Smallest Nontrivial Arenas of Double-Irreducible Simple Graphs

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Abstract

Defoliation serially removes all leaves alias degree-1 vertices from simple graphs. The end-product is a foliation irreducible alias cycle system. Homeomorphing serially removes all degree-2 vertices. In the context of simple graphs, cases in which this would create a multi-edge are to remain unchanged. The end-product is now a homeomorph irreducible. A double irreducible is the end-product of applying both. This models letting homeomorphing loose on any new degree-2 vertices resulting from defoliation.

We present an add-cycle modular lattice model for double irreducibles on up to 5 vertices. And an edge-scultping general bundle model for the same. We structurally analyze each, as well as demonstrating that 5 is minimum for this study to be nontrivial.

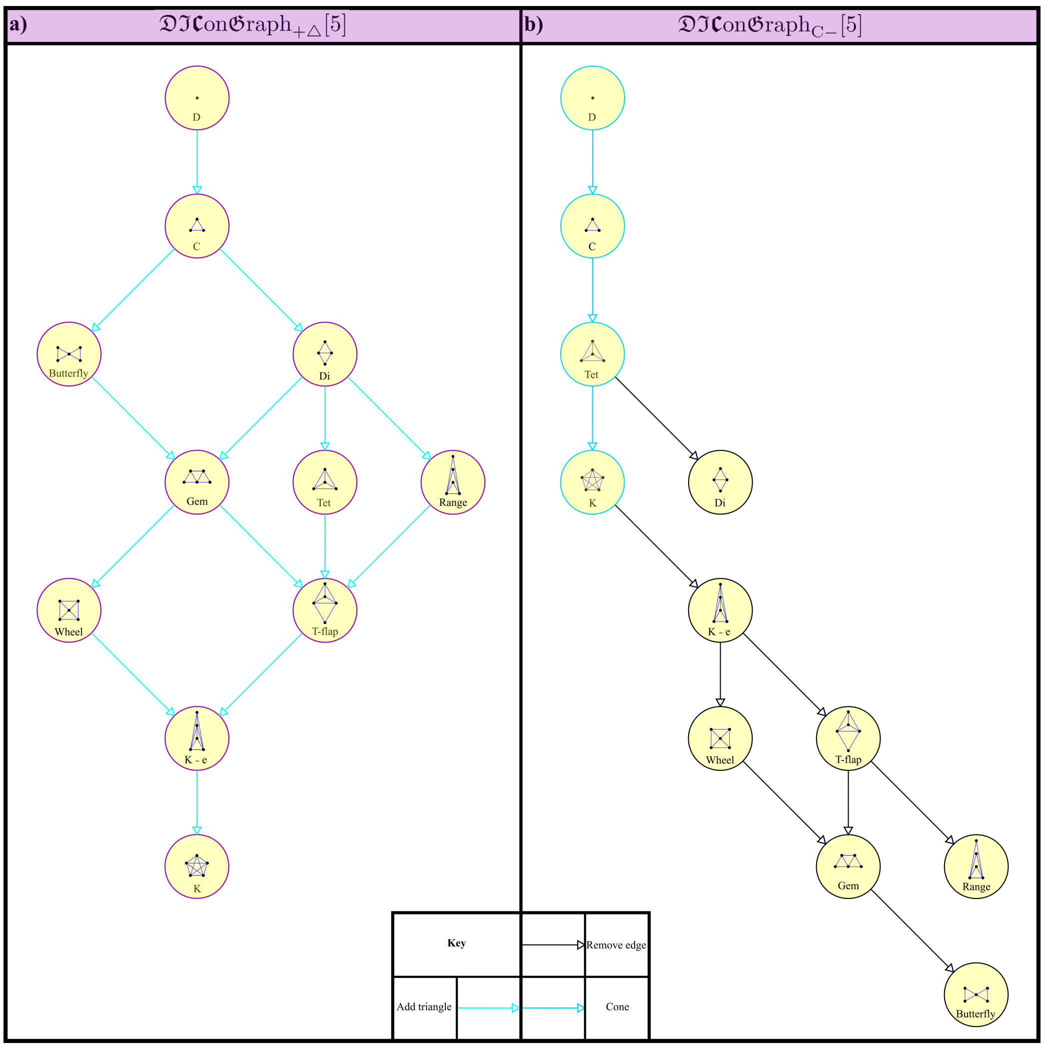

Figure 1:

This Article is ⟨3⟩: suitable for third-year undergraduates.
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1 Introduction

1.1 Notions of irreducibility for graphs

Remark 1 Given a type of Mathematical object, the corresponding *arena* [1, 6, 18, 20, 21, 27, 26] is the Mathematical space of the totality of these objects. The current Article considers simple graphs [3, 12, 20, 17] as objects. Let us denote the corresponding arena of these on N vertices by

$$\mathfrak{G}\text{raph}(N) \ .$$

And the cumulative arena of these on $\leq N$ vertices by

$$\mathfrak{G}\text{raph}[N] \ .$$

1.2 Notions of irreducibility for graphs

Definition 1 *Defoliation* [20] serially removes all leaves alias degree-1 vertices. This prunes off all side-chains, and, more generally, all side-trees. The end-product of this is a *foliation irreducible* (*FI*) alias *cycle system* [23]. Observe that if the incipient graph is a tree, then a single vertex survives this process: a terminal state since no leaves remain therein.

Definition 2 *Homeomorphing* [20, 14] serially removes all degree-2 vertices. In the context of simple graphs, cases in which this would create a multi-edge are not to be removed. The end-product of this is a *homeomorph irreducible* (*HI*) [4, 20].

Definition 0 A *double irreducible* (*DI*) [23] is the end-product of applying both. Including letting homeomorphing loose on any new degree-2 vertices resulting from defoliation.

Remark 2 All three of the above notions of irreducible are useful in structurally analyzing graphs and arenas of graphs and in naming graphs as well. The DI version of this is new.

Naming Remark 1 Homeomorphing is alias *subdividing* or *expanding* [5] in one direction. And *smoothing* or *series reduction* in the other. While [10] call HI trees *topological trees.*

Remark 3 In contrast, it is clear from definition 1 that even these topological trees all just collapse to a point in the study of DIs. This illustrates that, as a double quotienting notion, DIs shrink some kinds of graph arenas.

For all that (higher) quotients are also more likely to produce Mathematically harder arenas in ways other than overall size or measure. Such as inducing extra structural complexity. Or that quotients tend to kick one's objects out of their unquotiented case's category. For instance, quotienting manifolds by groups tends to produce non-manifolds, such as *orbifolds* or Whitney and Thom's notion of *stratified manifolds.* See [24] for an Order-Theoretic counterpart. In which for H and F quotients spoil incipient poset models of arenas of graphs by inducing forbidden triangles.

Aside 1 For comparison, some other notions of irreducible, atom and prime are as follows. In for instance respectively, Representation Theory. Next in Physics and Order Theory. And finally in Number Theory, its polynomial-factor counterpart, and various Topological analogues. All entail indivisibility, with some accompanying connotations of conceptual and structural primality.

1.3 Contents outline

In Sec 2, we present Sánchez' add-triangle modular lattice model for the DIs on ≤ 5 vertices. In Sec 3, we present the Author's alternative cone-sculpt general bundle model for the same objects. We structurally analyze each in Sec 4, as well as demonstrating that 5 is minimum for the two to be distinct.

2 The lattice model

2.1 The add-cycle operation for $N \leq 5$ simple graphs

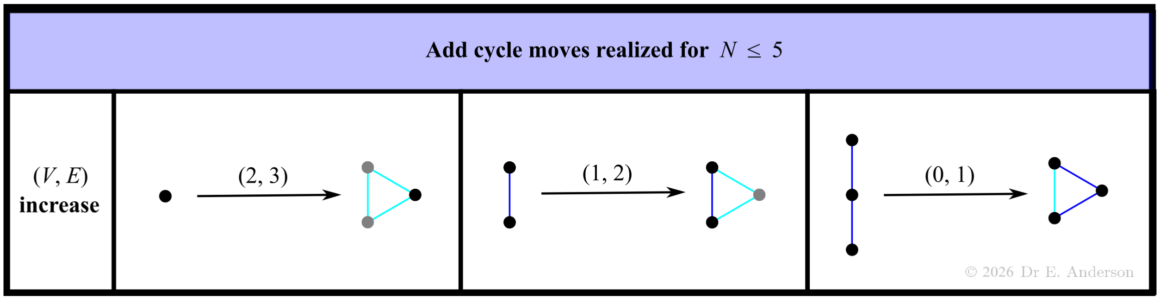

Figure 2:

Remark 1 For $N \leq 5$, a sufficient list of such moves is provided in Fig 2.

2.2 The ensuing lattice model

Structure 1 S. Sanchez' model is presented in Fig 1.a). This is a poset. That is a fortiori firstly up-down symmetric: reflection-symmetric about its horizontal midline. Secondly a lattice. And thirdly a modular lattice [13, 16].

It is not however a distributive lattice [13, 15]. For it contains a Didot alias $K_{3,2}$ subgraph (Fig 3 a). Which as a poset comes in the vertical 1 3 1 presentation (orange in this Figure). Being free from the C_5 subposet (Subfig b) suffices for modularity. While this and the preceding are the two forbidden subposets if a lattice is to be distributive.

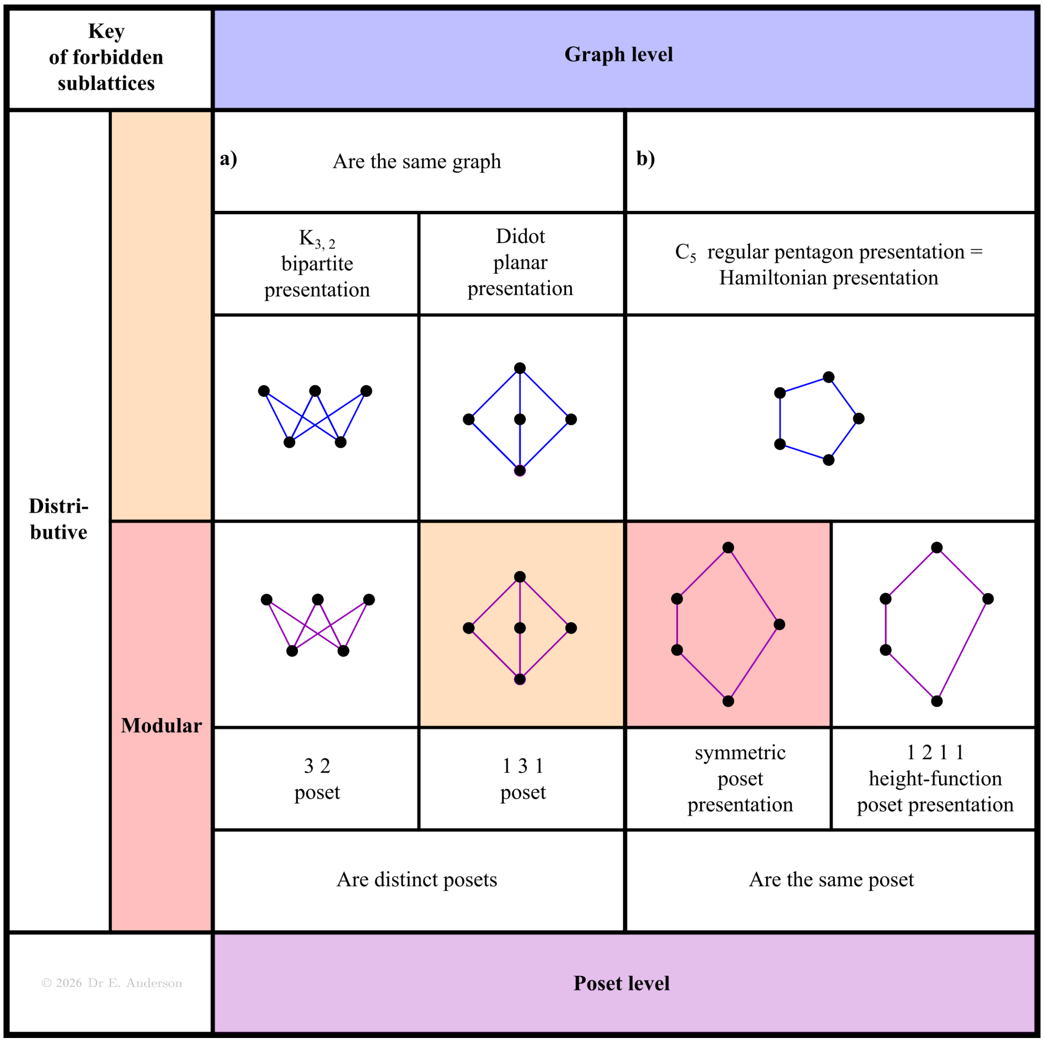

Figure 3:

Remark 1 Fig 4 shows how the natural edge-count depth function breaks the up-down symmetry. Relative to Fig 1.a).

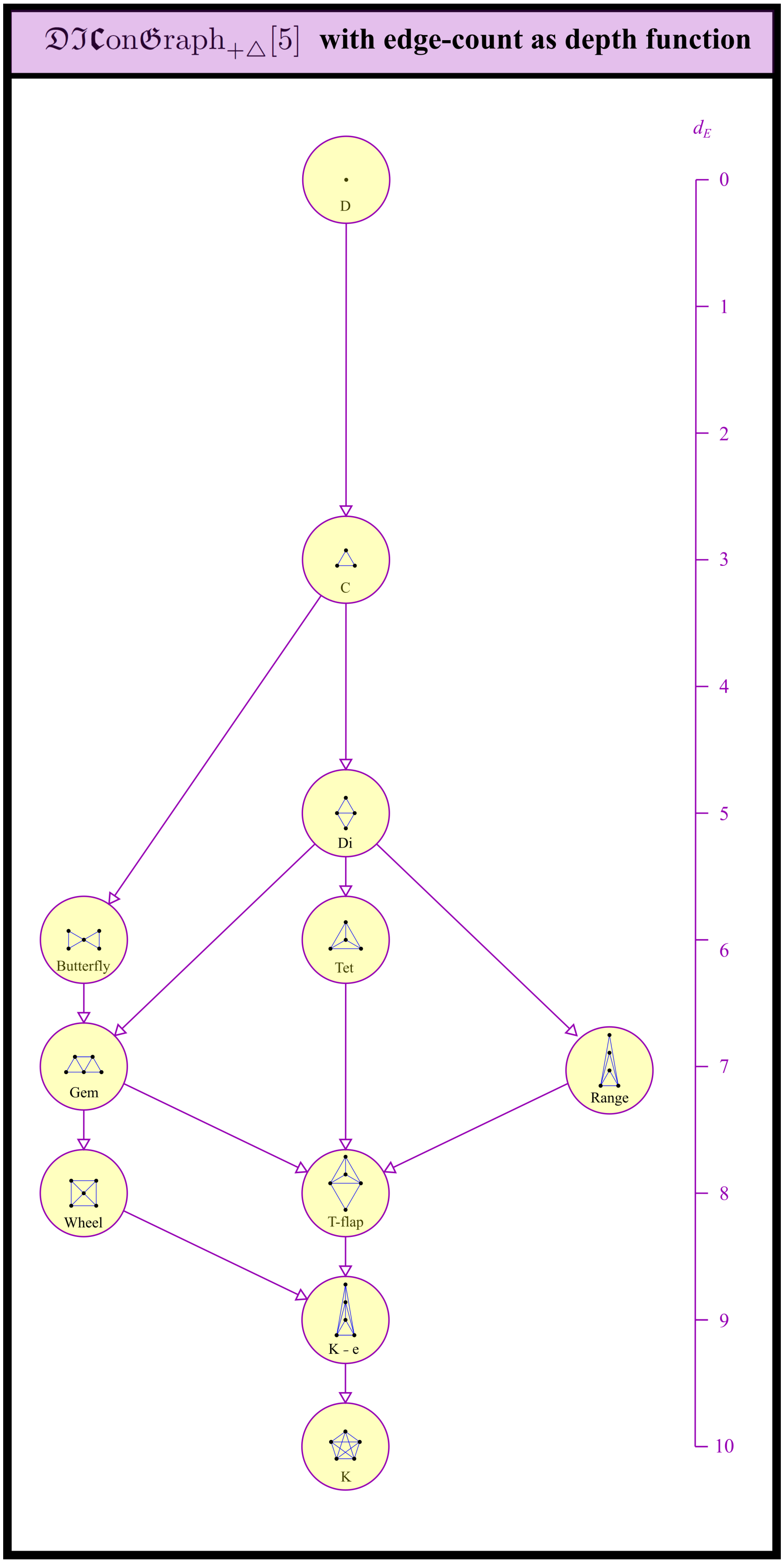
𝔇𝔗ℭon𝔊raph$_{+\triangle}$[5] with edge-count as depth function
d_E
0
1
2
3
4
5
6
7
8
9
10
D
C
Di
Butterfly
Tet
Gem
Range
Wheel
T-flap
K - e
K

Figure 4:

2.3 DI class analysis

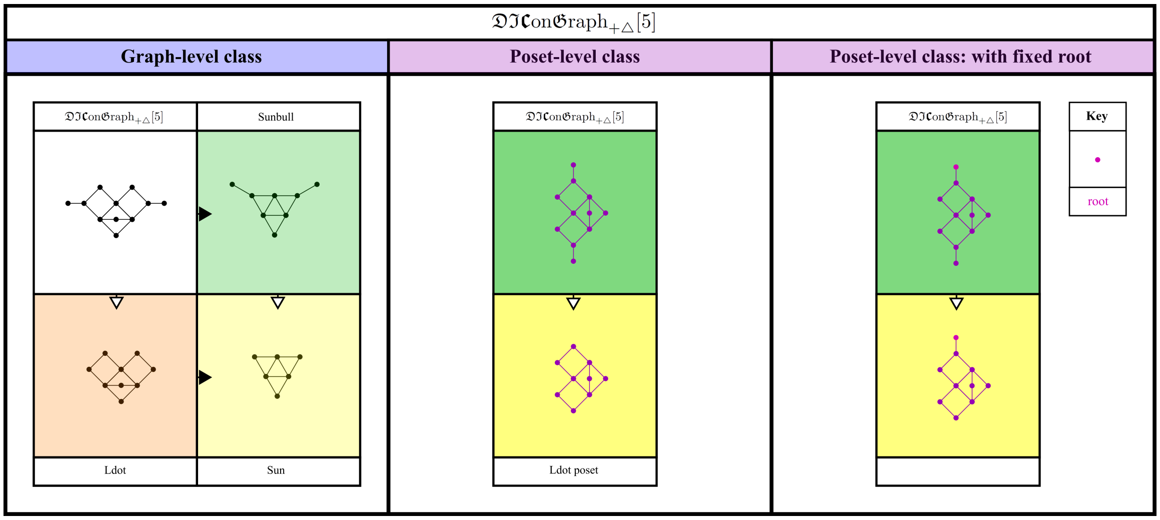

Figure 5:

Remark 2 The underlying graph is structurally analyzed in Fig 5.a).

Its HI is Sunbull .

Its FI is Ldot . I.e. the L-shaped grid alias first mixed-symmetry Young diagram [20]. With a further vertex –the dot – bolted into it. For all that there are 3 other ways in which this could be done.

Exercise 1$^-$ Draw these other ways out. Also explain how ours is 1 of 2 C_2-symmetric placings for the dot.

Remark 3 Finally its DI is the well-known Sun graph. This is one of the first few graphs derived from the equilateral triangle tessellation as fragments. It is furthermore the medial triangle graph [27].

2.4 Heat map

Remark 4 See Fig 6 for a heat map by vertex number for this model.

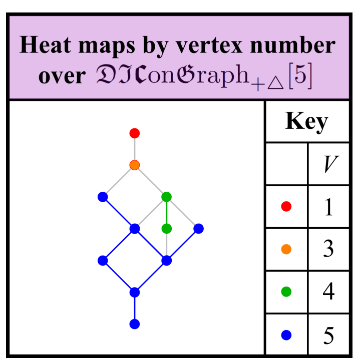

Figure 6:

3 The cone-sculpt general bundle model

3.1 The snout lemma

Remark 1 The Author chose instead to anchor a model on the complete graphs. This works because DIs fully populate a ‘densest snout’ for each N . [Other than $N = 2$, for which there is no DI at all.]

Snout Lemma $\mathrm{K}_n - [\ \leq (N - 2)$ edges are all DI. No matter which combination of edges are removed.

<u>Proof</u> $N - 1$ edge removals are required to sculpt down to a leaf. By picking all removed edges to emanate from a single vertex.

$N - 2$ edge removals do suffice to sculpt down to a degree-2 vertex by the same pick. But then no edge removals remain to free the configuration from the edge between its 2 neighbours. So our degree-2 vertex is necessarily trapped in a triangle. □

3.2 Examples

Example $N = 3$. Here the snout is just $\mathrm{K}_3 = \mathrm{C}_3 = \mathrm{C}$.

Example $N = 4$. Here it consists of the following. $\mathrm{K}_4 = \mathrm{Tet}$. And $\mathrm{K}_4 - e = \mathrm{Di}$ (more widely known as Diamond). See Fig 7.a).

Remark 2 There is subsequently always such a proxime in the arena of graphs and in our snout. Motivating the series notation $\mathrm{K}_N - e$

Example $N = 5$. Here it extends one edge-loss layer further, to the Wheel graph. Alongside the T-flap graph (previously alias Xhouse or Scotshouse [20]. See Subfig b).

Pointer 1 See [28] for the $N = 6$ snout.

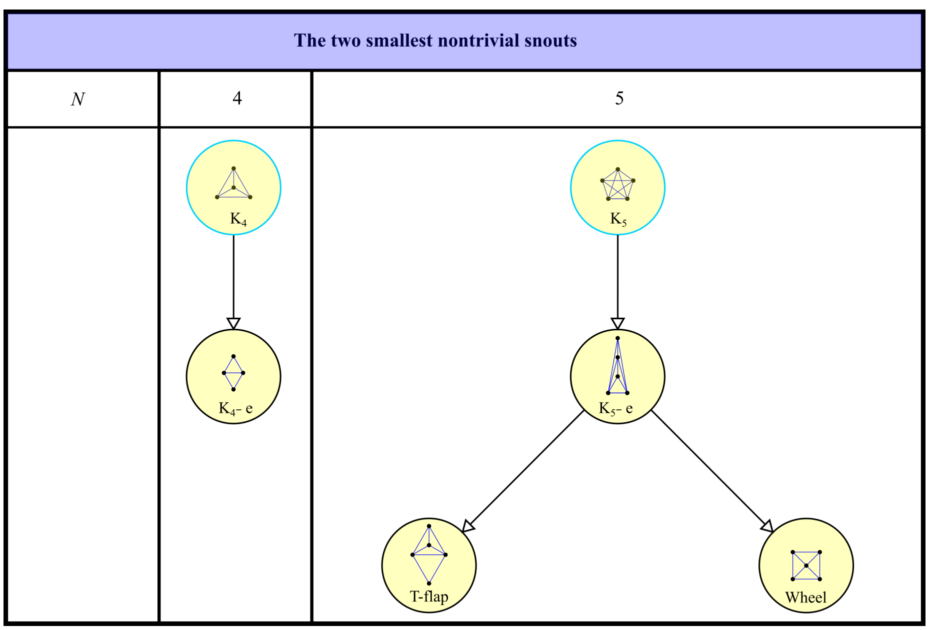

Figure 7:

3.3 The cone-sculpt model

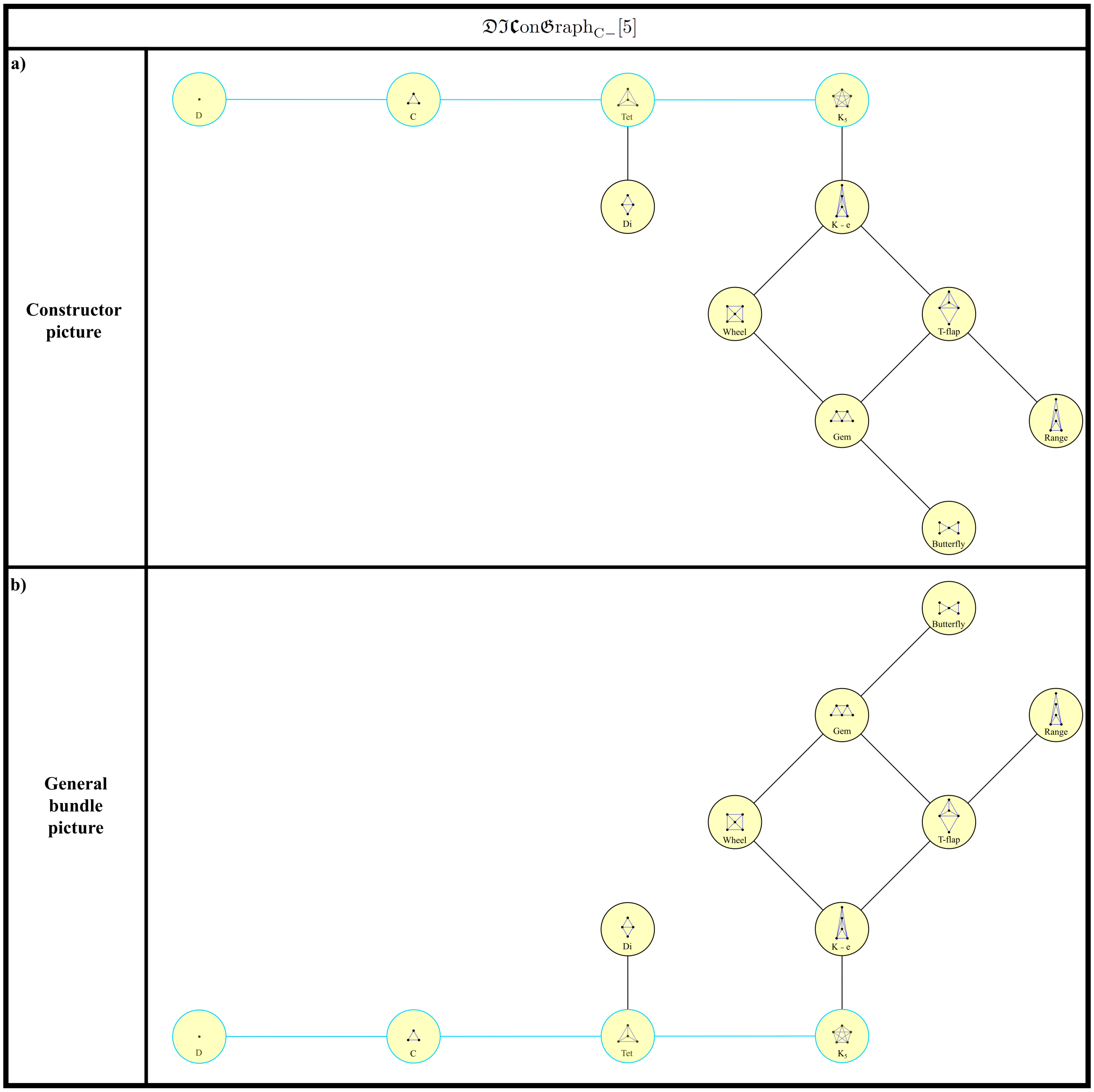

Figure 8:

Structure 1 The Author's approach is then to firstly construct all the complete graphs by serial coning [20]. With the one-off omission that $K_2 = P_2 = P$ is exceptionally not DI. We patch over this by allowing an exceptional double-coning move to jump from the point $Pt = D$ to $K_3 = C_3 = C$. Forming a chain of complete graphs. With the current Article omitting the ungraph as well, this is of length $n = N - 1$. Which is 4 for our main $N = 5$ case.

Structure 2 Secondly, to access the other DIs are accessed by serially removing edges: 'sculpting' [24].
To start off with, this is fully successful within the snout.

Past the snout's protective bound, some edge-removals maintain DI and some do not. For the purposes of the current Article, all $N \leq 5$ connected DI graphs can be found by serially removing edges in this way.

Pointer 2 For cases beyond this, [24]'s slightly more advanced *sculpting lemma* provides the guarantee that sculpting accesses all simple DI graphs.

Remark 3 Presenting this with the complete graph chain running horizontally along the bottom can be interpreted as a general bundle picture (Fig 8.b). With the complete graph chain as base space. And the graphs sculpted from a given complete graph as the elements of the object over that complete graph base space element.

These are not fibres since they are almost never isomorphic to each other. For instance, for $N = 4$, sculpting produces a 2-path $\mathrm{P}_2 = \mathrm{P}$. While for $N = 5$, it produces the 4-Sunlet minus a leaf. In fact, the only cases with isomorphic attached objects are $N = 1$ and $N = 3$, which each realize just a point.

In contrast, general bundles [9, 8] accommodate non-isomorphic attached objects.

3.4 Poset picture

Remark 4 This is provided in Fig 1

Remark 5 It is furthermore not a lattice, since it is finite. And finite lattices have unique top and bottom elements. But this poset contains 3 bottom elements!

3.5 DI class analysis

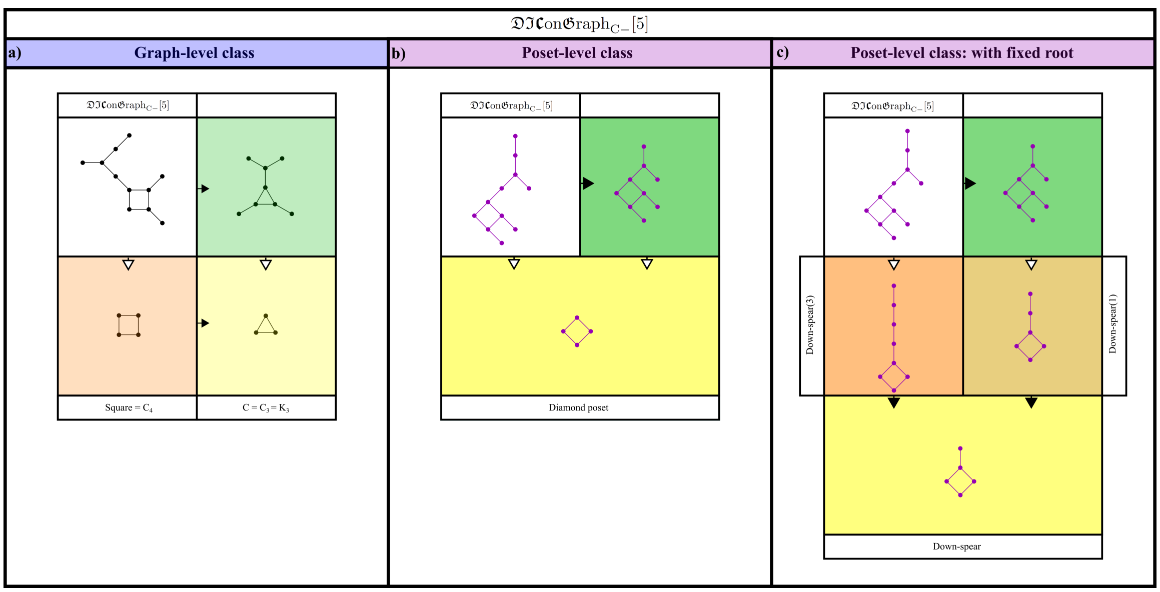

Figure 9:

Remark 6 The underlying graph is structurally analyzed in Fig 9.a).

Its homeomorph irreducible is the 2-tailed Snail.

Its foliation irreducible is just the square, C_4 .

Remark 7 And its double irreducible is just C .

3.6 Heat map

Remark 8 See Fig 10 for a heat map by vertex number for this model.

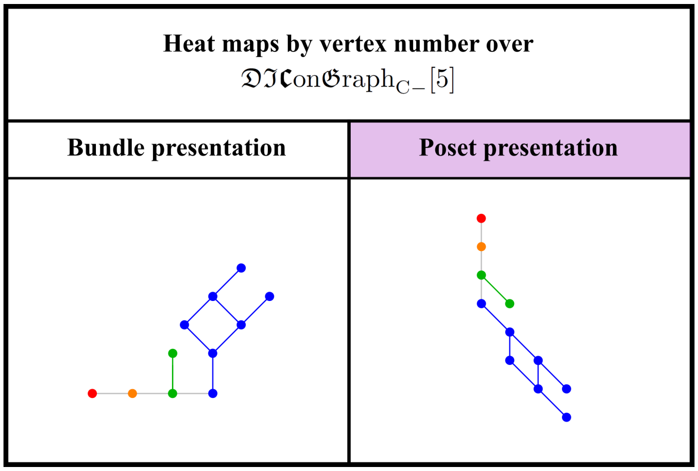

Figure 10:

4 Comparison of the 2 models

Advantage A.1 The above edge-sculpting bundle model A) is sparser than the add-triangle model, in the sense of having less edges. Which immediately translates to containing fewer elementary cycles. And in the cases considered in the current Article, contain just the 1 cycle.

Advantage A.2 A second benefit is trivialization of the heat map in A). In the sense that each base point and its fibre constitutes the totality of DI *SIT* graphs for a given value of N . I.e. a neat product space type separation (Fig 10), unlike in the cycle-adding order model (Fig 6) in S).

Advantage S.1 In contrast, S) entails further Order-Theoretic benefits. We claw one back by adapting our bundle into a poset. It does not however manage to be up-down symmetric. Or a lattice, because it is finite and with non-unique bottom.

Limitation S.1 S. Sánchez pointed out however that these niceties evaporate for $N \geq 6$. Details shall be reported in [29].

Advantage S.2 Also in S) up-down can be traded for a nice height function. While so far as the Author can tell, A) has but more contrived height functions.

Advantage A.3 A) also confers factorization benefits. In the sense that when treating a large number of cases, then gridding them into more manageable portions is advantageous where meaningful. S Sánchez' own sequel uses DI and then the HF combination of moves as a 2-d gridding. As such, it is reasonable to point out that the Author's model extends in this way to a 3-d gridding. And that the number of graphs grows steeply with N . By which for $N = 6$ or slightly more, a 3-d gridding remains practicable for slightly more N than a 2-d gridding.

Advantage A.4 Sculpting involves less types of move. $N = 5$ already involves 3 types of add-triangle. And [23] already pointed out that $N = 6$ DIs include elementary squares rather than just triangles.

Limitation A.1 Sculpting involves adding many edges and then removing a number of them. This is a 'U-turn', with some connotations of inefficiency. Both sides of this 'U-turn' are however methodical. The coning move is a regular bulk import of edges. While the sculpting other first gives a guaranteed snout. And then gives a systematic searching process for DI's transcending past this guarantee.

5 Pointers

Pointer 4 Cubic graphs are often further studied [2, 7, 22]. These are 3-regular: all vertices of degree 3. Thus no vertices of degree 1 or 2 are present. So the arena of cubic graphs is a subarena of the arena of DI graphs. However for now only the bottom element Tet is in play.

For complete $N = 4$ is minimum to realize a cubic graph.

And subsequently cubic graphs only exist for even N. This is by Euler's relation in degree expansion form,

$$2E = \sum_{i=1}^{N} d_i \ .$$

Which for cubic graphs becomes

$$2E = 3N \ .$$

So since 3 is not divisible by 2, N must be. Thus $N = 5$ adds nothing to this study.

In [28], we extend to $N = 6$'s cubic graphs. This still does not generate a structurally-significant arena. [11] follows cubic graphs for several further even N. One can infer that $N = 8, 10$ and 12 are minimum for a number of interesting examples. As well as to get up to size for the arena of cubic graphs to become structurally nontrivial in various ways. Within 1 to 2 years, small cubic graphs and corresponding arenas shall be covered in [26].

This is a simple example of a graph arena where the single add-edge operation is not an option. For this sends even-N to odd-N and vice versa. Nor is multiple use of add-edge intrinsic to the space of cubic graphs since it passes through intermediary odd-N graphs.

Pointer 5 Poset homeomorphs are distinct from graph ones. For instance, triangles are forbidden in posets, so homeomorphs sending 4-cycles to triangles are not poset homeomorphs. Also the idea of excepting the root from defoliation: it is a special labelled leaf that is not to be removed. This gives 2 poset parallels of the classification system in [23]. That triangles are forbidden in posets suffices to render it clear that the minimum cases for the posets are substantially different from those for the simple graphs.

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