Rohomeo and Rofeulliette.

And other easy variants of Double-Irreducible classes

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Abstract

We extend the recently revealed double irreducibles classification theorem from simple graphs to the following modelling situations. Rooted graphs, digraphs and rooted digraphs. And where double irreducibles are both homeomorph irreducibles and defoliation irreducibles. In the process, we extend these Topologically-significant notions to our three further modelling situations. For instance, rohomeomorph and rofeulliette are the root-preserving variants... We leave the lion's share of our material on this – minimum examples and their extent of nonuniqueness – as Exercises for around two years.

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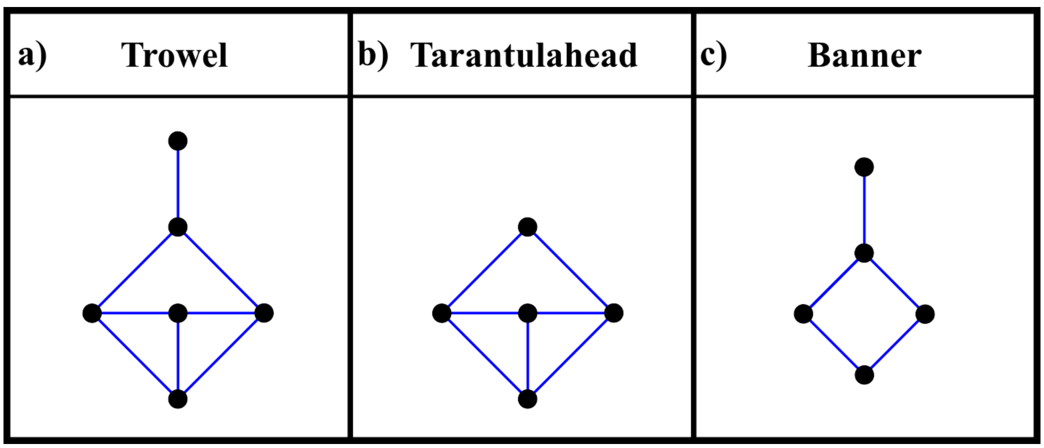

Figure 1:

1 Introduction

We extend the recently revealed double irreducibles classification theorem [10, 12] from simple graphs [2, 7, 9] to the following modelling situations. Rooted graphs [1], digraphs [5] and rooted digraphs (Sec 2). And where double irreducibles are both homeomorph irreducibles [3, 11] and defoliation irreducibles [11].

In the process (Sec 3), we extend these Topologically-significant notions to our three further modelling situations. For instance, rohomeomorph and rofeulliette are the root-preserving variants...

The original Theorem and our three counterparts are outlined in Sec 4, with our proof largely resting upon that in [12] We leave the lion's share of our material on this – minimum examples and their extent of nonuniqueness – as Exercises for around two years.

2 Modelling assumptions

Definition S A *simple graph* [2, 9], that are consists of a set of vertices. Alongside a set of edges, with at most 1 edge between each distinct pair of vertices.

Definition R *Rooted graphs* [1] additionally assign a special role to a single vertex called the *root*.

Definition D *Digraphs* [5] assign instead a single arrow to each edge. 'Di' stands here for *directed*.

Definition RD *Rooted digraphs* concurrently make both of the previous assignments.

Remark 1 None of these models contain multi-edges between vertices. Or loops: edges from a vertex to itself. See Part V of the freely-available [11] for an introductory text that makes active use of all of the current Subsection's notions.

3 Topological modelling

3.1 The simple graphs

Definition S.0 The *degree* of a vertex in a simple graph is the number of edges emanating from it. Chemists (and a few others) call this *valency*.

Definition S.1 *Homeomorphing* [11, 8] serially removes all degree-2 vertices from a simple graph. Excepting cases in which this would create a multi-edge. The end-product of this is a *homeomorph irreducible* (*HI*) graph [3, 11].

Naming Remark 1 Homeomorphing is alias *subdividing* or *expanding* [4] in one direction. And *smoothing* or *series reduction* in the other. While [6] call HI trees *topological trees*.

Motivation 1 Unaffected Topological properties include those involving forbidden subgraphs modulo homeomorphing. Such as *planarity* [7]: forbidden subgraphs K_5 and $\mathrm{K}_{3,3}$. Or *outerplanarity* [7, 11] : forbidden subgraphs K_4 and $\mathrm{K}_{3,2}$.

And some notions of connectivity [7, 11]. In which the difference between connection via an edge, and connection via an edge, then a vertex and then another edge, do not register. Such as *k-connectivity*. And the qualifiers that a graph contains a cut-edge or a cut-vertex. Though clearly not quantifiers of how many cut-edges or cut-vertices the graph possesses.

Definition S.2 *Defoliation* [11] serially removes all leaves alias degree-1 vertices. This prunes off all side-chains, and, more generally, all side-trees. The end-product of this is a *foliation irreducible* (*FI*) alias *cycle system* [12]. Observe that if the incipient graph is a tree, then a single vertex survives this process: a terminal state since no leaves remain therein. We finally also call the vertex adjacent to a leaf a *twig*.

Motivation 2 Topological graph properties unaffected by defoliation include, again, planarity and outerplanarity. For side-trees can always be deformed so as to avoid incurring any crossings. And a smaller subset of notions of connectivity. For 'branching points' – degree ≥ 3 vertices – contribute to some notions of connectivity. And pruning off side-trees is indeed capable of decommissioning some of these...

Definition S.3 A *double irreducible* (*DI*) [12] is the end-product of applying both of the preceding. Including letting homeomorphing loose on any new degree-2 vertices resulting from defoliation.

Motivation 3 All three of the above notions of irreducible are useful in structurally analyzing graphs and arenas of graphs [11] and in naming graphs as well [16]. The DI version of this is new [12, 13, 14]

3.2 Rooted graphs

Definition R.1 A rooted graph *rohomeomorph* H_{R} is a homeomorphing operation that is not however allowed to conflate the root with any non-root vertex.

Definition R.2 A rooted graph *rofeuilliette* F_{R} is a defoliation operation that is not however allowed to conflate the root with any non-root vertex.

Remark 2 The first of these is a relevant distinction whenever the root is assigned to a degree-2 vertex. That could elsewise be homeomorphed out. While the second of these is an immediately relevant distinction whenever the root is assigned to a degree-1 vertex. That could elsewise be defoliated out by a single defoliation. And relevant in the face of serial defoliation whenever the root is assigned to a side-tree vertex.

Structure R Rooted graphs support their own notion of double irreducibles, under the H_{R} and F_{R} operations.

3.3 Digraphs

Definition D.0 In a digraph, the *indegree* of a vertex is the number of edges emanating from it with arrows pointing into it. And the *outdegree* of a vertex is the number of edges emanating from it with arrows pointing away from it. A vertex with all arrows pointing away from it is a *source*. While a vertex with all arrows pointing into it is a *sink*. The only other possibility for a degree-2 vertex is a *flow-through*. This has one arrow flowing into the vertex and the other flowing out of it.

Definition D.1 A digraph *dihomeomorph* H_{D} is a homeomorphing operation in the following restricted sense. A flow-in arrow, flow-through vertex and flow-out arrow is replaced by a single arrow. Which points in the common direction of the 2 arrows being replaced. While any degree-2 vertices that are sources or sinks are to be left alone. All of this remains restricted by no multi-edges being created.

Definition D.2 A digraph *didefoliation* F_{D} is a defoliation operation that can conflate a leaf with its flow-through twig. But cannot conflate a leaf with its source or sink twig.

In the event in which a twig is a branching point of degree ≥ 4, finer distinctions are supported. One might or might not permit defoliation unto mixed twigs. Starting with degree-4 vertices with indegree 2 and outdegree 2. Let us refer to the case allowing mixed cases to be defoliated *strong didefoliation* F_{DS}. And to the case not allowing this *weak didefoliation* F_{DW}.

Structure D Digraphs support their own notion of double irreducibles, under the H_{D} and F_{D} operations. With a finer split into F_{DS} and F_{DW} versions.

3.4 Rooted digraphs

Definition RD.1 A rooted digraph *dorohomeomorph* H_{RD} is a homeomorphing operation to which both of the above restrictions apply.

All of this remains restricted by no multi-edges being created.

Definition RD.2 A rooted digraph *dorofeulliette* F_{RD} is a defoliation operation that to which both of the above restrictions apply. Again, this comes in weak and strong variants.

Structure RD Rooted digraphs support their own notion of double irreducibles, under the H_{RD} and F_{RD} operations. With a finer split into strong and weak versions.

4 Double irreducible classification theorems

4.1 For the simple graphs

Remark 1 We recently revealed the double-irreducible class theorem for simple graphs [12].

Theorem S The interplay between the H and F operations on simple graphs partitions their arena [11, 14] into the following. Precisely 8 double-irreducibility classes. Namely,
D, P, P', P_3, $\mathrm{C}_3, \mathrm{C}_4, \mathrm{Paw}$ and C_5.

Remark 1 The minimum examples of each of these 8 classes for simple graphs were also provided. Alongside the extent to which these minimum examples are unique.

4.2 Rooted and digraph versions

Theorem R Substituting 'rooted graph', H_{R} and F_{R} for 'simple graph', H and F respectively gives the same result.

Theorem D Substituting 'digraph', H_{D} and F_{D} for 'simple graph', H and F respectively gives the same result.

Theorem RD Substituting 'rooted digraph', H_{RD} and F_{RD} for 'simple graph', H and F respectively gives the same result.

<u>Proof</u> The Algebraic proof in [12]'s exhaustion carries through when the simple graphs are further structured or conditioned in the current Articles' ways. What these structurings or conditionings require is establishing that none of the 8 exhaustive classes are rendered empty by them. The list of minimum examples for each class – without multiplicity – suffice for this purpose. Thus a small sliver of the below Exercises finish off these 3 (or with weak to strong distinction 5) proofs. □

Remark 2 However, the minimum examples and the extent to which they are unique differ with the type of graph being modelled. Including with the finer weak versus strong distinction in the D and RD cases.

4.3 S. Sánchez' Exercise set

Remark 3 The simple graph's Trowel graph (Fig 1.a) is key in providing the minimum examples of the 3 hardest cases. I.e. P_3 , Paw and C_5 . This is based on the Tarantulahead cycle system (Subfig b).

Exercise 1 a) How many rootings does this support?

b) How does rooting affect the multiplicity of the minimum examples of these 3 classes?

c) Also show that 2 of the simple graph minimum examples need to be abandoned. Find their replacements.

d) Finally show that 1 of the remaining minimum graph examples picks up a new nonuniqueness.

Exercise 2$^-$ Find the number of distinct digraphs supported by the square. How many of these can be dihomeomorphed?

Exercise 3 Show that 16 of the 20 digraphs supported by the Banner graph (Subfig c) can be strongly dihomeomorphed.

Exercise 4$^+$ Carry out the corresponding analysis for Trowel and its first homeomorphs. Deduce the extent of nonuniqueness for each DI class for digraphs.

Exercise 5$^+$ Find the minimum examples for the 8 DI classes in the strong DR case.

Exercise 6$^+$ Find the weak counterparts of the previous two Exercises.

Acknowledgments A.F. and I thank S. Sánchez for setting these Exercises for us in 2019. We have solutions; we now give the world an around 2-year opportunity to solve these for yourselves. I also thank the Applied Combinatorics and Topology Discussion Group's members.

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