Simple Graphs' 8 Double-Irreducibility Classes

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Abstract

We prove that the interplay between homeomorphing and defoliation partitions the arena of simple graphs into 8 classes. And provide the minimum examples of simple graphs in each class.

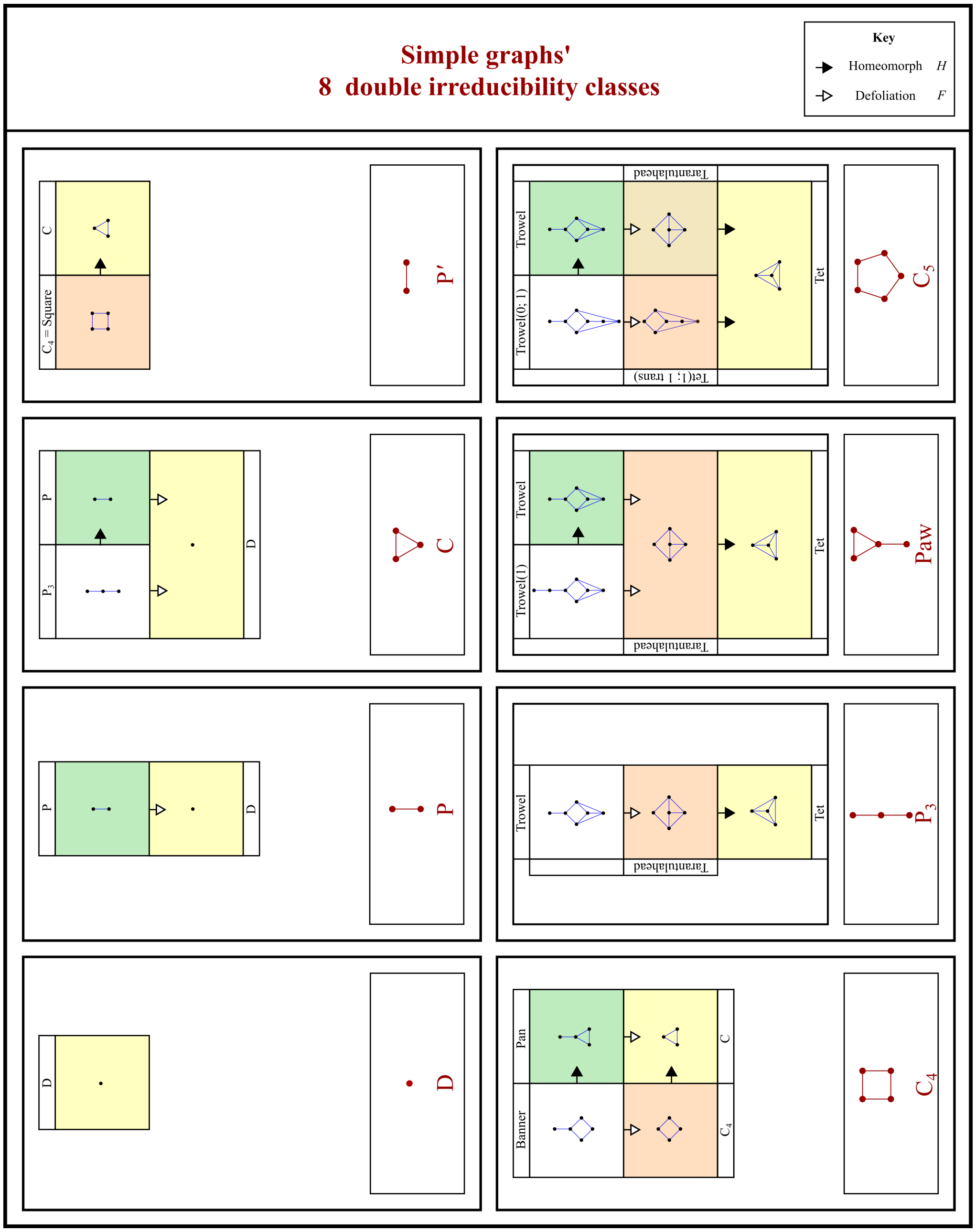

Figure 1:

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1 Introduction

1.1 Modelling assumptions

Structure 0 The current Article concerns *unlabelled simple graphs.* In the sense of consisting of unlabelled vertices, unlabelled edges between pairs of vertices, and nothing else. This 'nothing else' removes from contention for instance higher simplices and hypergraphs.

'Unlabelled edges' includes no edges being directed, oriented or arrowed in any way. By which our graph is not a di-graph, ori-graph or arrow-graph [15]. Whether in total: no graphs with all edges of one of these kinds. Or in some part: no graphs with any edges of any of these kinds: no *mixed graphs* [15] either.

Furthermore, the following restrictions on plain edges are to apply. A simple graph is not to contain any *loops*: edges from a vertex to itself. It is to have ≤ 1 edges between any given vertex pair: no *multi-edges.* It is thus neither to be a *looped graph* nor a *multi-graph* [15].

Definition 1 The *degree* of a vertex in a simple graph is the number of edges emanating from it.

1.2 2 notions of irreducible graph

Structure 1 Many Graph Theory [2, 5, 8, 6, 12, 10, 11] properties are unaffected by taking homeomorphs [15, 9].

Each *homeomorphing* operation replaces a simple graph's degree-2 vertex and the 2 edges leading out of it by a single edge. *Serial homeomorphing* continues this process until no further degree-2 vertices can be removed while staying within the simple graphs. We denote the corresponding homeomorphing operator by H. This process terminates at our incipient graph's *homeomorph ireducible* (*HI*) [3, 15]. Homeomorphing is alias *subdividing* or *expanding* in one direction, and *smoothing* or *series reduction* in the other.

Unaffected properties include those with forbidden subgraphs up to homeomorphs, such as *planarity* [8]. And some notions of connectivity [8, 15]. In which the difference between connection via an edge, and connection via an edge, then a vertex and then another edge, do not register. Such as *k-connectivity* [8].

Structure 2 Some Graph Theory properties are unaffected by defoliation [15].

Defoliation is the removal of ≥ 1 *leaves*: degree-1 vertices. *Serial defoliation* keeps on applying foliation until no leaves – original or emergent – remain. We denote the corresponding defoliation operator by F. This amounts to pruning off the entirety of every side-tree on the graph. Where *side-trees* are the obvious generalization to trees of side-chains. This process terminates at the graph's *foliation irreducible* (*FI*), alias *cycle system* [15].

Unaffected graph properties include, again, planarity. For side-trees can always be deformed so as to avoid incurring any crossings. And a smaller subset of notions of connectivity. For 'branching points' – degree ≥ 3 vertices – contribute to some notions of connectivity. And pruning off side-trees is indeed capable of decommissioning some of these...

1.3 Interplay between these notions

Motivation 0 The current Article investigates how homeomorphing and defoliation interact for simple graphs. The overall end-products are the *double-irreducibles* (*DI*=: graphs that are both HI and FI.

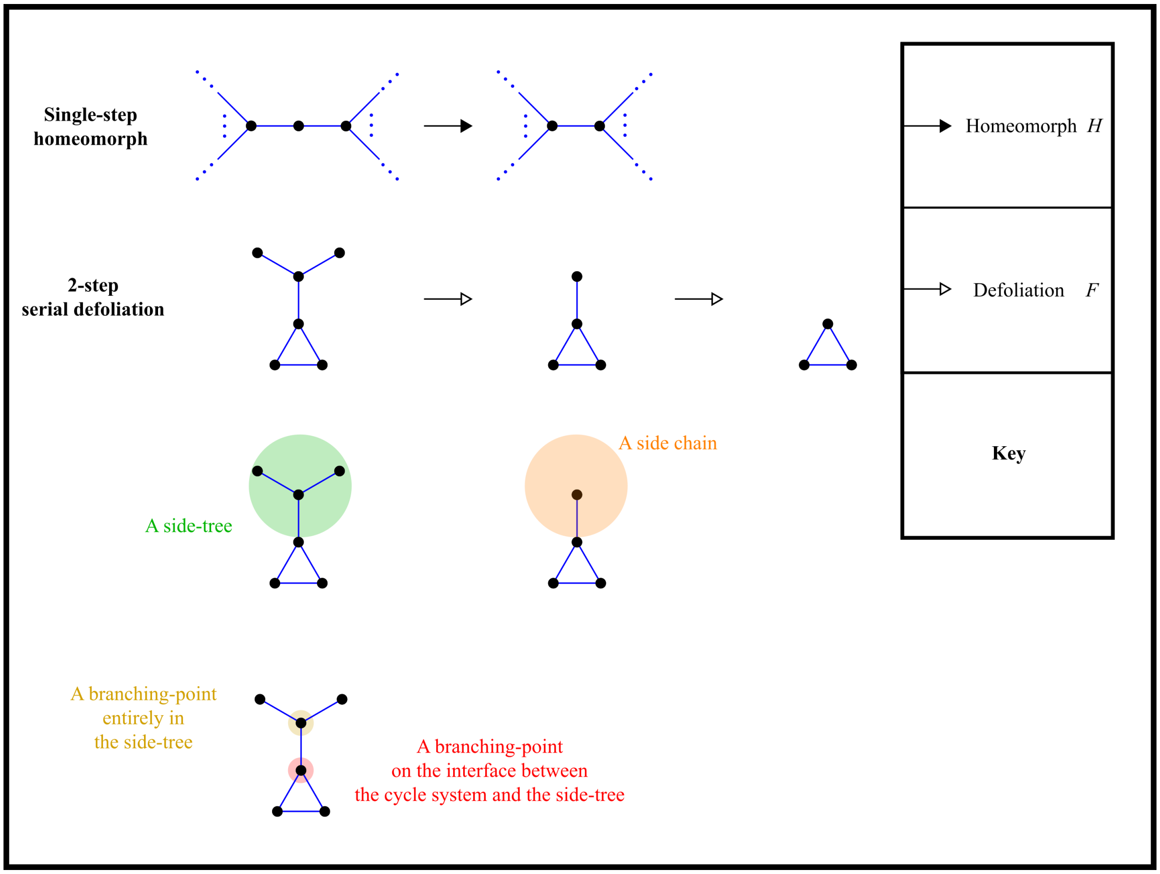


Figure 2:

Remark 1 In Sec 2.3, we prove that this interplay partitions the arena of graphs

$$\mathfrak{G}\text{raph}$$

into precisely 8 classes.

Motivation 1 Firstly, this provides us with a notion of genericity for simple graphs.

Motivation 2 Secondly, this funnels the study of the abovementioned latter aspects of simple graphs into a single class. I.e. the smaller arena of *DI graphs*,

$$\mathfrak{D}\mathfrak{I}\mathfrak{G}\text{raph} .$$

This is of some use in studying simple graphs of middling size. For instance, only a few sources [1, 4, 7, 14, 22] systematically study the 6-vertex graphs (6-graphs). Only the Graph Atlas [7] depicts and tabulates every 7-graph. And only the Graph Atlas provides a partial systematic study of the 8-graphs.

There are however various criteria under which arenas that are 1 to a few orders of magnitude smaller than the totality of p-graphs that remain systematically surveyable. Well-known ones include [7] trees, unicyclic graphs, asymmetric graphs, self-complementary graphs and regular graphs. The current Article's criterion provides a further such that [16] shall subsequently develop.

Remark 2 The current Article also provides the minimum graphs in each of the 8 classes (Sec 2.2). In the process, the 6-vertex *trowel graph* (Subfig 3.a) is the star of the show. Both directly and through its first homeomorphs (Appendix A). Its own DI graph is the tetrahedron graph Tet , alias the complete graph K_4 (Subfig b). Appendix B serves to establish that Sec 2.2's examples are indeed minimum. As well as documenting the extent to which each is uniquely minimum.

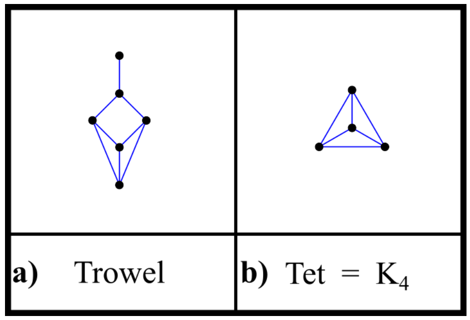


Figure 3:

1.4 Further motivation

Motivation 3 Homeomorphing and HIs were deployed in [15]'s structural analysis of graphs and of small arenas following from Basic Combinatorics that are modellable by graphs. Defoliation was separately used as well. Combined use has so far been limited: to more occasionally finding the cycle system of a significant graph and then its own HI. We henceforth wish to do this systematicaly. Including not only whichever of the HIs, cycle systems and HIs of the cycle systems that are distinct from the original graph. But also the class as per the current Article, and the class's diagram. Which aside from containing some subset of the distinct aforementioned cases generically includes a fifth structurally-significant subgraph as well (picked out with a brown background).

Motivation 4 Many small arenas following from basic Combinatorics, basic Linear Algebra [24, 21], basic Computer Science [16] and basic Geometry [23, 16] can be modelled by graphs. Many are moreover [13] furtherly usefully modelled by orders. We are thus also working on a poset counterpart of the current work [13], which shall make a practical debut with [21]. This increasingly useful nesting furthermore lies at the heart of why we have started an Online Encyclopaedia of Applied Graph and Order Theory [16].

2 DI Classification Theorem for Graphs

2.1 Statement

Theorem 1 Simple graphs are partitioned into precisely 8 DI classes, as follows.

$D, P, P', P_3, C_3, C_4, \text{Paw}$ and C_5 .

2.2 Small examples for each class

Remark 1 Let us first illustrate the 8 classes with small examples (Fig 1). Where we use green for HIs, orange for FIs, yellow for both, and white for initial graphs that are neither. The generic class' diagram C_5 uses a fifth colour: brown. This covers the defoliation of the HI in those cases in which this can be further homeomorphed.

2.3 Proof of the theorem

Proof I) We first exhaust the set of all possible words formed by concatenating H and F.

A) Both H and F are idempotent:

$$H^2 = H , \tag{1}$$

$$F^2 = F . \tag{2}$$

Thus all our words reduce to alternating strings of F and H.

B) F is furthermore *final* in our modelling context.

For the only other operator in play is H. And all this can do is remove degree-2 vertices. Thus it can never create any degree-1 vertices: leaves.

So suppose that we have applied F once so as to serially defoliate our graph. Then H has no means of producing any leaf. So the graph permanently remains defoliated. Overall, F's first action trivializes all subsequent actions by F. Thus

$$F \, (\text{ string containing } F \,) \;=\; (\text{ string containing } F \,) \, .$$

Consequently all distinct strings can be at most linear in F.

By A) and B) combined, ≤ 6 words are therefore possible. I.e.

$$I, \, F, \, H, \, H\,F, \, F\,H, \, H\,F\,H \, . \tag{3}$$

C) Claim: in all cases, the following holds.

$$H\,F\,H \;=\; H\,F \, . \tag{4}$$

For our graph G's set of degree-2 vertices $\mathfrak{a}_2$ partitions into the following. The set of $\mathfrak{S}_2$ those that can be removed solely by H. And the set of $\mathfrak{C}_2$ those that can be removed either by H or by F.

On the one hand, first applying F to G removes the set $\mathfrak{C}_2$. In the process, it is capable of creating a further set $\mathfrak{R}_2$, comprising the vertices whose degrees it reduces to 2. So subsequently applying H removes $\mathfrak{S}_2$ and $\mathfrak{R}_2$.

On the other hand, first applying H G removes $\mathfrak{C}_2$ and $\mathfrak{S}_2$. Next applying F creates the same $\mathfrak{R}_2$ as before. So finally applying H again, $\mathfrak{D}_2$ is excised.

Two paragraphs back, the sole H *would* remove $\mathfrak{C}_2$ as well, had this not already been annihilated by F. While one paragraph back, F *would* remove $\mathfrak{C}_2$, had this not already been annihilated by the first use of F. Finally each paragraph's uses of H eliminate whichever of $\mathfrak{S}_2, \mathfrak{C}_2$ and $\mathfrak{R}_2$ is at that stage present. So the end-product along either paragraph's path must be the same.

We are thereby down to ≤ 5 words: the first 5 in (3). Which constitute the truncation down to the inhomogeneous-quadratic combinations afforded by 2 idempotents. I.e. all terms up to quadratic order except the self products removed by idempotency. While maintaining distinction between the mutual products in either order.

D) In the Algebraically-generic case in which no further relations impinge, all 5 words are distinct. Forming the pentagonal Algebraic diagram: class C_5. Since no relations between these objects includes

$$0 \neq [\, F \, , \; H \,](\mathrm{G}) \;=\; F\,H(\mathrm{G}) \,-\, H\,F(\mathrm{G}) \, , \tag{5}$$

this corresponds to the generic pentagon of noncommutativity. Always within the current problem's special case that the closing operation must consist of 1 further use of H .

E) At the other extreme, it suffices to set

case 2)

$$F(\mathrm{G}) \;=\; \mathrm{G} \;=\; H(\mathrm{G}) \tag{6}$$

to obtain the maximum coincidence. Via

$$F\,H\,(\mathrm{G}) \;=\; F(\mathrm{G}) \;=\; \mathrm{G} \;=\; H(\mathrm{G}) \;=\; HF(\mathrm{G})\;.$$

This maximum coincidence clearly corresponds to the trivial-point Algebraic diagram: class D .

F) Case 2) and the following collectively exhaust.

Case 1)

$$F(\mathrm{G}) \;\neq\; \mathrm{G} \;=\; H(\mathrm{G})\;. \tag{7}$$

Case 1$'$)

$$F(\mathrm{G}) \;=\; \mathrm{G} \;\neq\; H(\mathrm{G})\;. \tag{8}$$

Case 0)

$$F(\mathrm{G}) \;\neq\; \mathrm{G} \;\neq\; H(\mathrm{G})\;. \tag{9}$$

Case 1$'$) $\Rightarrow$

$$H\,F(\mathrm{G}) \;=\; H(\mathrm{G})\;. \tag{10}$$

So at most $\mathrm{G}, H(\mathrm{G})$ and $F\,H(\mathrm{G})$ are distinct. But F is final, so $F(\mathrm{G}) = \mathrm{G} \Rightarrow$

$$F\,H(\mathrm{G}) \;=\; H(\mathrm{G})\;. \tag{11}$$

Thus class P' is forced.

Case 1) $\Rightarrow$

$$F(\mathrm{G}) \;=\; F\,H(\mathrm{G})\;. \tag{12}$$

So at most $\mathrm{G}, F(\mathrm{G})$ and $H\,F(\mathrm{G})$ are distinct.

The following then collectively exhaust.

Subcase i)

$$H\,F(\mathrm{G}) \;=\; F(\mathrm{G})\;. \tag{13}$$

Subcase ii)

$$H\,F(\mathrm{G}) \;\neq\; F(\mathrm{G})\;.$$

For subcase i), class P immediately follows. While for subcase ii), class P_3 immediately follows.

Case 0) forces ≥ 3 words. There are

$$\binom{4}{2} - 1 = 5$$

possible relations to check out. Which we denote below by a) to e).

This count corresponds to picking 2 out of 4 non-identity objects to equate. While precluding $F(\mathrm{G}) = H(\mathrm{G})$. For this enforces no leaves. Thus $F(\mathrm{G}) = \mathrm{G}$. And so we have transcended into case 2). While without any of these relations, we remain within the Algebraically-generic C_5 class.

Relation a) is (13). But then $H(\mathrm{G}) = \mathrm{G}$ is forced. Contradiction!

Relation b) is (10) But then $F(\mathrm{G}) = \mathrm{G}$ is forced. Contradiction!

Relation c) is (10).

$$\Rightarrow \; H\,F(\mathrm{G}) \;=\; H\,F\,H(\mathrm{G}) \;=\; H^2(\mathrm{G}) \;=\; H(\mathrm{G}) \,.$$

Which is yet another contradiction due to relation c).

Thus one or both of the following must hold. [If neither holds, then we are back to the above Algebraically-generic case: class C_5 .]

Relation d) is (12).

Relation e) is

$$H\,F(\mathrm{G}) \;=\; F\,H(\mathrm{G}) \;\Leftrightarrow\; [\,F\,,\;H\,](\mathrm{G}) \;=\; 0 \;:\; \text{ commutativity} \,. \tag{14}$$

When relation e) alone holds, the commuting square Algebraic diagram ensues, returning the C_4 class.

When relations d) and e) hold together, we are down to the commuting-triangle Algebraic diagram: class C .

Finally, when relation d) holds alone, $H\,F(\mathrm{G})$ is a further distinct state. So applying H to the bottom state of the arrowed triangle subgraph, we arrive at the Paw class.

II) The Reader is welcome to check that in all branches of the above argument in which (4) is not already given a part, it collapses to a mere identity. (**Exercise 1**!) Thus this relation does not wipe out, or further qualify, any of our 8 classes.

III) Finally, as regards existence of all 8 of our classes, Fig 1 constitutes a constructive proof that all are non-empty. □

2.4 The above examples are minimum

Remark 2 Appendix B establishes that Fig 1's examples are furthermore minimum. And that all bar the last are unique minimum examples; the last carries instead a 3-fold multiplicity. Since this working makes use of the small homeomorphs of Tet and Trowel , we place a pictorial prelude to these in Appendix A.

Pointer 1 See [18, 20] for first studies of arenas of DIs, and of simple graphs as centred about DIs.

Pointer 2 And [19] for rooted-graph, digraph and rooted-digraph counterparts of the current Article.

Acknowledgments We thank S. Sánchez for discussions, inspiration, and hope. K. Everard for proofreading an earlier version of the manuscript. And the Applied Combinatorics and Topology Discussion Group's members.

A Small homeomorphs of Tet and Trowel

Remark 1 See Fig 4 for Tet and Fig 5 for Trowel .

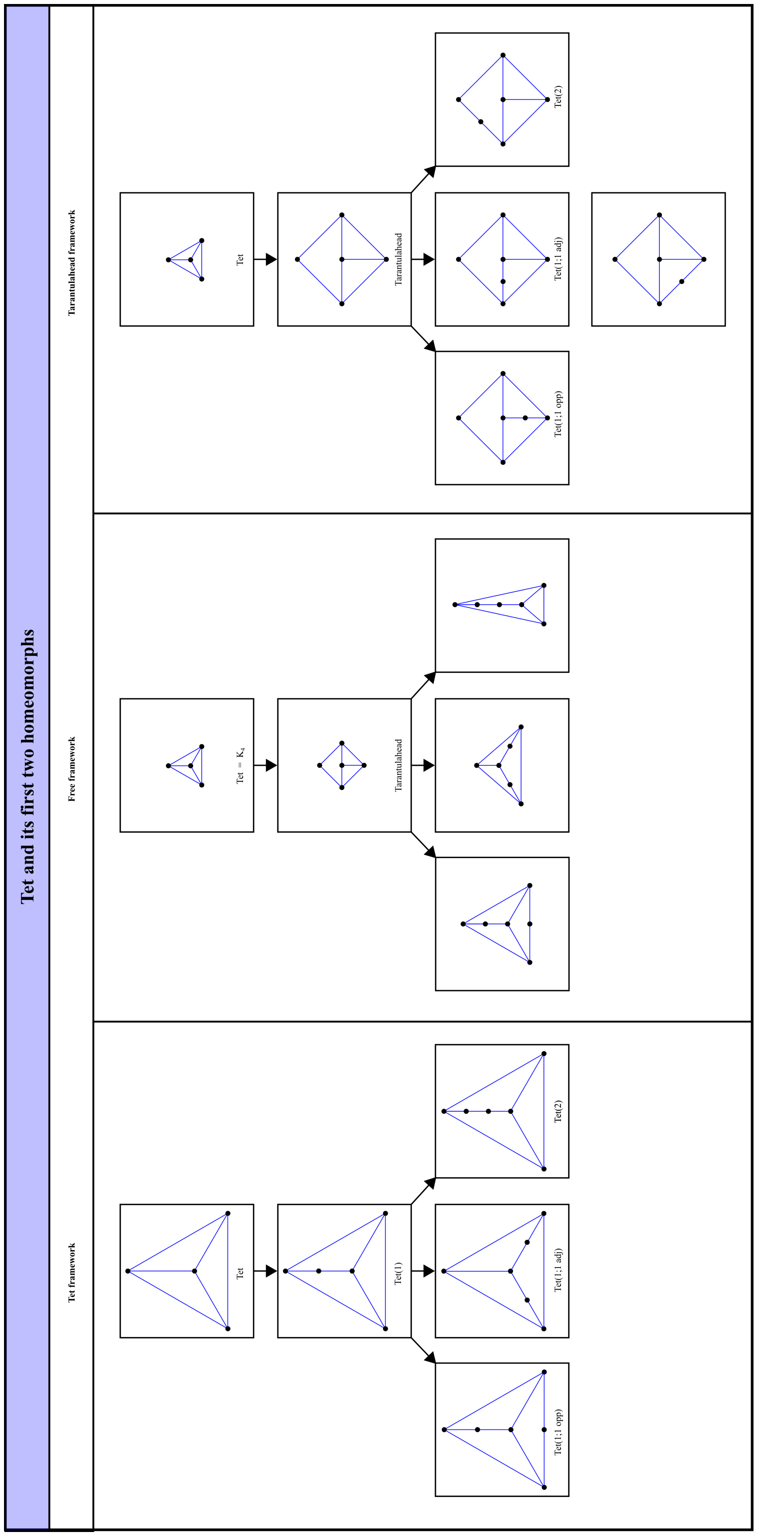


Figure 4:

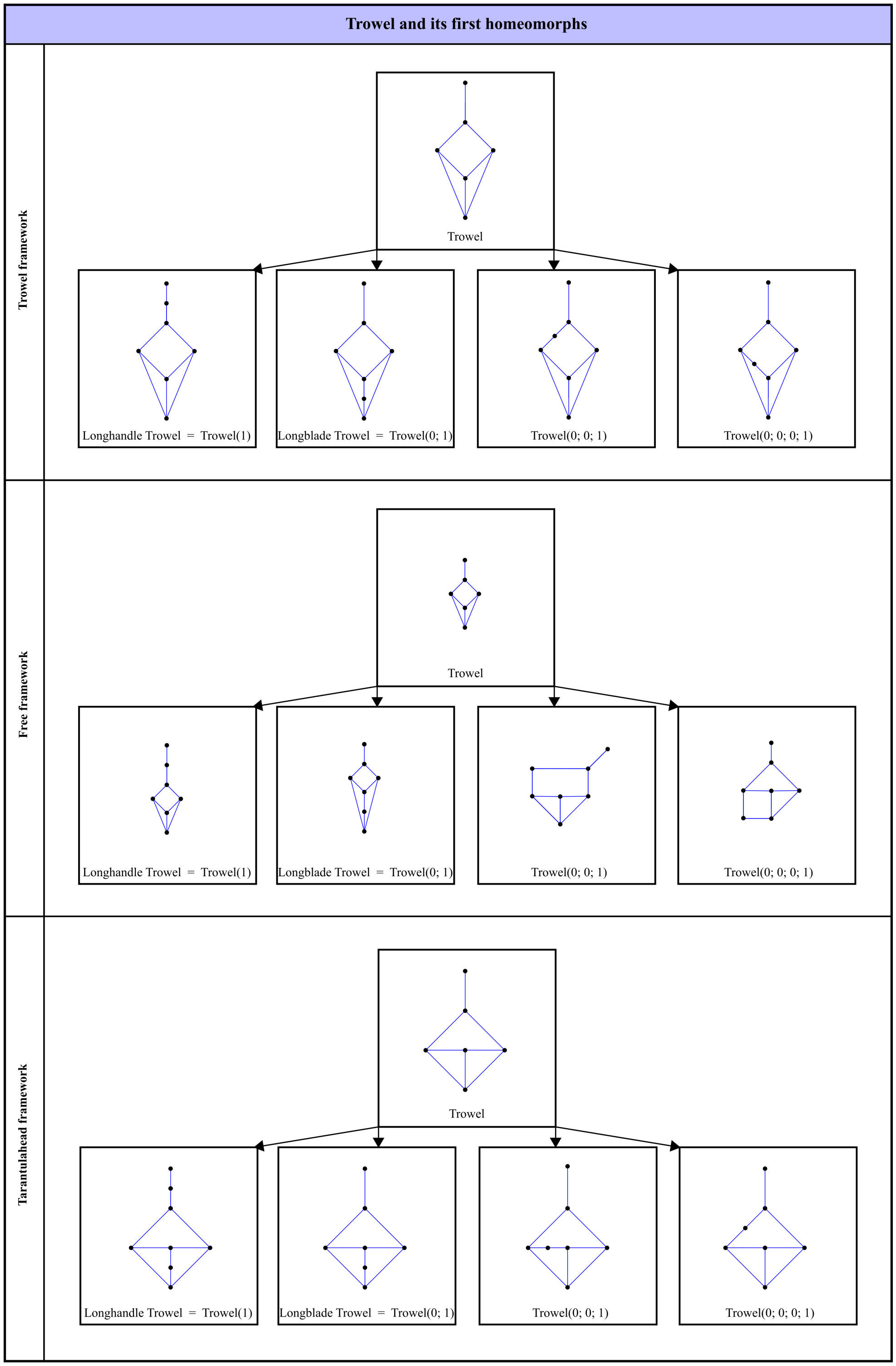


Figure 5:

Remark 2 We provide multiple presentations for each, as part of starting to build in [11]-level graph drawing and visualization into the current Encyclopaedia. See [17] for a systematic account of the various layers of notation for graphs used in the current Article.

B (Extent of) uniqueness

Figure 6:

Remark 1 We first sweep through the connected simple graphs in 0 to 5 vertices in Fig 6.

The smallest double-irreducible graphs, with some simple minimum properties

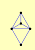
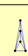
V	E	F when planar	Name	Doubly-irreducible graph	Degree sequence	Regularity	Triangle subgraph	2-d unit distance	Strong triangulation	Weak triangulation	Total stellar subdivision	Partial stellar subdivision	Compositionally weaker	Cut-vertex	Maximally-planar	Non-outerplanar	Non-planar
0	0	0	U	ϕ	ϕ	(0)									Separately trivially so		
1	0	0	D	·	0	0									Separately trivially so		
2 Only void V																	
3	3 Minimum to be edged	3	$C = C_3 = K_3$		2^3	2									Minimum nontrivial such		
4 Minimum not to be determined by just V	5	5	$Di = K_4 - e$		$2^2 3^2$	Minimum irregular									Minimum nonmaximally planar		
	6 Minimum not to be determined by just E	6 Minimum not to be determined by just E	Tet = K_4		3^4	3 = cubic: the minimum nontrivial kind		Minimum non-2-d unit distance								Is outerplanarity's minimum forbidden subgraph	
5			Butterfly		$2^4 4$												
	7 Minimum not to be determined by just V, E	7 Minimum not to be determined by just V, E	Range		$2^3 4^2$											Minimum to contain outerplanarity's only other forbidden subgraph, $K_{2,3}$	
			Gem		$2^2 3^2 4$												
	8	8	Wheel		$3^4 4$											Minimum not to be determined by just V, E	
			T-flap		$2\,3^2 4^2$												
	9	9	K_5 - e		$3^2 4^3$												
	10	10	K_5		4^5	4											Is planarity's minimum forbidden subgraph

Figure 7:

Remark 2 We next tabulate the DI connected simple graphs on 0 to 5 vertices in Fig 7. See eventually [22] for a 6-graph sequel.

Remark 3 Having found 5 of the DI classes, we look into where the remaining 3 are hiding among the slightly larger graphs (Fig 8). We exhibit a mechanism by which a graph is placed in the P_3 class. And 2 more for how adding in to the preceding a single vertex by homeomorphing yields Paw class and C_5 class graphs.

The P_3 mechanism requires a $\geq$ 4-cycle that is protected from homeomorphing. And has a side-tree 'fuse'. The defoliation of which removes the protection. To minimumly realize this, take specifically a single component, a 4-cycle, and a single leaf for its fuse. This uniquely picks out Trowel on 6 vertices as minimum example.

The 2 further mechanisms are then minimumly realized by its first homeomorphs. One uniquely, and the other as a triplet.

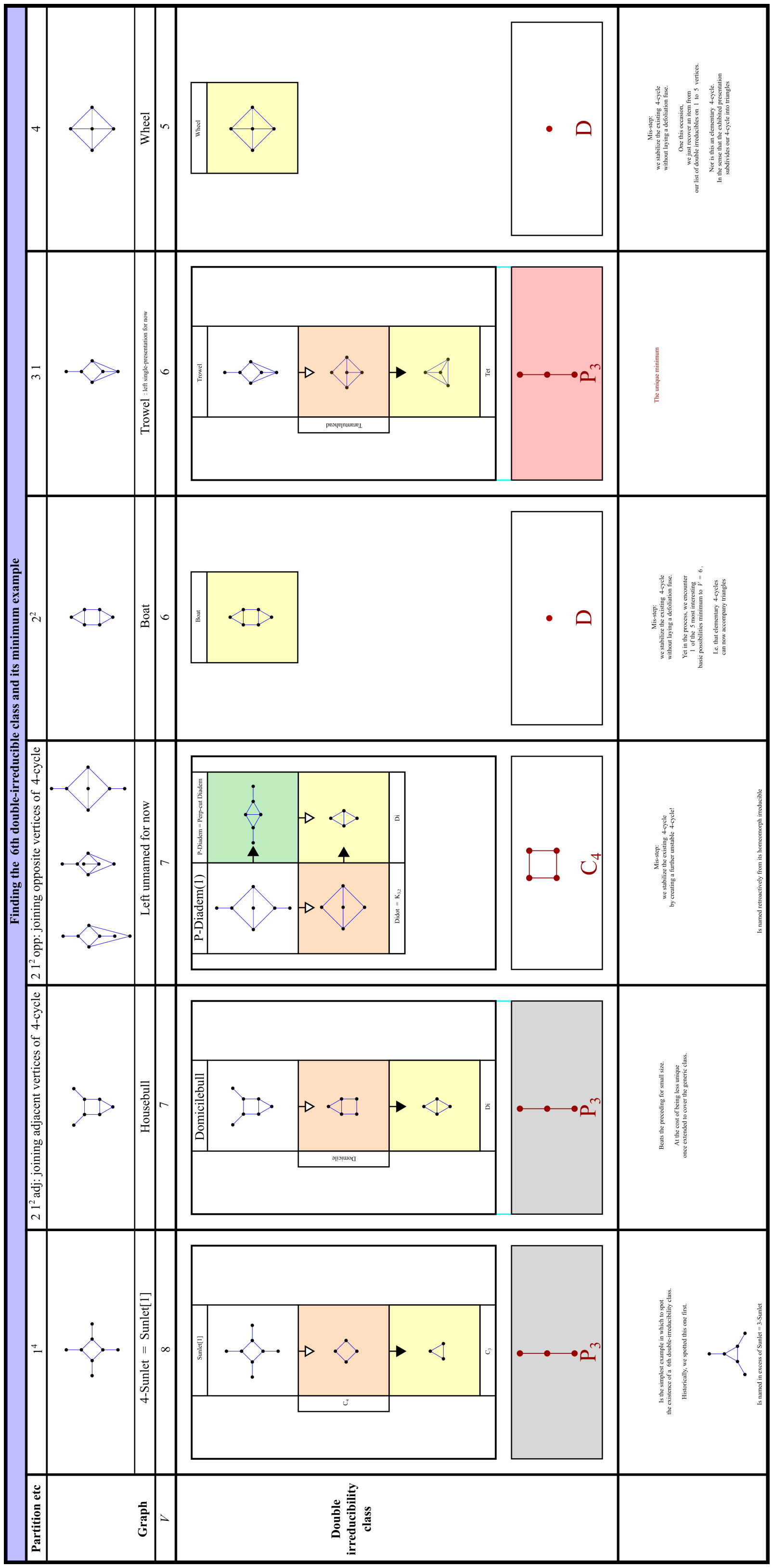

Finding the 6th double-irreducible class and its minimum example
Partition etc
1⁴
2 1² adj: joining adjacent vertices of 4-cycle
2 1² opp: joining opposite vertices of 4-cycle
2²
3 1
4
Graph
4-Sunlet = Sunlet[1]
Housebull
Left unnamed for now
Boat
Trowel : left single-presentation for now
Wheel
V
8
7
7
6
6
5
Double irreducibility class
Sunlet[1]
C₄
C₃
Domicilebull
Domicile
Di
P-Diadem(1)
P-Diadem = Perp-cut Diadem
Didot = K₁,₃
Di
Boat
Trowel
Tarantulahead
Tet
Wheel
P₃
P₃
C₄
D
P₃
D
Is the simplest example in which to spot
the existence of a 6th double-irreducibility class.
Historically, we spotted this one first.
Is named in excess of Sunlet = 3-Sunlet
Beats the preceding for small size.
At the cost of being less unique
once extended to cover the generic class.
Mis-step:
we stabilize the existing 4-cycle
by creating a further unstable 4-cycle!
Is named retroactively from its homeomorph irreducible
Mis-step:
we stabilize the existing 4-cycle
without laying a defoliation fuse.
Yet in the process, we encounter
1 of the 5 most interesting
basic possibilities minimum to V = 6 .
I.e. that elementary 4-cycles
can now accompany triangles
The unique minimum
Mis-step:
we stabilize the existing 4-cycle
without laying a defoliation fuse.
One this occasion,
we just recover an item from
our list of double irreducibles on 1 to 5 vertices.
Nor is this an elementary 4-cycle.
In the sense that the exhibited presentation
subdivides our 4-cycle into triangles

Figure 8:

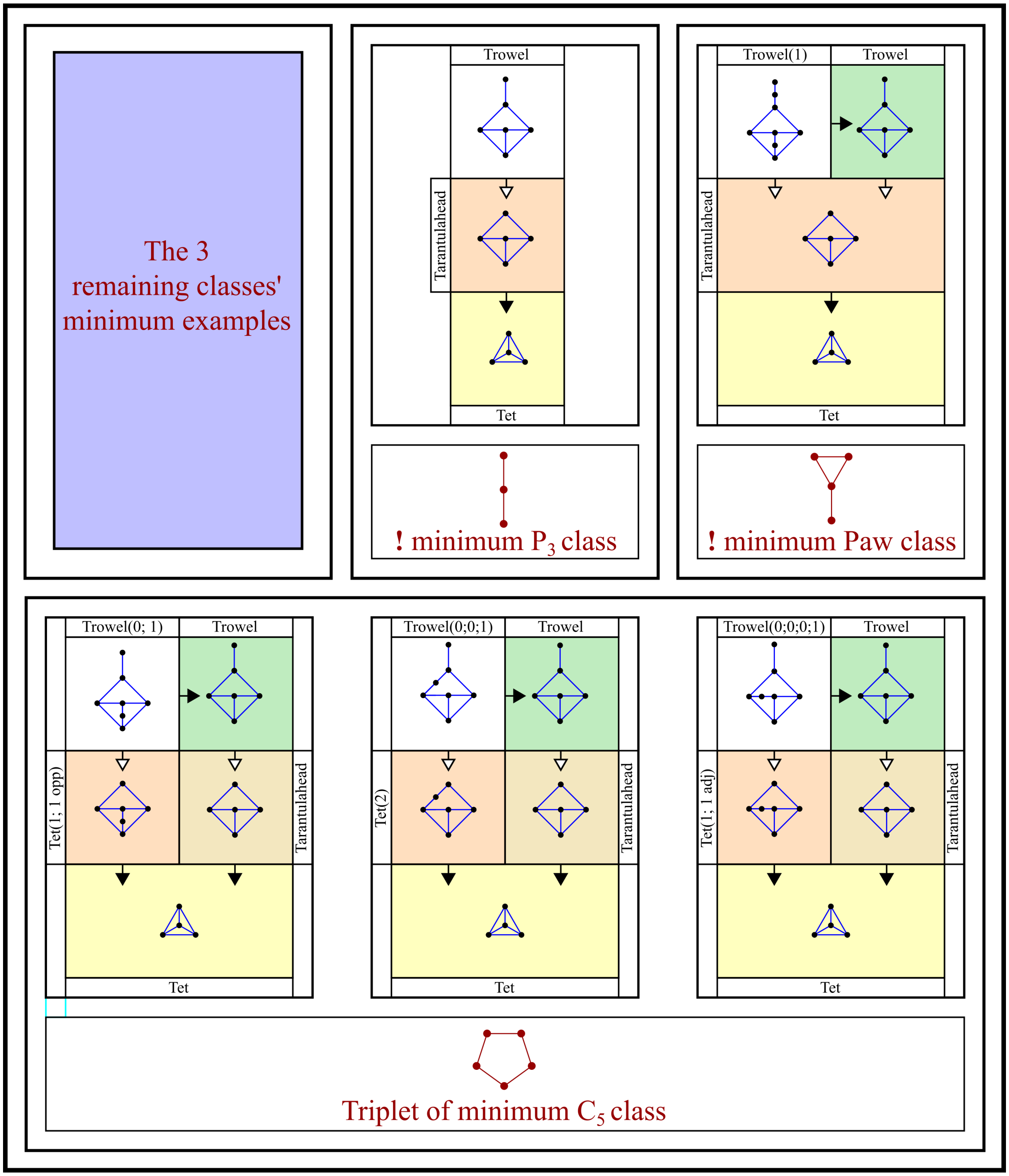


Figure 9:

Exercise 2 Use Fig 8's Sunlet[1] and Domicilebull to find further examples of the C_5 and Paw classes. Documenting in the process the extent to which the smallest such built upon these chassis are unique.

References

[1] F. Harary, *Graph Theory* (Addison-Wesley, Reading, MA 1969).

[2] R.J. Wilson, *Introduction to Graph Theory* (Longman, Edinburgh 2010). The first edition dates to 1972.

[3] F. Harary and E.M. Palmer, *Graphical Enumeration* (A.P., New York 1973).

[4] D. Cvetković and M. Petrić, *A Table of Connected Graphs on Six Vertices*, Discrete Math. **50** 37 (1984).

[5] V.K. Balakrishnan, *Graph Theory* (McGraw–Hill, New York 1997).

[6] B. Bollobás, *Modern Graph Theory* (Springer, New York 1998).

[7] R.C. Read and R.J. Wilson, *An Atlas of Graphs* (O.U.P., New York 1998).

[8] D.B. West, *Introduction to Graph Theory* (Prentice–Hall, Upper Saddle River N.J. 2001).

[9] J. Yellen and J.L. Gross, *Graph Theory and Its Applications* 2nd ed (Chapman and Hall, Boca Raton Fl. 2005).

[10] *Handbook of Graph Theory* 2nd ed., ed. J.L. Gross, J. Yellen and P. Zhang (Chapman and Hall, Boca Raton Fl. 2014).

[11] *Handbook of Graph Drawing and Visualization* ed. R. Tamassia (Taylor & Francis, Boca Raton, Fl. 2014).

[12] R. Diestel, *Graph Theory* 5th ed. (Springer, Berlin 2017).

[13] Combinatorial discussions between S. Sánchez, E.A. and A.F. (2017-2020).

[14] K. Clancy M. Haythorpe and A. Newcombe "A Survey of Graphs with Known or Bounded Crossing Numbers", Australasian J. Combi. **78** 209 (2020).

[15] E. Anderson, *Applied Combinatorics*, Widely-Applicable Mathematics Series. A. Improving understanding of everything with a pinch of Combinatorics. **0**, (2022). Made freely available in response to the pandemic here: institute-theory-stem.org/combinatorics/ .

[16] Online Encyclopaedia of Applied Graph and Order Theory, institute-theory-stem.org/online-encyclopaedia-of-graphs-and-orders/ .

[17] A. Ford, "The smallest simple graphs: naming and notation systems on ≤ 5 vertices", forthcoming (2026), for [16].

[18] E. Anderson, "Smallest Nontrivial Arenas of Double-Irreducible Simple Graphs", in [16] (2026).

[19] "Rohomeo and Rofeulliette. And other easy variants of Double-Irreducible classes", in [16] (2026).

[20] "Arenas of Simple Graphs on at most 4 Vertices", in [16] (2026).

[21] E. Anderson and K. Everard (she/her), "Eigentheory of Real-Symmetric Matrices from a Graph and Order Theory Point of View", forthcoming (2026).

[22] E. Anderson and A. Ford, *The* 6-*Graphs*, forthcoming (2026), for [16].

[23] E. Anderson, *The Structure of Flat Geometry*, in some places called additionally *Widely-Applicable Mathematics. A. Improving understanding of everything by dual-wielding Combinatorics and Linear Algebra.* **3**, forthcoming 2026. Expanded Geometrical Graph material is being assembled here: institute-theory-stem.org/geometrical-graphs/

[24] *Linear Mathematics. Vector Spaces, Linear Methods and Tensors throughout STEM.* Widely-Applicable Mathematics Series. A. Improving understanding of everything with a pinch of Combinatorics **1**, forthcoming 2027. Homepage with free sample: wordpress.com/page/conceptsofshape.space/3406