

Line and circle theorems from Euclid to Euler: underlying graphs

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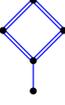
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Remark 1 For the first third of this account, we consider the cyclic quadrilateral family of Theorems. All of the ones presented were known to Euclid [1], bar one of Brahmagupta's [3, 5, 11]. The Geometrical configurations are in Fig 1. The ensuing multi-graphs [20] in Fig 2.

Circle and polygon Theorems from Euclid to Euler. Plate 1.1. Configurations				
1) \angle s in a Crossed Cyclic Quadrilateral \Leftrightarrow the Same Segment are Equal Theorem		2) \angle (at centre) = 2 \angle (at circumference) Theorem		
		Plain-acute case 	Critical case of Thales: diameter subtends \square 	Reflex-obtuse case
3) Opposite \angle s are Supplementary Theorem \Leftrightarrow \angle (internal) = \angle (opposite-external) Theorem		4) Alternate Segment Theorem		Angle Theorems
5) Chord Bisected by \perp from Centre Theorem <small>with a proving construct marked</small>	6) Intersecting Chords Theorem		9) Brahmagupta's other Theorem	Length Theorems
7) Equal Tangents Theorem <small>with a proving construct marked</small>	8) Power of a Point Theorem			
	$AX \cdot CX = (BX)^2 \Leftrightarrow$		BX tangent	© 2025 Dr E. Anderson

Figure 1:

Circle and polygon Theorems from Euclid to Euler. Plate 1.2. Graphs			
1) \angle s in a Crossed Cyclic Quadrilateral \Leftrightarrow are Equal Theorem		2) \angle (at centre) = 2 \angle (at circumference) Theorem	
without intersection point  K_4 with nonadjacent = pair		Plain-acute case	Critical case of Thales: diameter subtends \square
with intersection point  W_4 with opposite perimeter = pair		Reflex-obtuse case  Di with peripheral degree-2 P_3 of =	
without intersection point  K_4 with P_3 of =		with intersection point  W_4 with perimeter P_3 of =	
3) Opposite \angle s are Supplementary Theorem \Leftrightarrow		4) Alternate Segment Theorem	
\angle (internal) = \angle (opposite-external) Theorem			
 C_4 all in =		 Banner with C_4 all in =	
		 Ankh with C_3 all in =	
5) Chord Bisected by \perp from Centre Theorem with a proving construct marked		6) Intersecting Chords Theorem	
 K_4 with single =		 W_4	
		 W_4 with perimeter C_4 of =	
		9) Brahmagupta's other Theorem  Brahmagupta multi-graph	
7) Equal Tangents Theorem with a proving construct marked		8) Power of a Point Theorem	
without intersection point  K_4 with single =			
with intersection point  $K_5 - \{pt\}$		 Paw with C_3 all in =	
		 Dart with a C_3 all in =	

Angle Theorems

Length Theorems

Figure 2:

Circle and polygon Theorems from Euclid to Euler. Plate 1.3. Graphs with line intervals and circle arcs distinguished				
1) \angle s in a Crossed Cyclic Quadrilateral \Leftrightarrow are Equal Theorem		\angle s in the Same Segment are Equal Theorem		2) \angle (at centre) = 2 \angle (at circumference) Theorem
without intersection point		without intersection point		Plain-acute case
with intersection point		with intersection point		Critical case of Thales: diameter subtends \square
Opposite \angle s are Supplementary Theorem		\angle (internal) = \angle (opposite-external) Theorem		Reflex-obtuse case
Chord Bisected by \perp from Centre Theorem		Intersecting Chords Theorem		4) Alternate Segment Theorem
Equal Tangents Theorem		Power of a Point Theorem		9) Brahmagupta's other Theorem
without intersection point		without intersection point		
with intersection point		with intersection point		
K ₄ + nonadjacent cyan pair		K ₄ + cyan P ₃		
W ₄ + opposite perimeter cyan pair		W ₄ + cyan perimeter P ₃		
C ₄ doubled in cyan		Banner + C ₄ all in cyan		
K ₄ with single =		W ₄ + perimeter C ₄ in cyan		
K ₅ - {pt} with high-degree C ₃ in cyan		Paw + C ₃ in cyan		
K ₄ with single =		Dart + C ₃ in cyan		
Brahmagupta multi-graph with degree-5 cyan square				

C₄ + C₃ in cyan

Angle Theorems

line intervals

circle arcs

Length Theorems

Figure 3:

Remark 2 In Fig 3, these are further decorated with line-segment to circular-arc distinction. By using blue for the first and cyan for the second. Theorems featuring just lines are then ‘purely blue’; in particular all projectively meaningful Geometrical configurations are ‘purely blue’.

Remark 3 One substantially new graph arises in this first third. Namely the *Brahmagupta graph* of Fig 4. As a 7-graph, we can lean on many of its properties having already been tabulated in the systematic account [14]. That this graph models the Topological content of ‘Brahmagupta’s other theorem’ [5, 11] is quite possibly its most distinctive feature. Thus accounting for our coining this name for it.

With this simple graph skeleton having 14 edges, its complement has just 7. Since the Brahmagupta graph also has a maximum-degree vertex, this complement contains a loose point. We furthermore then have the good fortune that the remaining component has a standard name: the Domino 6-graph. Thus another name for the Brahmagupta graph is $\overline{\text{Domino} \amalg D_1}$.

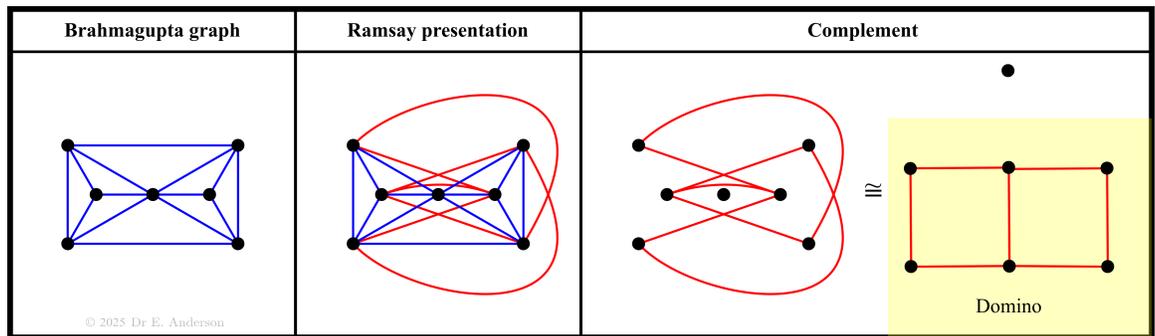


Figure 4:

0.1 Triangle and circle graphs: case study of the circumcircle

Remark 1 Euclid already considered circumcircles in [2], from forming perpendicular side bisectors and finding that these concur at the thus-defined circumcentre. Including the diametric cutting points and all intersection points gives a model that illustrates well that topological graphs of Geometrical configurations can be rich. Indeed, 11 qualitatively distinct cases emerge. Fitting together in the manner of Kendall’s spherical blackboard [7, 8, 10, 13] (Fig 9.a) with Carroll [4] decor [12, 19, 18] (Fig 9.b). See Fig 5.

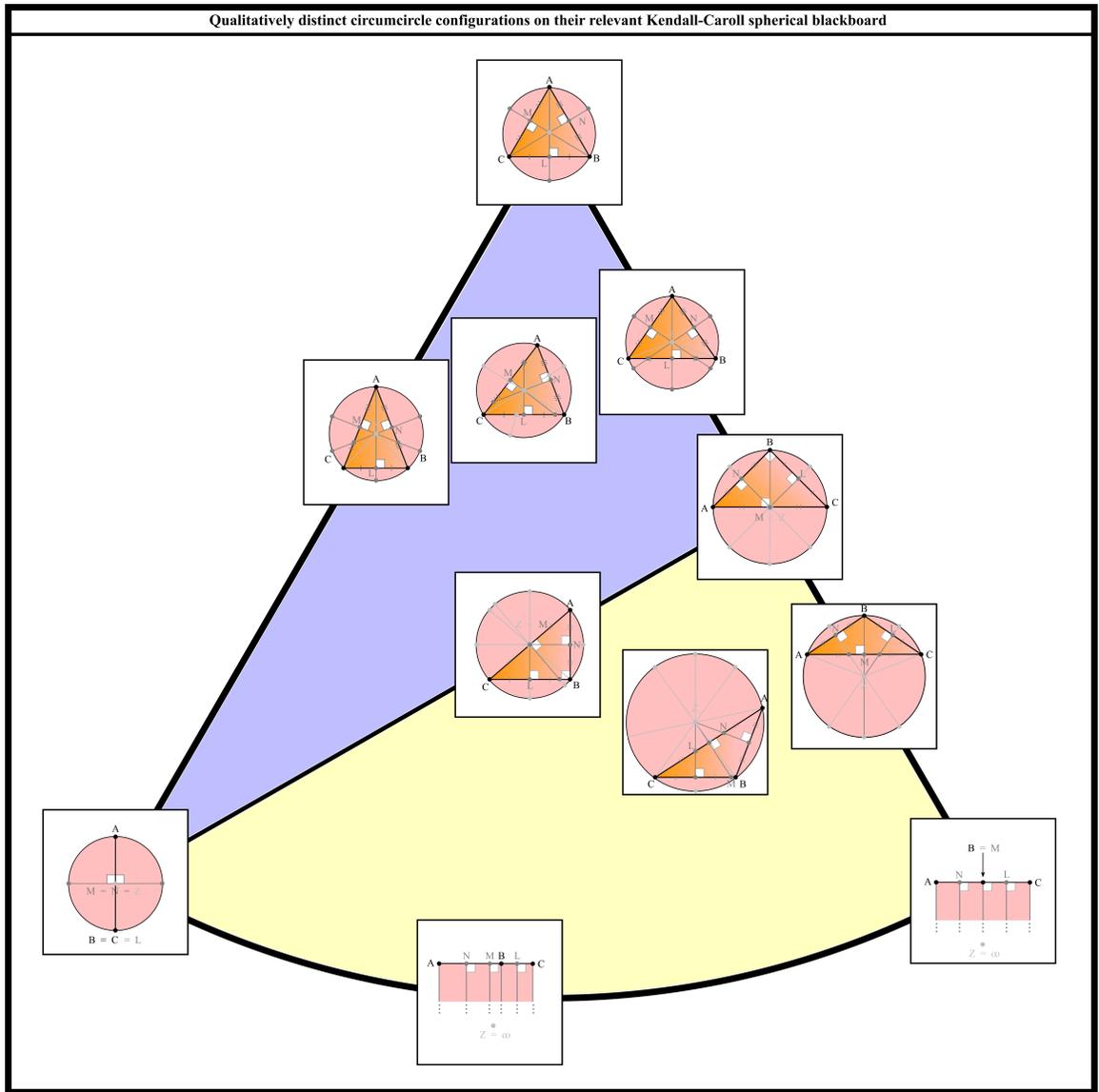


Figure 5:

Remark 2 As per Fig 6, the corresponding graph models form the adjacency graph (Fig 9.e-f) [17, 18] of the corresponding [18] topological blackboard (Fig 9.c-d). This is now not only an adjacency graph like Fig 9.e-f). But also a *perstroika* of tearings corresponding to topological rearrangements that transmute between the constituent graphs. This is a simple Combinatorial analogue of Arnol'd's use [9] of 'perestroika' in the analysis of dynamical systems arenas. Fig 7 continues our custom of further picking out the circle arcs in cyan. Finally, Fig 8 shows using the ZIPHoN theorem [20] (alias Grinberg's theorem [6]) that this perestroika succeeds in preserving the Hamiltonian property.

The above paragraph's 3 figures have been rotated so as to have the binary B at their pole. This is due to the underlying symmetry of the topological blackboard and its adjacency graph.

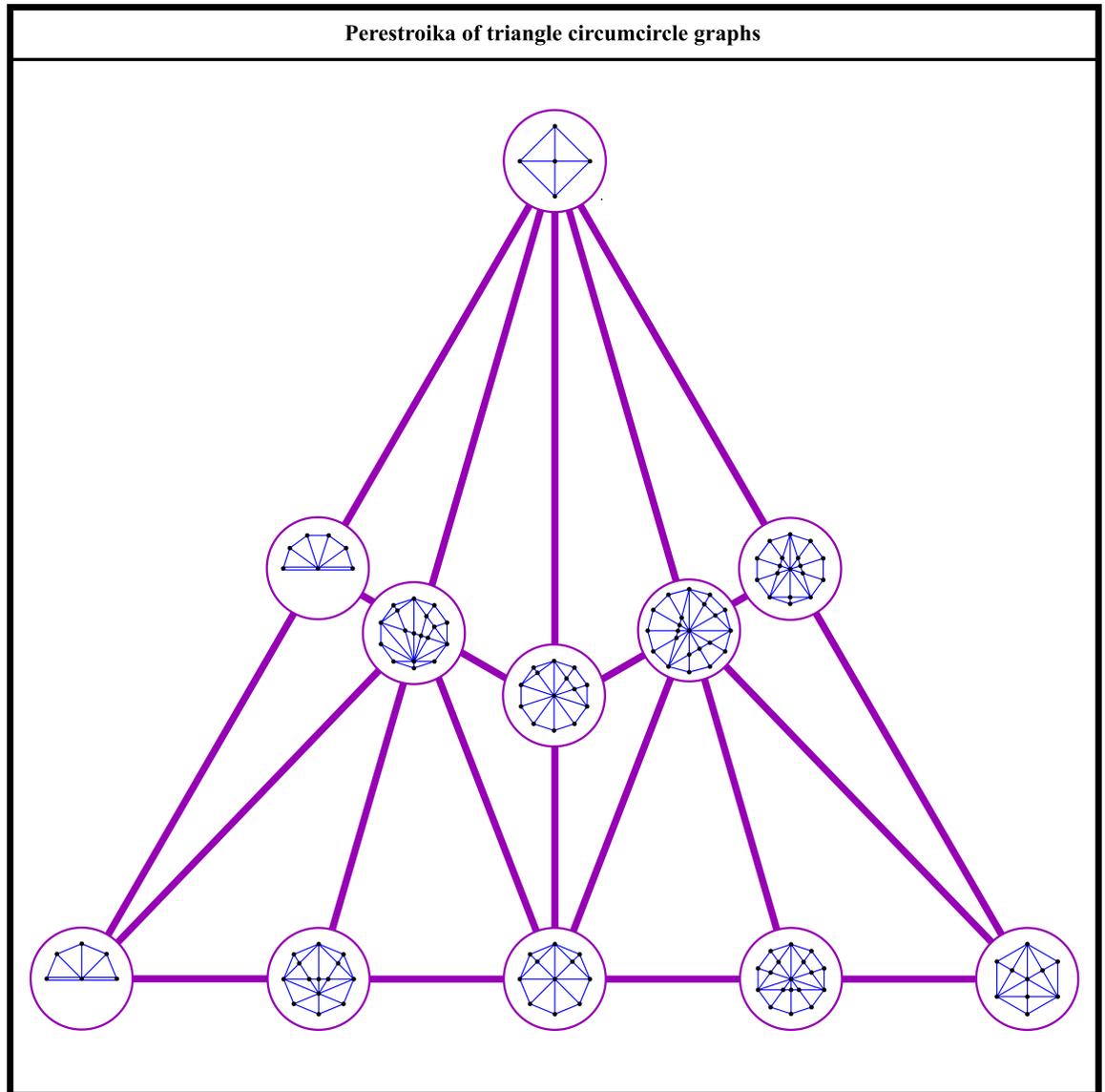


Figure 6:

Remark 3 One idea then is to find all qualitatively-distinct variants of all Geometrical configurations, constructs and theorems of note. As encoded using graphs, to be displayed in the inter-relation afforded by the underlying shape space [13, 12, 17, 15, 21]. Not all such qualitative-type analyses are as rich or insightful as our circumcircle example. Which decor of shape space [17, 18, 21] is relevant also varies between configurations, constructs and theorems.

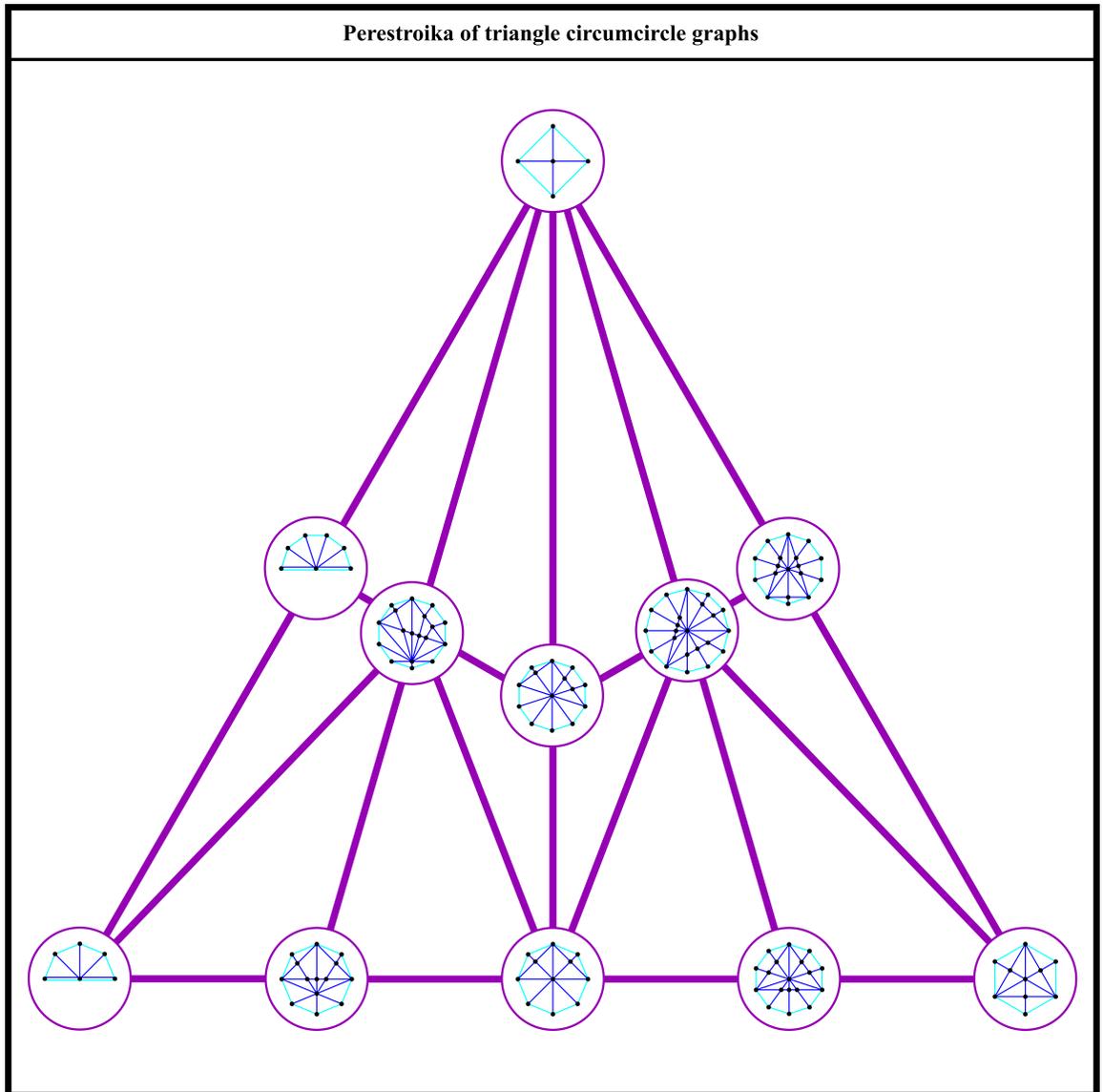


Figure 7:

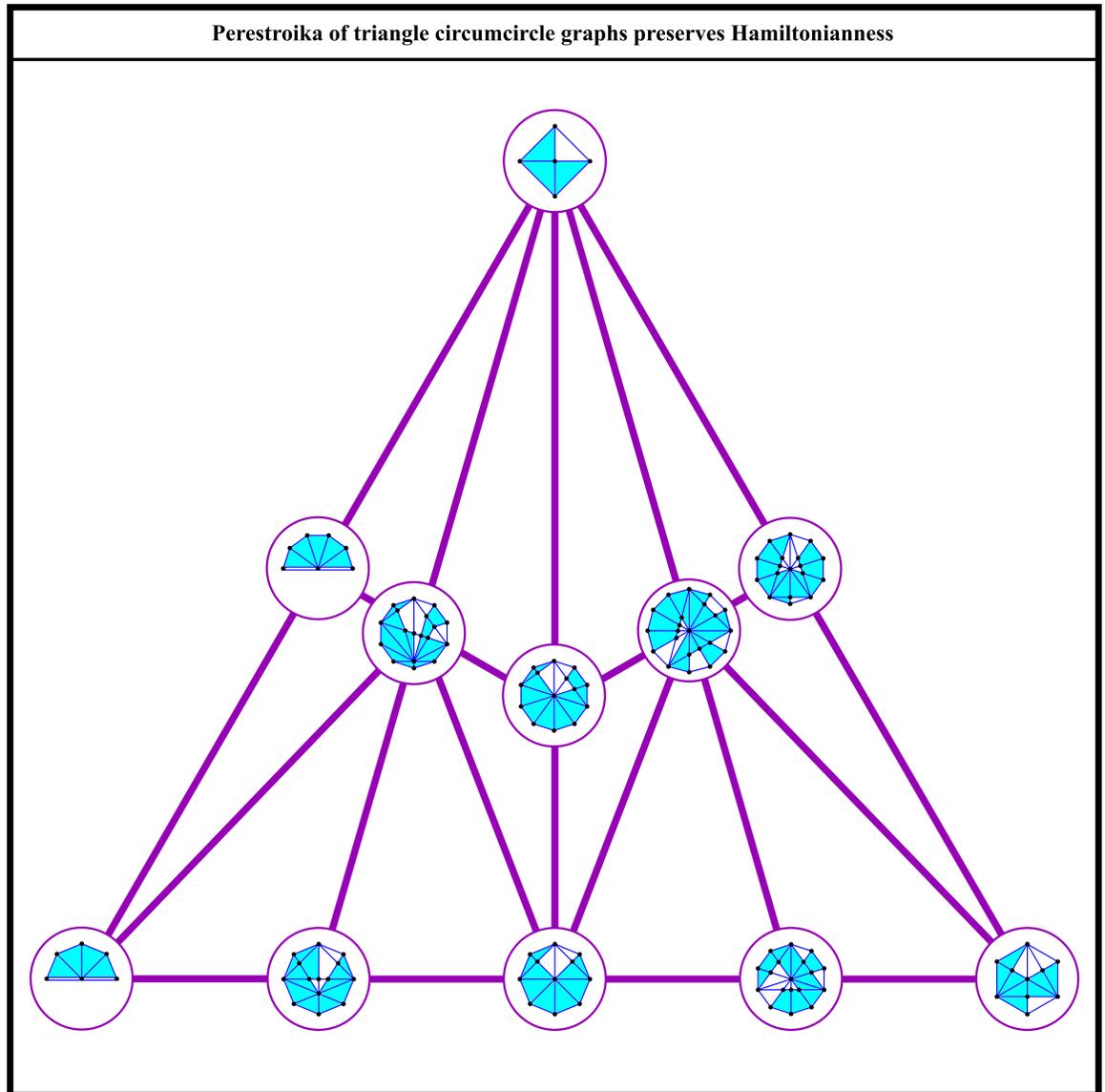


Figure 8:

0.2 Appendix

Structure 1 We present here Kendall's spherical blackboard of unlabelled triangle shapes modulo similarity. This is Metric-Geometrically as well as Topologically valid, and is given in the bird's eye view presentation [16]. Including its right decor splitting between acute and obtuse regions. The topological blackboard counterparts of each of these. And their corresponding adjacency graphs.

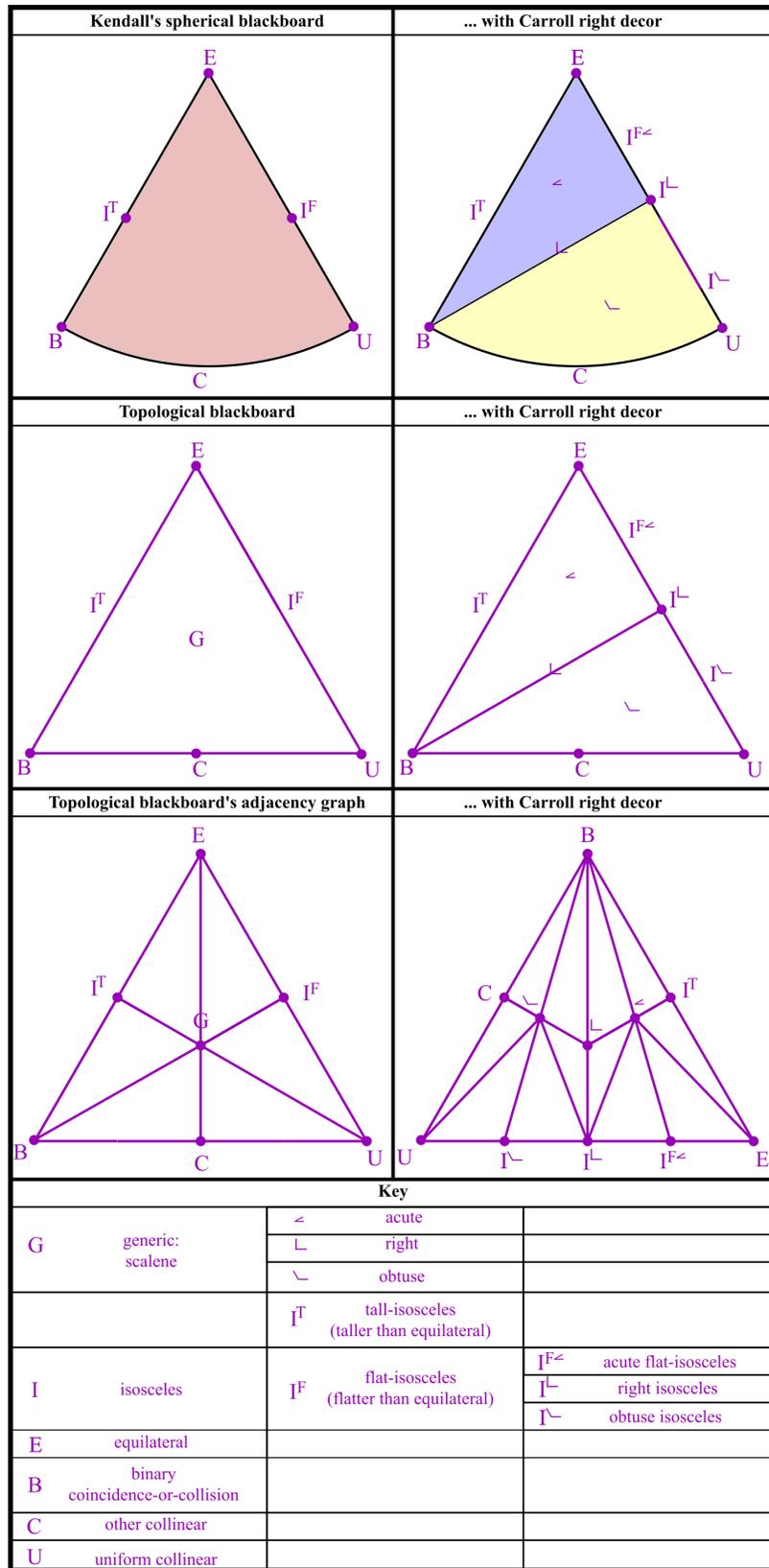


Figure 9:

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