# The Pappus Incidence Graph

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**Abstract**

Pappus' Theorem is one of the two main structural theorems of Projective Geometry. Its configuration can be modelled as a graph. Projective Geometry is furthermore the study of incidence, and incidence graphs are themselves meaningful. This brings Pappus' incidence graph to the forefront. We study its basic Graph Theory properties. We also use it to give further examples of Graph Theory drawing and visualization, including use of shelling and tessellation methods.

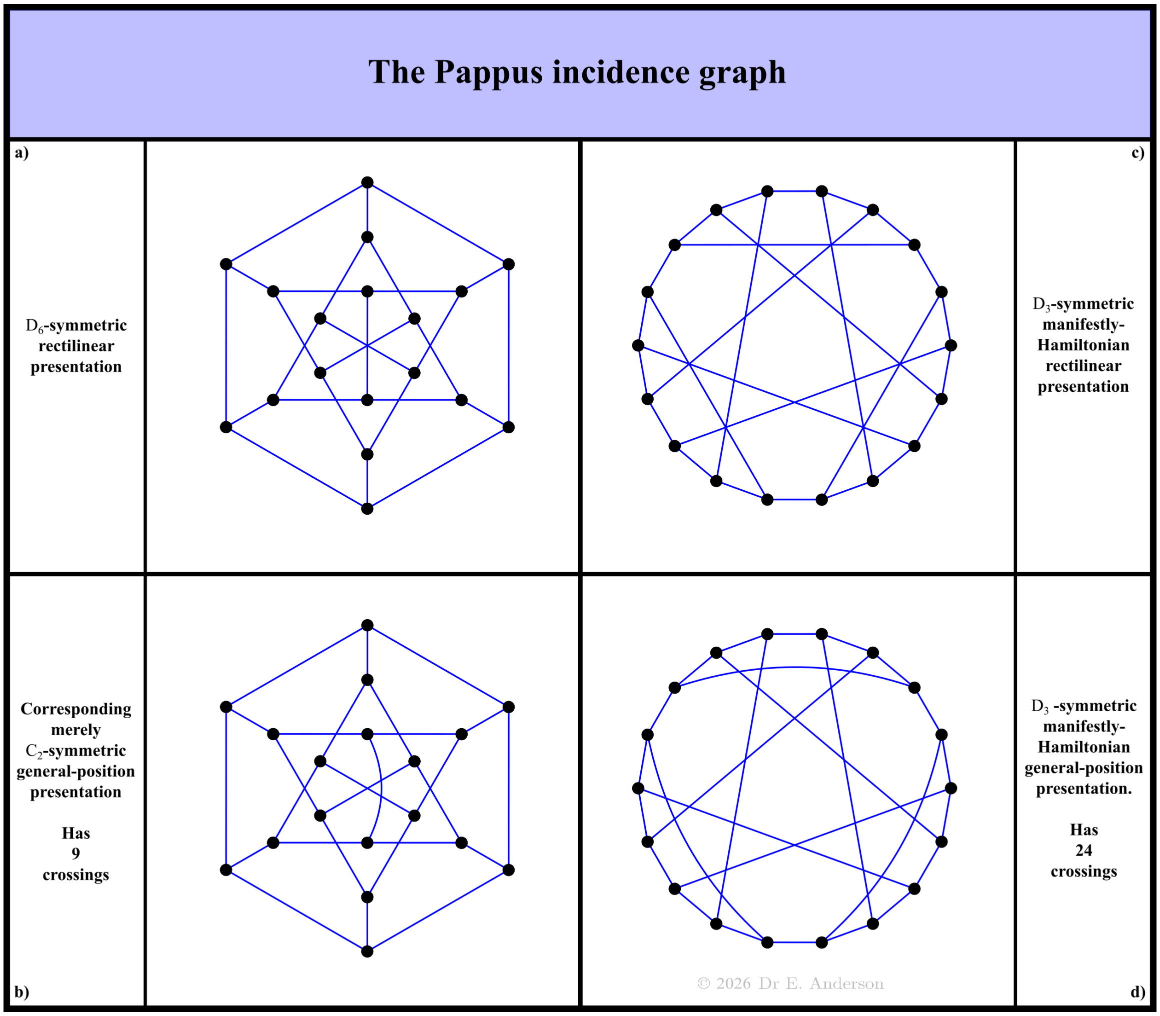


Figure 1:

This Article is ⟨3⟩: suitable for third-year undergraduates.

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# 1 Introducing Pappus' Incidence graph

## 1.1 Motivation

**Motivation 1** This is a further Article on abstracting interesting graphs from *Projective Geometry* [20, 10, 11, 2, 8, 12] : the study of incidence. We previously considered [35] Pappus' configuration and theorem [1, 9, 11, 8, 28, 22]. And the *Pappus graph* modelling the 9-point configuration itself.

**Remark 1** We now consider instead the corresponding incidence graph.

**Definition 1** Suppose that we are given an Incidence Geometry configuration. Then the corresponding *incidence graph* alias *Levi graph* [4, 6, 17, 18, 21] has as its vertices (blue) each primary vertex (grey) and line in the configuration. Pairs of these blue vertices are furthermore accorded edges whenever the corresponding primary vertices or lines are incident.

**Motivation 2** Since Projective Geometry is the study of incidence, it makes particular sense both to consider incidence graphs for Projective configurations and to use this name in this context.

**Remark 2** While we call the current Article's object of study [6, 21, 17, 23] the *Pappus incidence graph* be warned that many other sources [36, 37] refer to this one as the Pappus graph...

## 1.2 A $D_6$-symmetric presentation

**Structure 1** Let us start by giving a widely used presentation of this famous graph: Fig 1.a). This manages to exhibit a bit more symmetry than the others in this Figure. This presentation features for instance in [37] and, with 18 edges curved into arcs, in [36].

**Remark 3** This said, the Pappus configuration and incidence graphs have rather more symmetry than this presentation manages to muster. For its automorphism group is of order 108 , or, including dualities as well, of order 216 . See [13] ⟨6-8⟩ if interested in which specific group this is. In contrast, the current presentation's dihedral group $D_6$ is just of order 12 .

**Remark 4** Subfig b) deals with this presentation having a common Topological problem, namely that its vertices are not in general position. This is manifested by 3 edges meeting at a non-vertex point, which we place at the centre of symmetry. We deal with this by bending one edge. This is a presentational technique that is already familiar to most Readers from the presentations of the also famous Utilities graph alias $\mathrm{K}_{3,\,3}$ . This comes at the cost of breaking most of Subfig a)'s presentation's manifest symmetry.

**Remark 5** Among the many possible $D_6$-symmetric presentations, the displayed configuration has the following characteristics. It consists of 3 concentric hexagonal shells. These are equally radially spaced both mutually and relative to the centre of symmetry. See Appendix A.1 for a brief account of shell structure in graph presentations.

## 1.3 Some basic counts

**Remark 6** Having given one presentation so as to be able to display our graph, let us make the following basic counts for future use. Its number of vertices is

$$V(\mathrm{I}(\,\mathrm{Pappus}\,)) \;=\; 18 \;. \tag{1}$$

Its number of edges is

$$E(\mathrm{I}(\,\mathrm{Pappus}\,)) \;=\; 27 \;. \tag{2}$$

And its degree sequence is

$$\deg(\mathrm{I}(\,\mathrm{Pappus}\,)) \;=\; 3^{18} \;. \tag{3}$$

So all of its vertices' degrees are the same, which Graph Theorists call *regular* [30]. And their common value is here 3 , which Graph Theorists call *cubic*, and cherish [3, 5, 7, 14, 31]. This

$3$ carries Projective significance; in further detail, the underlying Projective configuration is $9_3$. Signifying $9$ points arranged to enjoy collinearity in $3$'s. At the level of the incidence graph, then, this enforces the cubic subcase of regularity...

**Remark 7** There is no practical point to considering the complement of our graph, since this has far more edges than our graph itself. This occurs whenever

$$E << E_{\text{max}} = \frac{V(V-1)}{2} .$$

For us,

$$E_{\text{max}} = \frac{18 \times 17}{2} = 153 >> 24 = E . \quad (4)$$

So

$$E\left(\overline{\text{I}(\text{Pappus})}\right) = 153 - 27 = 126 .$$

**Exercise 1**$^-$ Check this number using regularity instead.

**Remark 8** A finer consideration, for drawing and naming purposes, is to entertain whichever of $\text{G}$ or $\overline{\text{G}}$ is of smaller size ( $=$ edge number). Which is to be gauged against the critical value

$$E_{\text{crit}} := \frac{E_{\text{max}}}{2} = \frac{V(V-1)}{4} . \quad (5)$$

For us, this is

$$E_{\text{crit}}(\text{I}(\text{Pappus})) = \frac{18 \times 17}{4} = \frac{153}{2} = 76.5 . \quad (6)$$

Then indeed

$$27 < 76.5 < 126 . \quad (7)$$

So we go with drawing, and naming, the $27$-edge version!

**Remark 9** Projectively significant graphs span both sides of the divide as regards whether complements are worthwhile. For the Fano graph [34], they are. For the Pappus graph [35], the count is the critical value itself. For all of the Desargues graph [39], and the $3$ corresponding incidence graphs ([38, 40] and the current Article), however, using the complements is less advantageous. Considerably so for this trio of incidence graphs. The counting-level technicality here is that these graphs are both too large and too sparse to have useful complements.

# 2 Further properties

## 2.1 Structural analysis

**Remark 1** The Pappus incidence graph has no side-trees and is thus a cycle system alias foliation irreducible. It also has no vertices of degree $2$, and so is a homeomorph irreducible. Thus it is a double irreducible: DI class D [33].

## 2.2 Metric properties

**Definition 1** The *girth* $g(\text{G})$ of a graph $\text{G}$ is the length of its smallest cycle. While the *circumference* $c(\text{G})$ of a graph $\text{G}$ is the length of its longest cycle.

**Definition 2** The *diameter* $diam(\text{G})$ of a graph $\text{G}$ is as follows. The maximum over all vertex pairs of the minimum lengths between each vertex pair.

**Remark 1** Throughout the above definitions, length refers to path-length as measured by the number of edges in that path.

**Exercise 2** Show that

$$g(\,\mathrm{I}(\,\mathrm{Pappus}\,)\,) \;=\; 6\;, \tag{8}$$

$$d(\,\mathrm{I}(\,\mathrm{Pappus}\,)\,) \;=\; 4\;. \tag{9}$$

Why are we not explicitly noting down $c(\,\mathrm{I}(\,\mathrm{Pappus}\,)\,)$ in the current Article?

## 2.3 Traversibility properties

**Remark 1** The Pappus graph is clearly not Eulerian. For by (3) the Pappus graph has all vertices of odd degree. But Eulerian graphs must have all vertices of even degree!

**Remark 2** Consider again the symmetric presentation of Fig 1.a). In Fig 2, we mark upon this in emerald one of the Hamiltonian cycles that is conceptually simplest to describe. I.e. view the vertices as lying on 3 hexagons. The inner 2 of which have overlapping edges and are depicted as triangles with marked midpoints. First go around the outer hexagon as far as possible. Next cut in to an inner hexagon and sweep out all of its vertices. Then cut across into the other inner hexagon and sweep out all of its vertices. This finally has the good fortune of leaving us exiting back to the outside in the right place to close up the overall cycle.

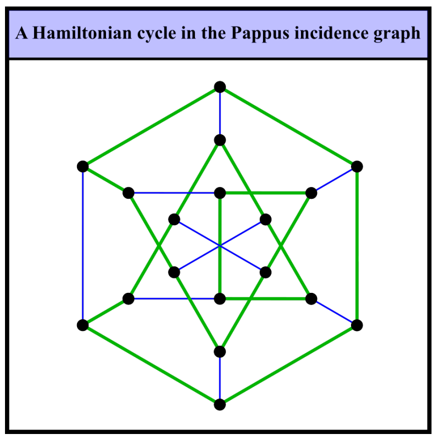



Figure 2:

**Remark 3** Subfig c) instead manifestly exhibits that the Pappus incidence graph is Hamiltonian. On this occasion, one has the good fortune that most of Subfig a)'s manifest symmetry can continue to be exhibited. This presentation features for instance in [37] and, without centering about a symmetry axis, in [36]. It features, centred and yet upside-down relative to our presentation, in the world's first review of Projectively significant graphs: Coxeter's [6].

**Remark 4** Subfig c) however suffers from a considerably more severe case of not being in general position. Now no less than 6 separate cases of 3 edges meeting at a non-vertex point are present. Rather neatly, one only needs to bend 3 edges to handle this, nor is any presentational symmetry is lost in the process! See Subfig d).

**Exercise 3$^-$** Retain a larger chunk of symmetry by bending more than one edge in Subfig a) in the process of getting into general position.

**Exercise 4** Investigate whether Fig 1.a)'s presentation permits other shapes of Hamiltonian cycle. Are these distinct as cycles within presentation-free graphs? In the process, figure out where Subfig c)'s nicely symmetric Hamiltonian presentation can be taken to reside within Subfig a)'s presentation...

# 3 Minimum-crossing presentations

## 3.1 Preamble

**Remark 1** All presentations considered so far contain rather a lot of crossings. In fact, the Pappus incidence graph's crossing number is $5$. While this is too hard to prove in these early stages of writing up a small Encyclopaedia, why various simple techniques fail does lie within our grasp.

**Exercise 5**$^-$ Find a $K_{3,3}$ subgraph within. Why is the other forbidden subgraph for a planar graph, $K_5$, irrelevant to our analysis?

**Definition 1** The *crossing number* $Cr$ is the minimum over all presentations of the number of crossings $cr$ exhibited in each presentation.

**Remark 2** By Exercise 5, $Cr \geq 1$. By Fig 3.a), $Cr \leq 5$. Thus what we can say at present is that

$$1 \leq Cr(\mathrm{I}(\mathrm{Pappus})) \leq 5 . \qquad (10)$$

For all that finding a 5-crossing presentation, such as the one that we exhibit in Subfig a), is not particularly straightforward either... ⟨6⟩

**Remark 3** In simple cases, we can either draw the graph with even less crossings to tighten the upper bound. Or use an inequality to tighten the lower bound. In the present context, the first is impossible, while an incipient inequality one can always try out for the second is follows.

$$Cr \geq E - E_{\mathrm{planar}} = E - 3V + 6 = E + 3(2 - V) . \qquad (11)$$

**Exercise 6** a)$^-$ Show that this is not useful for $\mathrm{I}(\mathrm{Pappus})$.

b) Show that a nontrivial amount of girth[1] $g > 3$ improves (11) to the following.

$$Cr \geq E + \frac{g}{g-2}(2 - V) . \qquad (12)$$

c)$^-$ Show that the good fortune of (8) on this occasion fails to be enough to turn the tide.

d) Show that the following more advanced *crossing number inequalities* [16, 26] ⟨6-7⟩ also fail to help out.

$$Cr(\mathrm{G}) \geq \frac{E^3}{64V^2} \quad \text{for } E > 4V . \qquad (13)$$

$$Cr(\mathrm{G}) \geq \frac{4}{135}\frac{E^3}{V^2} - \frac{9}{10}V . \qquad (14)$$

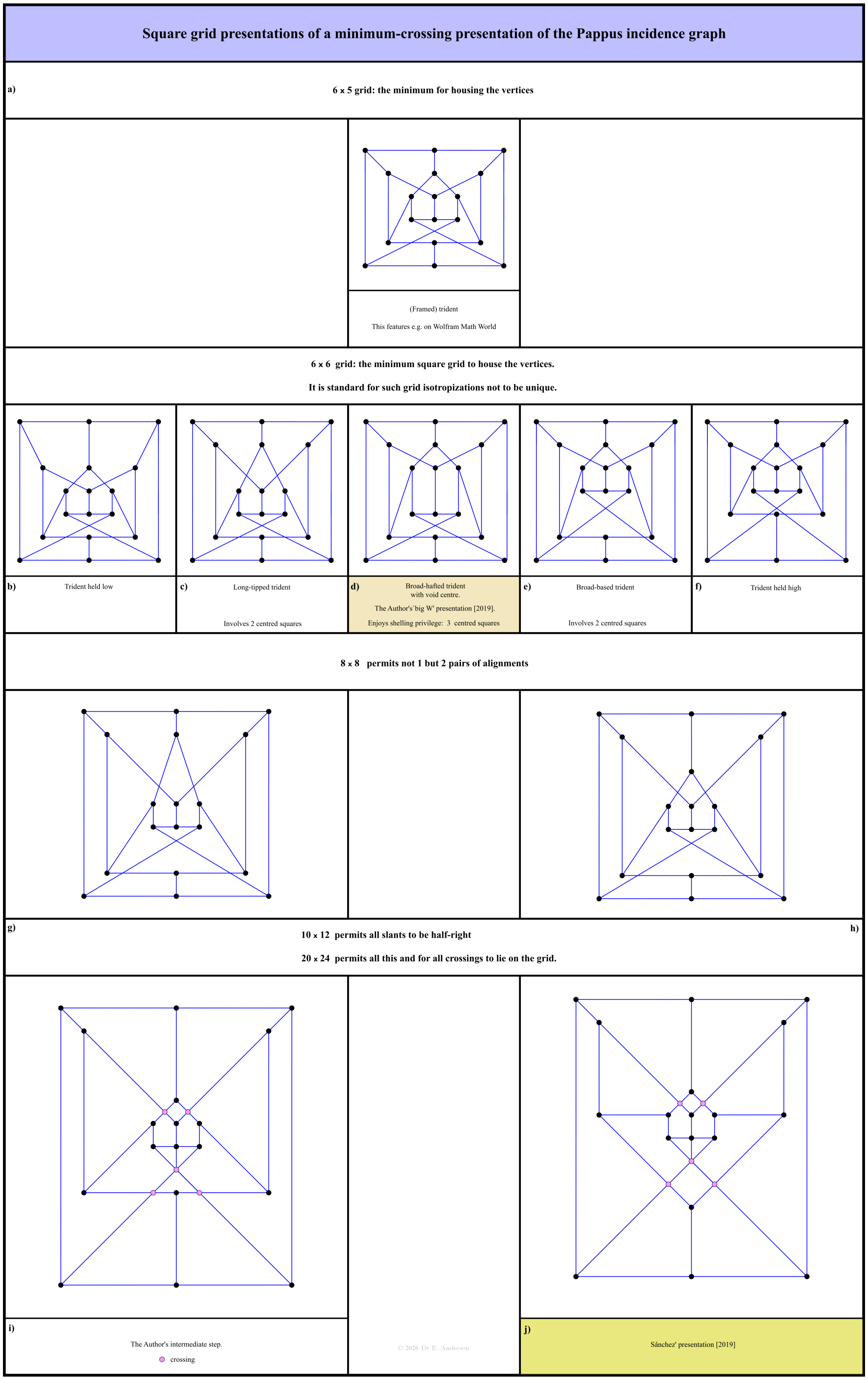
In the process, compute and blame the Pappus incidence graph's sparseness. Use whichever of the 'per unit vertex' or 'relative to the maximum edge density' quantifications of sparseness that best suit the problem at hand.

## 3.2 Square grids

**Remark 4** The graph at the top of Fig 3 is quite a well-known 5-crossing presentation. For instance, [37] contains it, though without emphasizing its $5$ crossings or its general presentational type [25]. Which is a *grid graph presentation*, with reference to a grid of squares. And is more specifically a $6 \times 5$ grid graph. Which is minimum to get all vertices on integer grid coordinates.

**Remark 5** In the next row, I isotropize this to obtain the smallest square grid graph presentations. There are $5$ places where one can slot in an extra row to isotropize $6 \times 5$ to $6 \times 6$. Thus this procedure entails a 5-fold redundancy. Such redundancies are always a feature of grid isotropizations...

[1] Ignoring the infinite girth convention for trees!



Square grid presentations of a minimum-crossing presentation of the Pappus incidence graph
a)
6 x 5 grid: the minimum for housing the vertices
(Framed) trident
This features e.g. on Wolfram Math World
6 x 6 grid: the minimum square grid to house the vertices.
It is standard for such grid isotropizations not to be unique.
b)
Trident held low
c)
Long-tipped trident
Involves 2 centred squares
d)
Broad-hafted trident
with void centre.
The Author's 'big W' presentation [2019].
Enjoys shelling privilege: 3 centred squares
e)
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8 x 8 permits not 1 but 2 pairs of alignments
g)
h)
10 x 12 permits all slants to be half-right
20 x 24 permits all this and for all crossings to lie on the grid.
i)
The Author's intermediate step.
crossing
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j)
Sánchez' presentation [2019]


Figure 3:

**Remark 6** On this occasion, there is however a selection principle for the middle case. For the grid points here all lie on $3$ centred squares: another instance of shelling.

**Remark 7** The third row shows that expansion to $8 \times 8$ improves the amount of collinearity exhibited. The fourth row shows that using an even larger grid permits all angles to be right or half-right [29]. This necessarily forces our presentation of the Pappus graph to have anisotropic perimeter. However, a second benefit can be accumulated, namely that the crossings themselves lie on the grid. Due to this, at least out of the selection of 5-crossing presentations currently on offer, this one takes the 'gold medal'.

## 3.3 Using the hexagonal shelling and equilateral tessellation

**Remark 8** All 5-crossing presentations exhibited so far have a rectangular perimeter. But with $2$ opposite sides' midpoints as vertices in addition to the $4$ corners. Thus using a regular hexagon perimeter should also be entertained.

**Remark 9** Many of our presentations have furthermore $6$ points in their second shell, and again in their third, final and innermost shell. This suggests using not a piece of the square tessellation of the plane. But rather a piece of the hexagonal shelling, or of an underlying equilateral-triangle tessellation of the plane. See Appendix A.2 for a brief account of tessellation structure in graph presentations.

**Pointer 1** ⟨6⟩ This is part of the program [29, 32] of passing from using not only square grids in drawing and visualizing graphs ([27] involves a subcase) but also increasingly general tessellations of the plane [15]. The first port of call here are the tessellations by a single regular polygon. And then by a single less regular tile, or by $2$ regular ones.

**Remark 10** For the Pappus incidence graph, we now take the incipient square grid (Subfig 3.a) to $3$ layers' worth of hexagons. Unfortunately, this necessarily brings in further crossings (Subfig b). However, relocating a particular innermost-shell vertex to the hitherto unoccupied centre cures this (Subfig c). The relative angles are nice as well. However, the crossings are now not at nice places (Subfig d). In the sense of being on the tessellation's intersection points or some integer multiples thereof. Due to this, on this occasion our hexagonal foray only manages to take the 'silver medal' in the particularly distinguished presentation stakes.

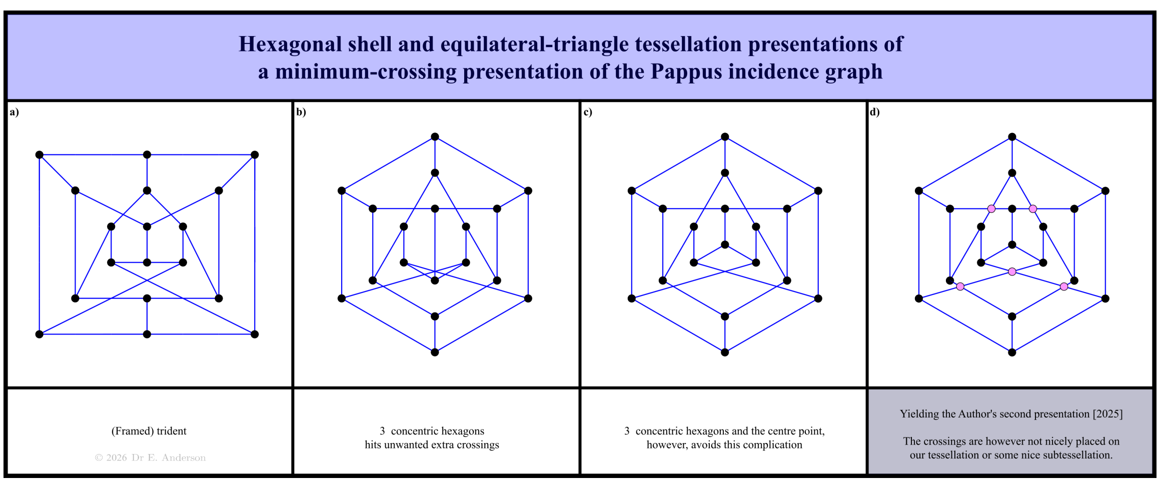


Figure 4:

# A Manifest presentations

## A.1 Shellings

**Remark 1** We exhibit here the square and hexagonal shells, alias layers, alluded to in the main text. Occasionally the centrepoint can furthermore be marked in as an innermost zero-radius shell.

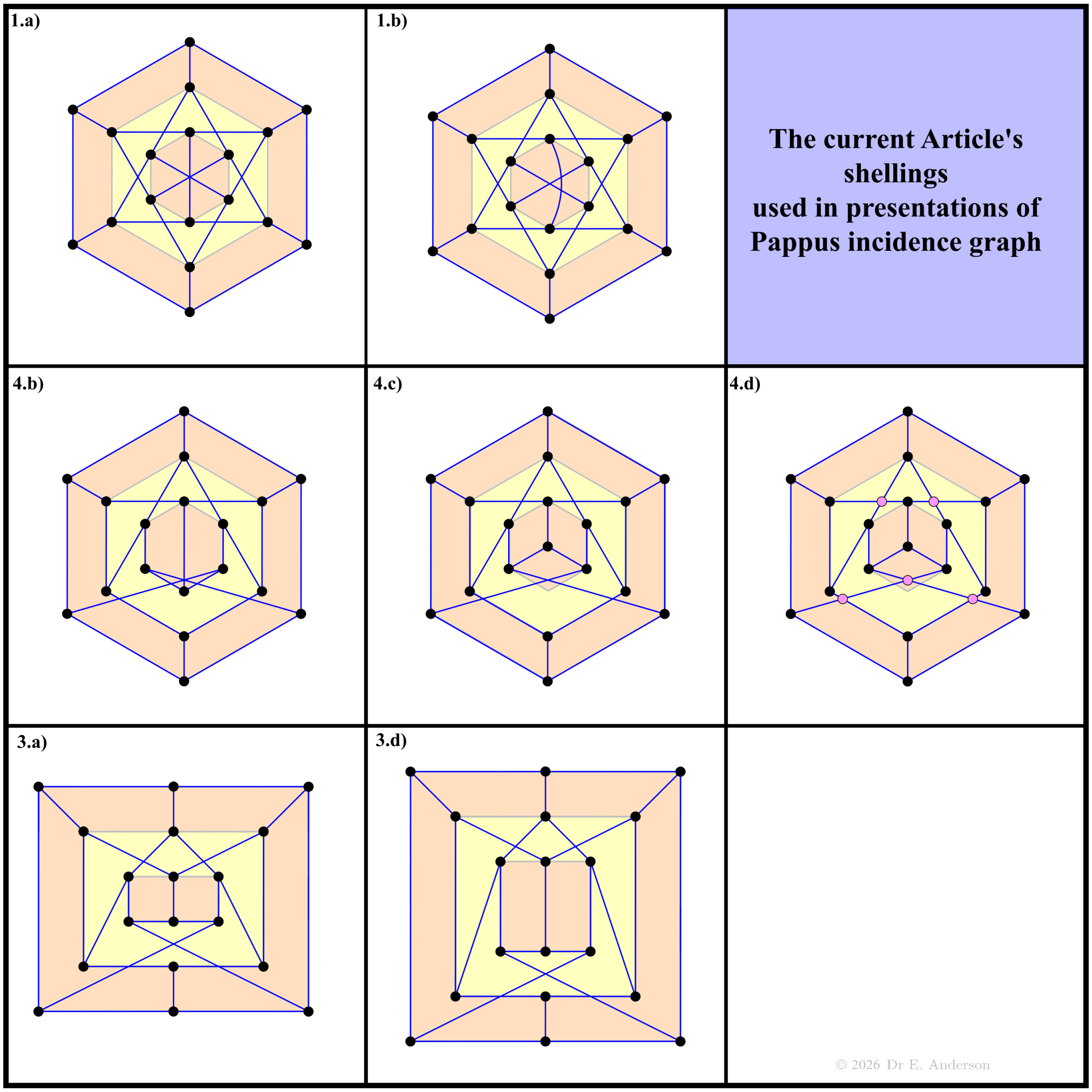


Figure 5:

## A.2 Tessellations

**Remark 1** We finally exhibit here the equilateral-triangle and square tessellations alluded to in the main text.

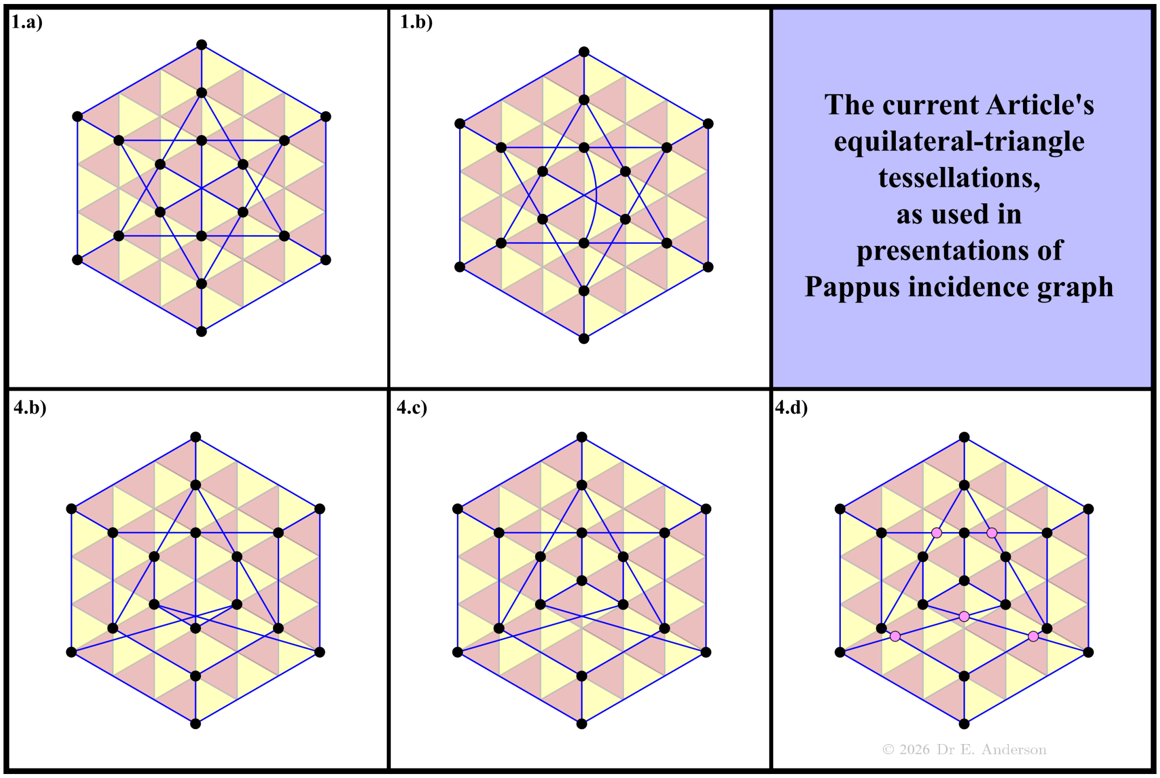


Figure 6:

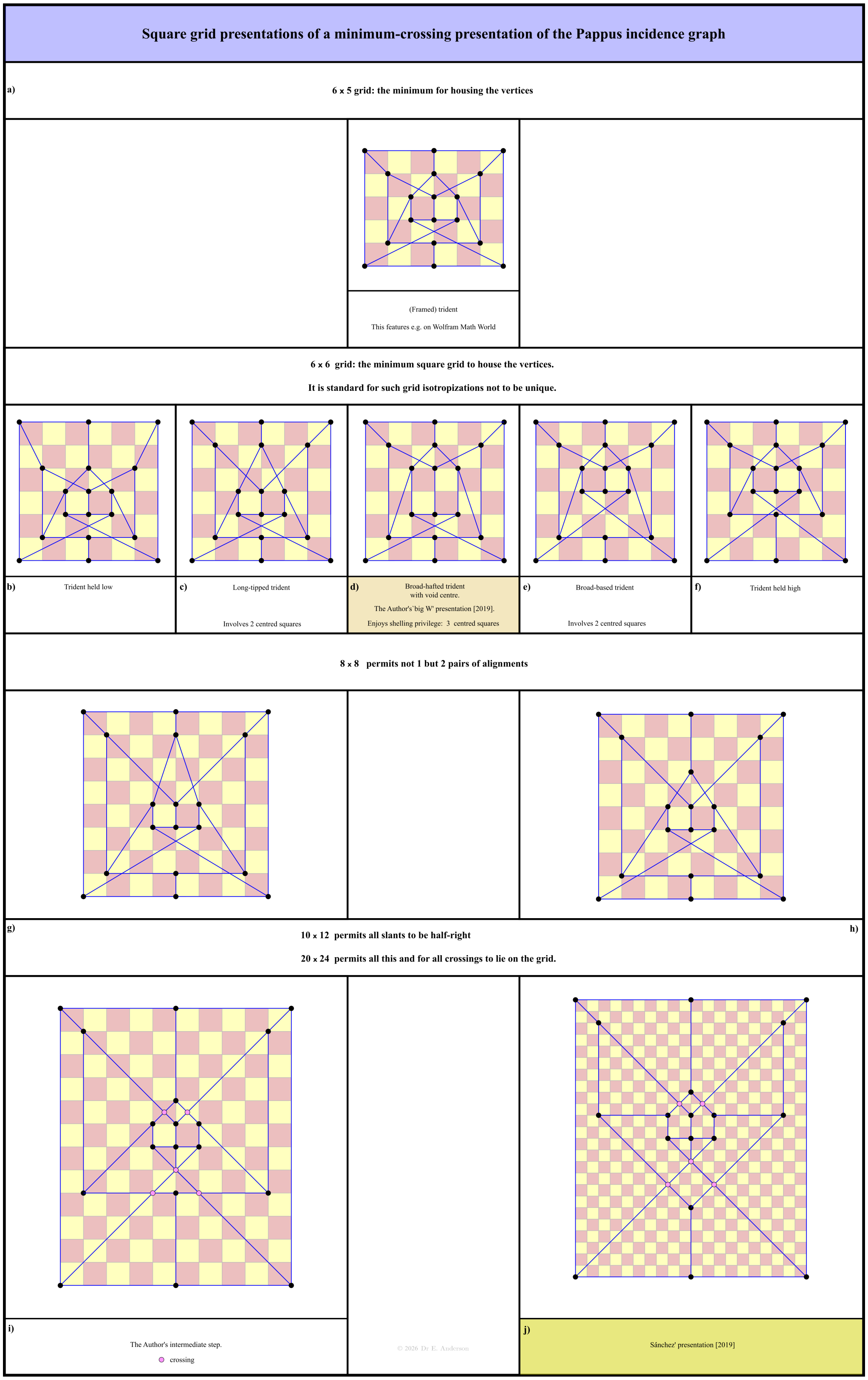

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Figure 7:

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