,2}$ is the smallest heterogeneous complete bipartite graph that is not just a star.

Example 3 $K_{3,3}$ is one of the current Article's cubic graphs, alias the Utilities graph.

Remark 1 Suppose that the utilities problem is posed for whichever of just 1 house and just 1 utility. Then the ensuing stars are trivially planar. For 2 of each likewise, now by uncrossing the

4-cycle as per row 1 column 2. And for 3 of the one and 2 of the other. Now by embeddings such as the $C_2{}^2$-symmetric one in row 2 column 2.

Remark 2 We also give a construct in the last entry of row 2 by which $\mathrm{K}_{3,2}$ does not admit a unit-distance presentation in 2-d . For 2 rhombi with 2 shared sides must be congruent. Throwing 1 unshared vertex from each into the same position, which we pick out in fuchsia and rose. Thus breaching general position at the level of vertices.

A.4 Arenas of presentations of a graph, and corresponding notions of crossing

Definition 1 There are various versions of arena of presentations of a given graph G . For instance, the arena of embeddings $Emb(G)$ of graph G in $\mathbb{R}^2$,

$$\mathfrak{E}\mathrm{mb}_2(\mathrm{G})$$

Which are less general than embeddings

$$\mathfrak{E}\mathrm{mb}_3(\mathrm{G})$$

in $\mathbb{R}^3$. And so on, including, in the present Article, in the Möbius strip.

Definition 2 A *crossing* in the embedding of a graph is an intersection of 2 edges at other than a vertex.

Structure 1 Making use of a small perturbation if necessary, no further edges are to pass through a crossing point. This is a simple example of *general-position argument*. More generally, Topology is replete with general-position arguments.

Definition 3 Let

$$cr(\, Emb(\mathrm{G})\,) \;:=\; \#\,(\,\text{crossings in } \; Emb(\mathrm{G})\,)\;.$$

The *crossing number* [62, 70, 72] of a graph G is then

$$Cr(\mathrm{G}) \;:=\; \stackrel{\text{min}}{Emb(\mathrm{G}) \,\in\, \mathfrak{E}\mathrm{mb}_2(\mathrm{G})} \;\; cr(\;\; Emb\;(\;\mathrm{G}\;)\;\;)\;\;. \tag{6}$$

A.5 Some planar graph theorems

Theorem 1 (Kuratowski's Planarity Theorem) [11, 48, 58, 77]. A graph is planar iff it does not contain a homeomorph of the complete graph K_5 (Subfig c) or of the complete bipartite graph $K_{3,3}$.

Remark 1 See Fig 35 for a 1-crossing presentation for K_5 . And Fig 6.e) or f) for a 1-crossing presentation of $K_{3,3}$.

Theorem 2 (Wagner–Fáry–Stein Rectilinear-Representation Theorem) [13, 18, 20, 48, 58]. Every planar graph admits a rectilinear edge presentation.

Theorem 3 (Steinitz Representation Theorem 1916) [10, 107]. The convex polyhedra correspond to the planar 3-connected cubic graphs.

Structure 1 Where 3-*connected* means that 3 vertices need to be removed in order to disconnect the graph. Which is a simple notion of network robustness.

Remark 1 This provides a way in which Euler's formula (Sec 1.1) can be reinterpreted in terms of planar graphs.

A.6 Graph Colouring

Structure 1 An n**-Colouring Theorem** refers to being able to colour in 'connected countries in a geographical map' in no more than n colours. Most usually, this refers to a geographical map in the plane. For which the strongest possible such theorem, which is also the hardest to prove in the plane, is Appel and Haken's 4-*colour theorem* [45, 46].[2] For all that the current Review also refers to the 7-colour theorem on the torus. And [8]'s title to the 6-colour theorem on the Möbius strip.

A.7 Some simple crossing number inequalities

Definition 1 The *girth* of a graph G is

$$g := \min_{\text{cycles } S \in G} \text{length}(S) . \tag{7}$$

Exercise A.1 a) Show that for a planar graph,

$$E \leq E_{\text{max-planar}} = 3N - 6 = 3(N-2) . \tag{8}$$

b) Show that

$$Cr \geq E - E_{\text{max-planar}} = E - 3(N-2) . \tag{9}$$

c) But that for a graph of girth g ,

$$Cr \geq E - \frac{g}{g-2}(N-2) . \tag{10}$$

Exercise A.2 a) Establish that $K_{3,3}$ and K_5 have crossing number 1 .

b) Show that this method does not however place any useful bounds on the crossing number of the current Article's main 2 other nonplanar graphs. I.e. Wagner and △Utilities .

[Here we proceed instead by picking out a $K_{3,3}$ subgraph and then wheeling out Kuratowski's Theorem...]

Acknowledgments I thank S. Sánchez for discussions, and A. Ford for proofreading an earlier draft. I also thank the Applied Combinatorics and Topology Discussion Group's members.

[2] See rather [33] or [62] if you are a beginner seeking an introductory reference!

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