The Fano Graph: Properties and some Presentations

Edward Anderson* -

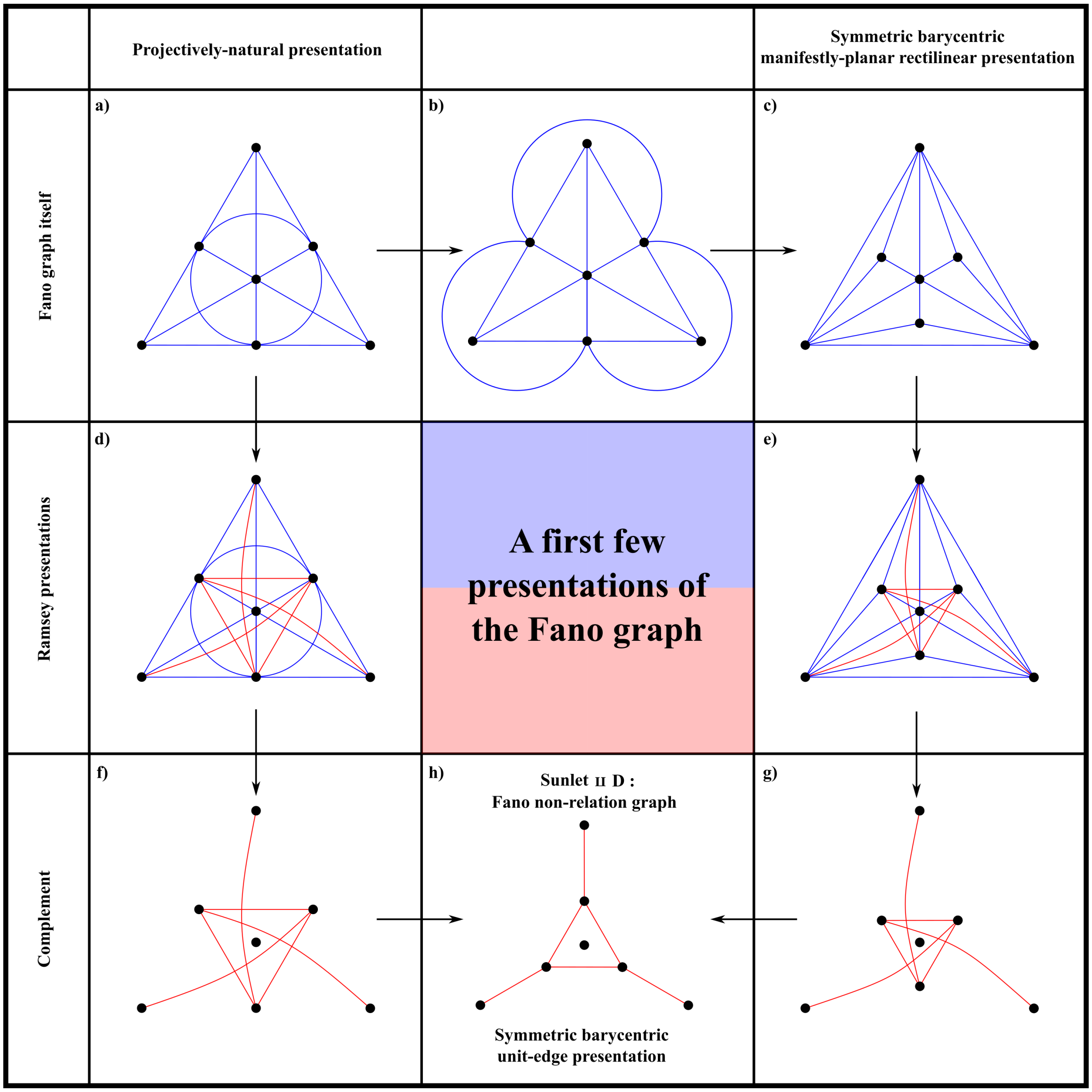

Figure 1:

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1 The Fano configuration

1.1 Projective planes

Subject 0 *Projective Geometry* [25, 11, 12, 6, 8, 15] is the study of *incidence*. At the level of planes, this is a binary relation on points-and-lines.

Structure 0 Axioms for a *projective plane* are as follows [21].

Projective Plane 0 Any 2 distinct points are incident with ('lie in') a unique line.

Projective Plane 1 Any 2 distinct lines are incident at ('intersect at') a unique point.

Projective Plane 2 ≥ 4 points are to be present such that no 3 of them are collinear.

Remark 1 0) and 1) form a *projectively-dual* pair of axioms. For planes, this is in the sense of exchanging points and lines. More generally, this aspect becomes dimension and codimension dependent. Notions of join and meet are also to be interchanged.

Naming Remark 1 The above 'lie in' amounts to being joined: by a line. While the above 'intersect' is a notion of meeting: at a point. Using 'are incident' in place of whichever of these automatically builds in the join-and-meet dual. 'Lines-and-points' can then be interpreted as a concomitant automatically-dual phrasing in the case of planes.

Remark 2 In contrast, 2) gives content to notions of plane exceeding mere notions of line. For without this, one would still be axiomatizing a Projective structure, just one that is a Projective line, rather...

1.2 Smallest projective plane route to the Fano plane

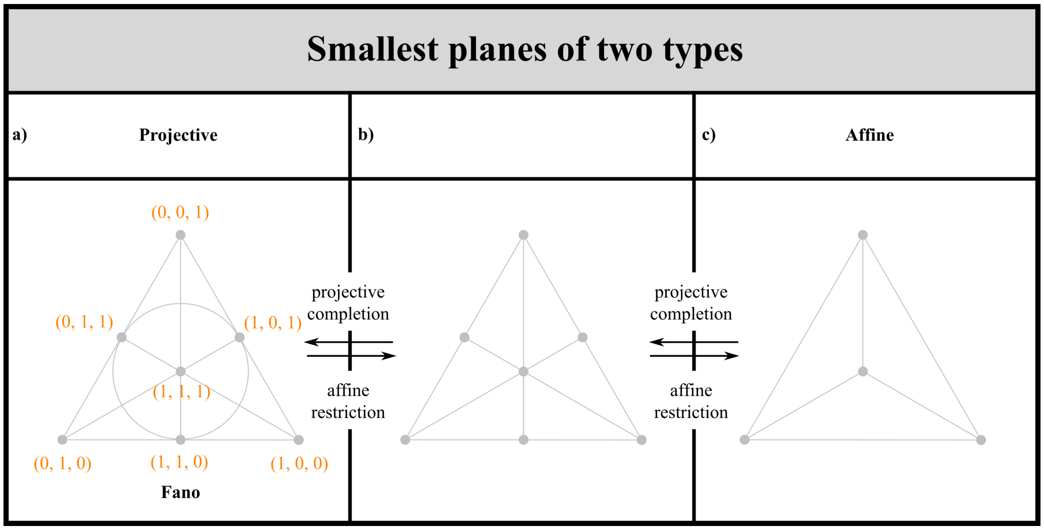

Figure 2:

Route 0 A zeroth route to the Fano plane [5, 7, 21, 28] (Fig 2.a) is setting out to find the smallest *Projective plane*.

Remark 3 The circular arcs here represent line segments that are no different from this Subfig's others.

Remark 4 The labels here are *homogeneous coordinates*: Projective Geometry's analogue of Cartesian coordinates. Homogeneous coordinates however exceed the space in question's dimension d by 1 in number. The interconnection is that one is to form ratios out of homogeneous coordinates. And to have d independent ratios, we require $D := d + 1$ quantities [13]! So the 2-d real-projective plane has 3 independent homogeneous coordinates to the 2-d real plane having 2 independent Cartesian coordinates.

For the Fano plane, these are binary-valued. $(0, 0, 0)$ is not however among them. For no ratios can be defined between a set of numbers consisting of just zeros! Modifying the count of binary triples from

$$2^3 = 8$$

to

$$8 - 1 = 7 = |\,\text{Fano}\,| \,.$$

1.3 Completing the smallest affine plane route to the Fano plane

Subject 1 *Affine Geometry* [9, 18, 16] is the study of *parallelism*. Historically this largely preceded Projective Geometry, from Euclid's parallel postulate to Euler's 18th century abstraction of Affine Geometry. In contrast, while the first Projectively significant result was found by Pappus in the 4th century, this did not arise in a Projective context. The next were found by Desargues and Pascal in the 17th century. But synthesis into a coherent picture of Projective Geometry had to await the early 19th century, with work of Carnot and Poncelet. Projective Geometry was furthermore found to simplify many Affine results and proofs, and to complete various Affine structures.

Remark 5 Another route to the Fano plane is setting out find the smallest Affine plane (Subfig c). And then completing this to form a Projective plane.

To make sense of this approach, we first need to present the Affine plane axioms...

Affine Plane 0 and 2 These are the same as Projective Plane 0 and 2. Indeed, they build up from a notion of line to a notion of plane, without reference to what Geometrical kind of plane we mean.

Affine Plane 1 A point P and line $\mathfrak{L}$ support a unique line which contains P And is parallel to $\mathfrak{L}$.

Remark 6 This has various equivalent formulations corresponding to considerably different conceptualizations. Among these, the following due to Playfair is somewhat of a stepping stone toward Projective Geometry.

Affine Plane $1'$ Suppose that we start with a line $\mathfrak{L}$ and any point P not lying on $\mathfrak{L}$. Then there is a unique line containing P while not meeting $\mathfrak{L}$.

Remark 7 For it can be couched in Projective language as follows.

Affine Plane $1''$ Suppose that we start with a line $\mathfrak{L}$ and any point P not incident with $\mathfrak{L}$. Then P is incident with a unique line that is not incident with $\mathfrak{L}$.

Remark 8 Passage to Projective Geometry moreover involves ditching this for the dual version of axiom 0. Thus forming a more structured first pair of axioms.

Structure 1 We next need to explain what we mean by the form taken by *Projective completion* when applied to an incipient Affine plane. This amounts to adding in *points at infinity*. So as to serve as where parallel lines extend to intersect. Alongside adding in a line at infinity incident with the points at infinity. Which is subsequently to be treated no differently from the others.

Proceeding in the opposite direction – *Affine restriction* – amounts to ripping out a line assigned to serve in this role. Alongside its incident points and the other line segments incident with these points.

So consider the Fano plane as presented in Subfig 2.a). Then it is the 'line depicted as a circle' that is assigned the role of the line at infinity. And is thus ripped out in passing to the smallest Affine plane.

While starting from the minimum affine plane, 3 points at infinity are required. In our equilateral presentation, extending to these and placing new points at the intersections forms the *barycentric subdivision* of Subfig b). Finally joining these up necessitates representing this line as a circle.[1]

[1]Projective Geometry does not in any case distinguish between lines and circles... We might as well use 'cline' for circle-or line. Though it appears to be slightly less convenient to portmanteau 'line segment' and 'circular arc'. 'Cline

2 Introducing the Fano Graph

2.1 The projective route

Structure 1 The underlying *Fano graph* is in Subfig 1.a). By its matching the Projective Geometry configuration, let us refer to a) as the Projectively-natural presentation of the Fano graph.

Remark 1 The Fano graph is readily established to be planar, by taking the circle's arcs outside of the vertices' hull (Subfig b). Subfig c) is then a corresponding rectilinear presentation; it is always possible to find such for planar graphs by the Fáry–Stein–Wagner Theorem [24].

Remark 2 Among all the possible rectilinear presentations, this particular one is uniquely fixed by firstly manifesting the graph's full symmetry. This is $S_3 = D_3$: the symmetry group of the equilateral triangle. And so entails working within an equilateral triangle perimeter. And secondly by barycentrically placing the remaining vertices. Which is a stronger restriction as regards the middle layer of vertices than just maintaining the above symmetry group...

2.2 The stellar subdivision route

Structure 2 Indeed, the Fano graph is also the result of twice *totally stellating* a triangle. Each individual stellation move replaces an input triangle subgraph $C = C_3$ (3-cycle) with a Tet subgraph. Where Tet is the *tetrahedron graph*, alias complete 4-graph K_4 . Coincidentally, this is also the graph underlying the smallest affine plane (Fig 2.c). And the projective completion sending this to the Fano plane turns out to be the same as this particular total stellation!

For the first stellation of a triangle there is no difference between stellation and total stellation. But the second stellation is minimal for these to be distinct notions. For one now has 3 smaller triangles, and one could choose to stellate just 1 , 2 or all 3 . See Fig 3 for the first two. While in the case of *all* three, a total stellation is being conducted.

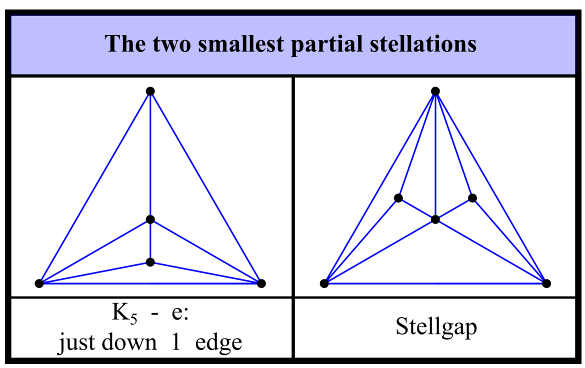

Figure 3:

Remark 3 Barycentric placings are natural in considering stellations. Using an incipient equilateral triangle gives the most symmetric embedding of the graph into the plane for each subsequent total stellation. All in all, let us refer to Subfig 1.c) as the unique *symmetric-barycentric* specialization among the manifestly-planar rectilinear presentations!

2.3 Ramsey presentations and complement graphs

Structure 3 A *Ramsey presentation* places related and unrelated on the same footing. Consider this in the context of graphs whose edges mark related pairs of vertices and are coloured in say blue.

sarc' is our hitherto unaired suggestion [30].

Then unrelated pairs of vertices are not to be left unhighlighted. But are rather to be placed on the same footing as related pairs, by bringing in edges of a second colour, say red.

Remark 4 Applying this to the projective and symmetric-barycentric presentations of the Fano graph returns the corresponding Ramsey presentations in Subfigs 1.d-e).

Remark 5 One can then peel off the blue edges to reveal the complement of the Fano graph (Subfigs 1.f-g). Which can readily be straightend out to reveal the Sunlet graph alongside a loose point, D (Subfig h).

The loose point here corresponds to the Fano graph having a cone point: of degree $N - 1$ in a graph of size N. So that it is linked by an edge to every other vertex in the graph. In both of our presentations of the Fano graph, symmetry is maximized by placing the cone point innermost. Where it can be placed to coincide with the centre of symmetry.
Exercise 1 Show that the Fano graph's degree sequence uniquely specifies it.

2.4 The Contact Geometry route

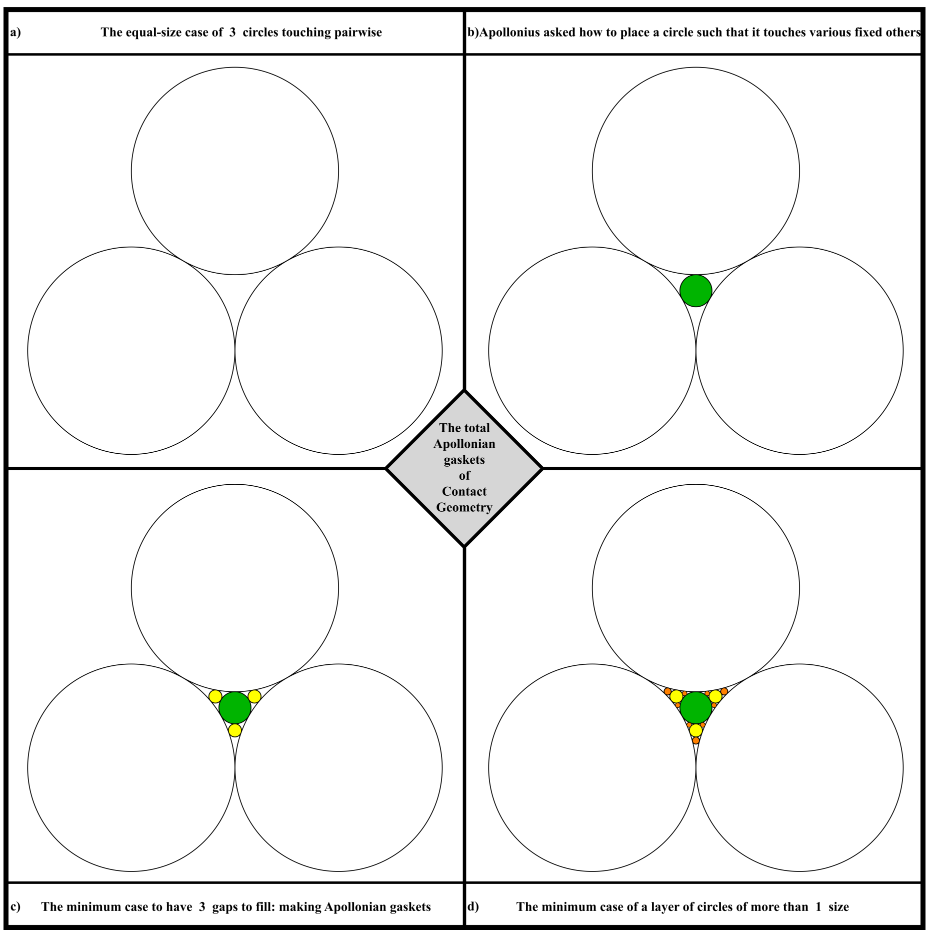

Figure 4:

Structure 4 We can also arrive at Tet by taking 3 touching circles and trapping a fourth between them [Subfigs a) and b) of Figs 4–6]. For sake of simplicity, we take these incipient 3 to be of the same size. We can then repeat this process of inserting touching circles into the new gaps created. Then joining up the centres of all pairs of touching circles produces the C, Tet, Fano ... sequence. See now Subfigs c)-d) of the above Figures.

Historical Remark 1 This Contact Geometry route for arriving at the 'Fano' graph in fact dates all the way back to a study of Leibniz [2]. Of a simple recursively-defined subcase of Apollonius' problem [1] of placing a circle to be in contact with multiple other circles. This has higher significance through being one of the first of what came to be known as *fractal structures* to be studied.

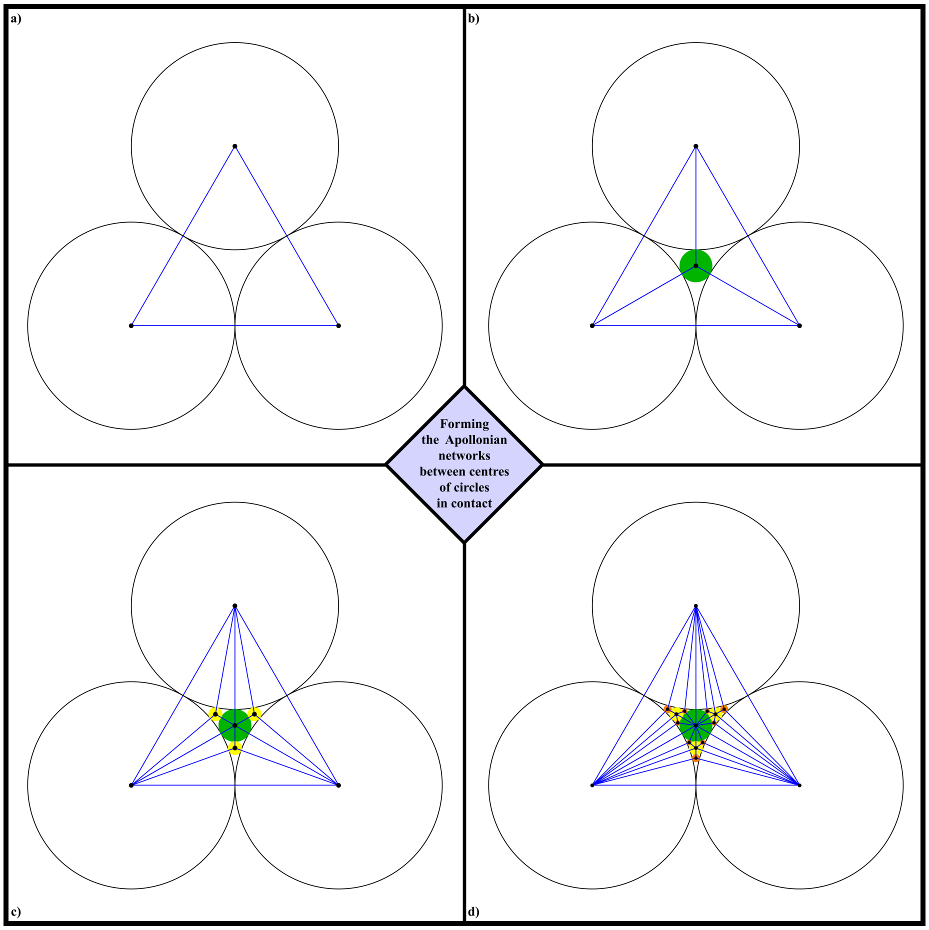

Figure 5:

Naming Remark 2 In the process, a widely used alias for the stellated triangles' graphs arises: *Apollonian networks*. Though some authors extend the meaning of one both of these from out total use to partial use as well. This corresponds to inserting touching circles into some gaps but not others.

Remark 6 Contrast also Fig 6's Contact Geometry presentation with Fig 7's barycentric stellar subdivision presentation!

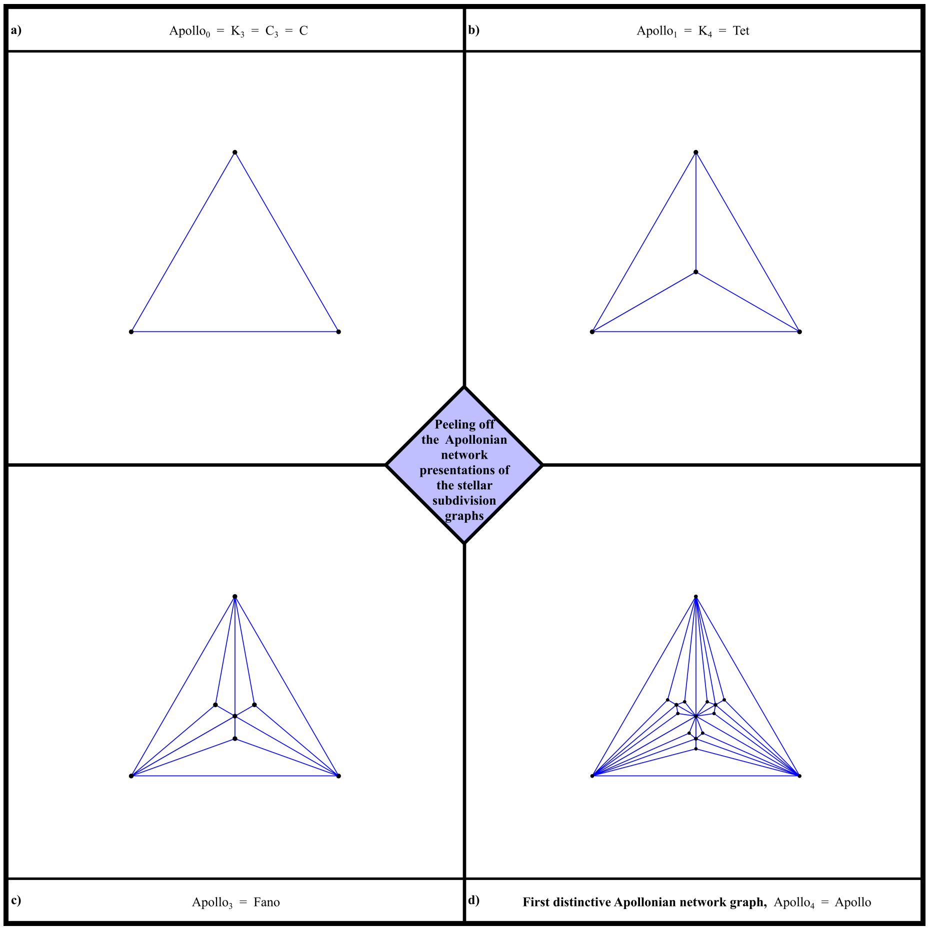

Figure 6:

2.5 Strong triangulation route

Sturucture 4 Since all faces of a triangle that has been subjected to stellation are triangles, stellations are also examples of *triangulation* [10, 26, 27, 33]. Let us here distinguish between cutting up a polygon into triangles by edges emanating from its vertices. And cutting up a triangle specifically. In this second case, the *outer* face is also a triangle, and so *all* the faces are triangles: a *strong triangulation*. While in the first case for a $\geq$ 4-sided polygon, the outer face is not a triangle: a *weak triangulation*.

Remark 7 Then the Fano graph, and more generally the Apollonian graphs alias stellations of the triangle – in each case total or partial – are strong triangulations.

Exercise 2 Find the smallest strong triangulation that is not some stellation of the triangle.

Remark 8 This establishes that the converse is false, by which strong triangulations cover more than just a reconceptualization of stellations... So on the one tip of a trident, the smallest Affine plane and its completion the Fano plane are a terminating series. On another tip of a trident, stellation or the Apollonius gasket have these feature near the start of an infinite series. And on the final tip of a trident both, are additionally strong triangulations, which constitute *more than* just series structure!

Naming Remark 3 The Fano graph thus starts to fulfill the many routes condition for a Mathe-

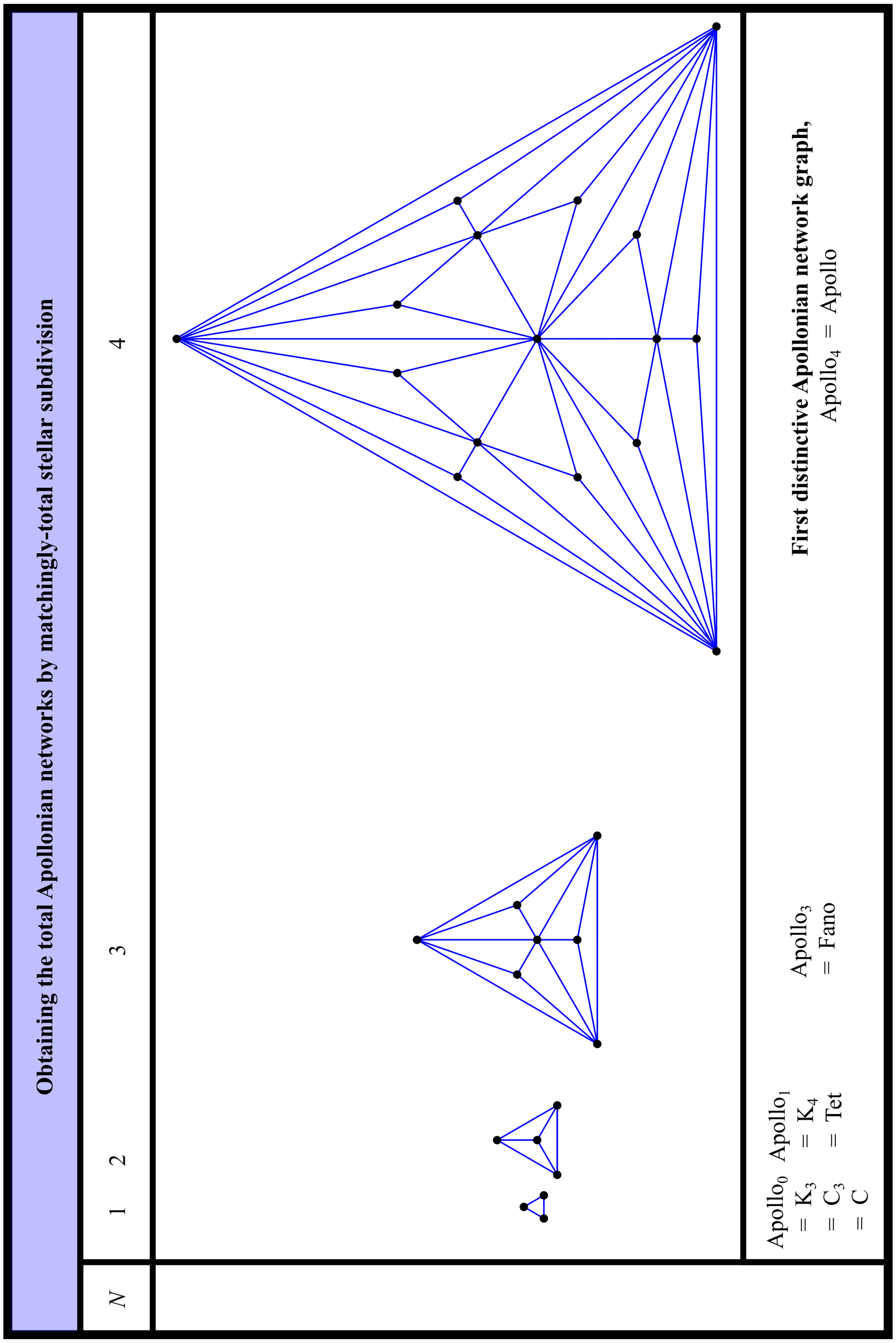

Figure 7:

matical object or structure to be a citizen of Kallista [30, 31]. As we shall outline at the end of the current Article, both it and the Fano configuration are rather stronger citizens of Kallista than just this.

3 Notions of traversability

3.1 Eulerian and Hamiltonian graphs

Definition 1 A graph is *Eulerian* if out of all of its edges, one can form a *circuit*. I.e. a closed loop, with no repeat uses of any edge. But with permission granted to pass multiple times through vertices if needs be.

Definition 2 A graph is *Hamiltonian* [17, 19, 24] if one can form a cycle that passes through every vertex. Unlike a circuit, a *cycle does not* grant permission to pass multiple times through any vertex.

Structure 1 These are respectively a notion of edge traversability and a notion of vertex traversability. For each of vertices and edges, graphs indeed support other distinct notions of traversability.

Remark 7 The Fano graph is immediately obviously non-Eulerian since it possesses vertices of odd degree. But Euler [3] initiated this subject with the result that Eulerian graphs must contain solely even-degree vertices.

Remark 8 The Fano graph is small enough that it is easy to espy Hamiltonian cycles therein.

3.2 Planar Hamiltonian graphs

Remark 9 For a planar graph, a more systematic way of establishing Hamiltonianness is to split the graph into precisely 2 regions. Such that each contains the same number of triangles (weakly triangularizing up any larger polygons present). This is widely known as *Grinberg's theorem* [14, 19, 24].

Though in our institution, we call it the *ZIPHoN theorem* [30, 31, 32]. Standing for *zero-index planar Hamiltonian Necessity!* Where the zero index in question is the precise balance of the inner and outer triangles. Though the literature elswewhere has more confusingly proceeded by defining 'face strengths' instead of by triangulating and then just plainly counting. And have also not realized that it is a zero-index theorem in the plane. Or, indeed, an actual (non-zero) toy index theorem when conducted on surfaces [30, 32].

Remark 10 The Fano graph is already fully triangulated, outer face included! This situation – simplifying its ZIPHoN treatment – is equivalent to Sec 2's construction of the Fano graph as a strong triangulation. One choice of white-and-cyan regions with equal counts of triangles is then exhibited in Subfig 8.a) The Hamiltonian cycle is then the boundary between these 2 regions.[2]

Exercise 3 Does the Fano graph support inequivalent Hamiltonian cycles? [Ones related by a symmetry transformation count as the same.]

[2]For planar Hamiltonian graphs, each of the inner and outer regions must furthermore be *outerplanar*! This observation gives the following nice formulation [30]. Suppose that a planar graph can be split up into precisely 2 outerplanar strips. Then there must necessarily be a Hamiltonian cycle between them!

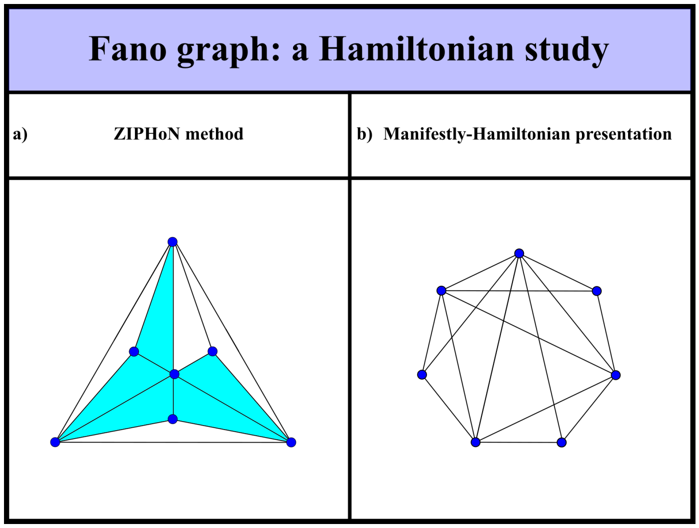

Figure 8:

3.3 A manifestly-Hamiltonian presentation of the Fano graph

Remark 11 Since the Fano graph is Hamiltonian, for some purposes it is useful to form a manifestly-Hamiltonian presentation. One such is provided in Subfig b). A subsequent version of the current Article shall look into variants of this, such as minimizing the presentation's crossings. And looking into whether any symmetry elements can be exhibited within the straightjacket of a Hamiltonian-and-rectilinear presentation...

3.4 Further presentations of the Fano graph forthcoming!

Pointer 1 More generally, over the next few years we will be applying the full force of the Handbook of Graph Drawing and Visualization [29] to a collection of particularly exhalted graphs, the Fano graph included. The idea here is to give *optimized* rather than just random presentations of these graphs. And to give not just individual such presentations for each graph of interest. But rather large mosaics of what they look like within the confines of a large number of competing simplicity, optimization, manifestness and aesthetic criteria.

4 Colourability properties

4.1 The basic ones

Definition 1 [17, 24, 20] A (*vertex*) *colouring* of a graph involves colouring in its vertices. According to the rule that no 2 adjacent vertices – joined by some edge – share the same colour. An *edge-colouring* of a graph involves colouring in its edges. According to the rule that no 2 adjacent edges – meeting at some vertex – share the same colour. The (*vertex-*)*colouring number* of a graph is the minimum possible number of colours in a (vertex) colouring of it. The (*edge-*)*colouring number* of a graph is the minimum possible number of colours in an (edge) colouring of it. (*Vertex-*)*chromatic number* and *edge-chromatic number* are widely used aliases.

Remark 1 In these colourings, the colours assigned are modelled to be mutually-distinguishable and yet individually meaningless. This means that permuting the colours assigned in actually drawing a presentation of a colouring is not taken to affect that colouring. So that all colouring presentations related by such permutations are taken to be one and the same.

Remark 2 The Fano graph is 6-edge colourable: one less than the maximum possible for a simple 7-graph. For it is a cone graph. So its cone-point vertex is adjacent to all other 6 vertices. So each of the 6 edges emanating from it must be of a different colour. See Fig 9.a).[3]

Finally see Subfig b) for a subsequent completion of the edge-colouring scheme that re-uses 5 out of 6 colours. Which is a sharp lower bound. For the Fano graph also contains a vertex of degree 5 (in fact 3 such).

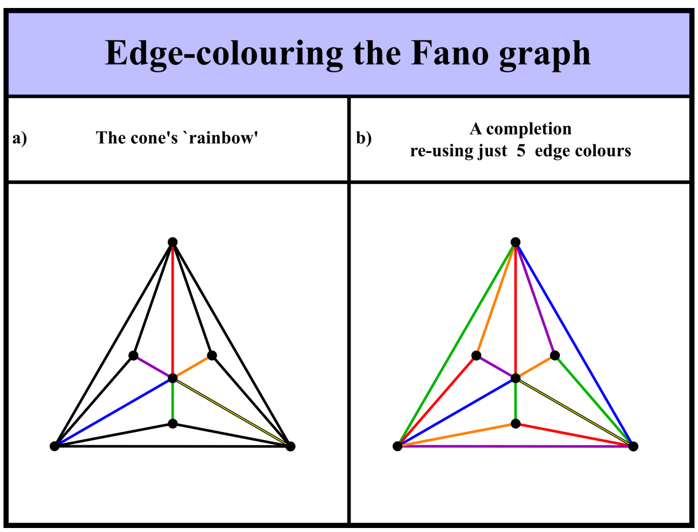

Figure 9:

Remark 3 The Fano graph is 4-colourable – the maximum possible for any planar graph by the famous 4-colour theorem. This readily follows from the below analysis that establishes a rather stronger property.

[3]The complete graph K_7 illustrates how bringing in even more edges can force a seventh edge colour. No simple 7-graph can have more edges than this, so its edge number places an upper bound on all the others.

4.2 Unique colorability

Remark 4 *Unique colourability* is a significant enough property to be tabulated in *An Atlas of Graphs* [23].[4] Unique colourability is indeed up to the abovementioned permutation of vertex colour labels. A standard characterization for unique 4-colorability is as follows. Colouring in any triangle's vertices fixes the colours of all remaining vertices. This 4 is relevant as firstly the maximum possible vertex chromatic number for planar graphs, by the famous 4-colour theorem [24]. Secondly, as the generic value of the vertex chromatic number in planar graphs. And thirdly today indeed as the vertex chromatic number of the Fano graph!

Remark 5 By symmetry, the Fano graph has 4 choices of an initial triangle to colour. As characterized by each triangle's vertices' degrees; see the top floor of Fig 10 Via the forcing moves indicated by this diagram's arrows, each of these initial probes leads all the way down to the Fano graph acquiring the same vertex colouring.

4.3 Colourability-forcing posets

Structure 2 This figure is our first public exhibition of S. Sánchez' notion of a *colourability-forcing poset*. For our Fano example, this is a fortiori a rooted tree. Which we present 'botanically', i.e. with the root at the bottom of the figure. This poset carries furthermore a natural height function: the *colouring-in number* h_{Ci} .

4.4 The unique colourability route to Fano and friends

Pointer 2 In fact, unique 4-colourability and being a stellation of the triangle turn out to be equivalent criteria [22]. Giving yet another route to (the more general) Apollonian graphs. Among which the Fano graph is the first nontrivial exemplar with various stronger properties. Such as being a strong triangulation, a total stellation, and having more symmetry elements.

[4] Indeed, it, and the vertex and edge chromatic numbers, are the only small colour-related items to make it into these tables. The larger colour-related item tabulated there is the *chromatic polynomial*.

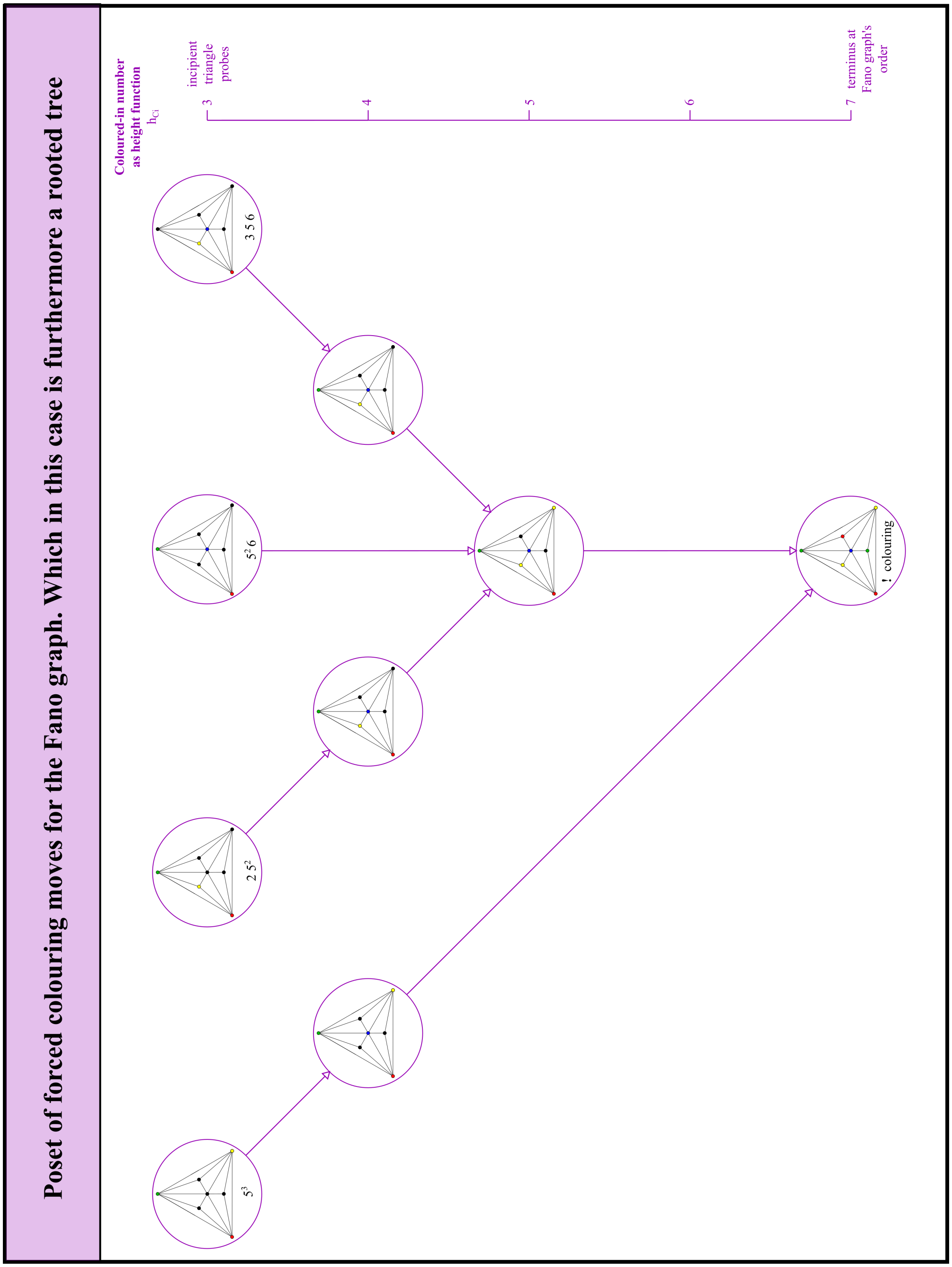
Poset of forced colouring moves for the Fano graph. Which in this case is furthermore a rooted tree
Coloured-in number as height function
h_{Ci}
3
4
5
6
7
incipient triangle probes
terminus at Fano graph's order
3 5 6
5^2 6
2 5^2
5^3
! colouring

Figure 10:

5 The Fano configuration as a citizen of Kallista

Pointer 3 It is a *Steiner triple system.*

Pointer 4 It is a *block design.*

Pointer 5 It is a *matroid*, and furthermore distinguished as a minimum binary-and-non-regular such. Which honour it shares with its own dual...

Remark 1 All of the above structures correspond to large active research fields within Combinatorics.

Pointer 6 The Fano configuration has a large Projectively significant-symmetry group: $PGL_2(3, \mathbb{R})$ of order 168. Which aspect is unfortunately lost by the Fano graph, whose symmetry group is just $S_3 = D_3$, of order 6 [23].

$PGL_2(3, \mathbb{R})$ furthermore manages to be a simple group: it contains no nontrivial normal subgroups. Simple groups are key in Group Theory's classification of the finite groups. $PGL_2(3, \mathbb{R})$ is quite famous as marking the end of the first 'simple group desert'. For it is the smallest simple group after the basic and long-known alternating permutation group A_5, of order 60 ... On the one hand, Group Theorists have a neat way of constructing this group that is intrinsic to their subject On the other hand, more complicated groups often happen to be found via their action upon some object or structure. And for $PGL_2(3, \mathbb{R})$, this role is played magnificently by the Fano plane!

Naming Remark 3 It remains unclear what could be used as a truer name for the Fano configuration. Firstly in a routes-unbiased manner, which the very strong property *smallest projective plane* does not attain. Secondly, since not all of the routes leading to it may have been found yet (though our notion of *truer* name is adaptable in the face of new discoveries!) For the Fano graph, for now we {cS recommend *doubly-stellated triangle graph.* Generalizing to *n-fold-stellated triangle graph* for the stronger – totally stellated – Apollonian graphs.

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