On Ford's Extension of Smale's Little Theorem

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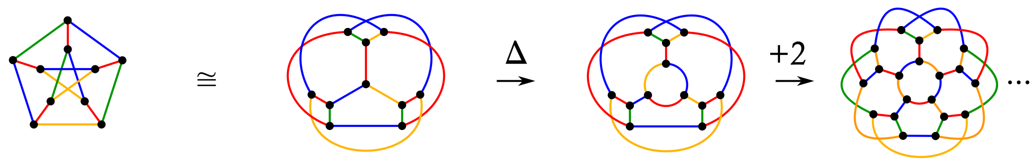
Abstract

Smale's little theorem is that the arena of unlabelled triangles modulo similarities is the sphere. I recently reproved this from the eigentheory of Heron's formula viewed as a quadratic form. The current Article reports A. Ford's generalization of this proof to the arena of all 3×3 Combinatorial matrices. For which I coin the name 'Ford–Smale little theorem'.



ANNALS OF THE INSTITUTE FOR THE THEORY OF STEM

Ann. Inst. Theory STEM Volume 1 Number 8 (2025)

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Date stamps: preprints v1 03-10-2025, v2 11-12-2025. Publication 11-12-2025.

1 Introduction: Smale's little theorem from Heron's formula

Structure 1 Consider Heron's formula [1, 3, 9, 6, 10, 18]

$$Area \;=\; \sqrt{s \prod_{\text{3-cycles}} (\,s \;-\; a\,)}\;. \tag{1}$$

This is for a triangle with side-lengths a and 3-cycles. And semi-perimeter

$$s \;:=\; \frac{1}{2} \sum_{\text{3-cycles}} a\;. \tag{2}$$

Re-express the square of this as a quadratic form [2, 7, 11],

$$T^2 \;=\; ||\boldsymbol{S}||_{\boldsymbol{F}}{}^2 \;:=\; \underline{\boldsymbol{S}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \overline{\boldsymbol{S}}\;. \tag{3}$$

For *tetra-area* T . Side-lengths-squared vector

$$\boldsymbol{S} \;:=\; \begin{pmatrix} A \\ B \\ C \end{pmatrix}\;. \tag{4}$$

Where

$$A \;:=\; a^2 \;\text{ and }\; \text{3-cycles}\;. \tag{5}$$

And *fundamental triangle matrix*

$$\boldsymbol{F} \;:=\; \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\;. \tag{6}$$

Structure 2 The eigenexpansion version of this quadratic form reads [11, 18]

$$T^2 \;=\; \sum_{i \,\in\, \mathfrak{C}\mathrm{spec}(\boldsymbol{F})} \lambda_i \; ||\boldsymbol{v}_i||^2 \;=\; 1 \;\times\; R^2 \;+\; (\text{-}2) \;\times\; \left(Aniso^2 \;+\; Anelp^2 \right)\;. \tag{7}$$

Where $\mathfrak{C}\mathrm{spec}$ denotes eigenspectrum. Lone eigenvalue $\lambda_1 \;=\; 1$ and 2-fold degenerate eigenvalue $\lambda_2 \;=\; \text{-}2$. The lone eigenvector quantity

$$R \;:=\; \frac{A \;+\; B \;+\; C}{\sqrt{3}}\;. \tag{8}$$

Which is proportional to the square of the radius of gyration of our triangle.

And the following choice of eigenvector quantities for the degenerate 2-d eigenspace. *Anisoscelesness*

$$Aniso \;\;:=\;\; \frac{B \;-\; A}{\sqrt{2}}\;. \tag{9}$$

Alongside its orthonormal completion the *departure from equilateral proportion*

$$Anelp \;\;:=\;\; \frac{A \;+\; B \;-\; 2\,C}{\sqrt{6}}\;. \tag{10}$$

Structure 3 Non-dimensionalizing by dividing though by R^2 , the eigenexpansion (7) becomes

$$\mathcal{T}^2 \;=\; 1 \;-\; \mathcal{A}\mathrm{niso}^2 \;-\; \mathcal{A}\mathrm{nelp}^2\;. \tag{11}$$

For

$$\mathcal{T} \;:=\; \frac{T}{R}\;, \quad \mathcal{A}\mathrm{niso} \;:=\; \sqrt{2}\,\frac{Aniso}{R}\;, \quad \mathcal{A}\mathrm{nelp} \;:=\; \sqrt{2}\,\frac{Anelp}{R}\;. \tag{12}$$

Where our definitions have also absorbed the magnitude of the corresponding eigenvalue via their factor of $\sqrt{2}$.

(11) then readily rearranges to

$$\mathcal{T}^2 + \mathcal{A}\text{niso}^2 + \mathcal{A}\text{nelp}^2 = 1 . \tag{13}$$

Thus realizing the on-$\mathbb{S}^2$ condition. By which *Heron's formula provides a new proof of Smale's little theorem* [5]. I.e. that the arena of unlabelled triangle shapes modulo similarity transformations is Topologically a sphere, $\mathbb{S}^2$. Whose original proof bore no relation to Heron's formula. Let us coin the expression '*Smale–Heron formula*' for (13).

2 The Combinatorial matrix arena

Structure 1 Ford [14] pointed out that $\boldsymbol{F}$ and the other triangle matrices in [13] are Combinatorial matrices [4]. I.e. size-K square matrices of the following form.

$$\boldsymbol{C} = \begin{pmatrix} x + y & x & \dots & x \\ x & & & \vdots \\ \vdots & & & x \\ x & \dots & x & x + y \end{pmatrix} . \tag{14}$$

Which Ford summarized by the *irreducible symbol*

$$(x, y)_K . \tag{15}$$

For whichever $K \geq 2$, these form [14, 18] a 2-d vector space

$$\mathfrak{C}\mathrm{M}_K(2) . \tag{16}$$

Question 1 Ford posed and answered the question of what happens to the above proof of Smale's little theorem in the following context. Upon generalizing the

$$\boldsymbol{F} = (1, 1)_3 . \tag{17}$$

at the core of our quadratic form to the arbitrary $K = 3$ Combinatorial matrix $\boldsymbol{C}$.

Naming and Notation Remark 1 To access this, let us first unbridle the above notions, names and notations of Geometric triangle significance. Our notation for a Combinatorial matrix $\boldsymbol{C}$'s quadratic form is

$$Combi := ||\boldsymbol{K}||_{\boldsymbol{C}}{}^2 = \underline{\boldsymbol{K}}^{\mathrm{T}} \cdot \overline{\underline{\boldsymbol{C}}} \cdot \overline{\boldsymbol{K}} . \tag{18}$$

I.e. with the Combinatorial counts vector $\boldsymbol{K}$ in place of $\boldsymbol{S}$. On the one hand, R, *Aniso* and *Anelp* remain a valid choice of eigenvectors. On the other hand, we now denote the first of these by U , standing for the unit –normalized– average-count eigenvector. The second by *Ind*: induced from a 2-d precursor. And the third by *Comp*: the subsequent orthonormal completion.

Remark 1 Subsequently normalizing by U , and rescaling at will, yields the next Section's 'Ford–Smale little theorem'. 1 of whose 4 generic cases returns the on-$\mathbb{S}^2$ condition.

Remark 2 The current Article's original conceptualization of the theorem comes with the following subtlety. Under what circumstances does it make sense to model our quadratic forms by our choice of squared variable $Combi^2$? Explaining why this is a priori questionable, justifying that this is modelling something meaningful, and finding alternatives that do not use this, are all best left to the longer companion article.

3 Ford–Smale little theorem

Theorem 1 The eigenexpansion of the quadratic form corresponding to a $K = 3$ Combinatorial matrix takes the form in Fig 1. With the case names parametrized by the sign of the 2-fold eigenvalue and then minus the sign of the lone eigenvalue.

Where, in terms of the corresponding eigenvalues,

$$\mathcal{C}ombi \ := \ \frac{Combi}{U} \ , \ \ \mathcal{I}nd \ := \ \sqrt{\lambda_2}\,\frac{Ind}{U} \ , \ \ \mathcal{C}omp \ := \ \sqrt{\lambda_2}\,\frac{Comp}{U} \ .$$

And

$$\widetilde{\mathcal{C}ombi} \ := \ \frac{Combi}{\sqrt{\lambda_1}\,U} \ , \ \ \widetilde{\mathcal{I}nd} \ := \ \sqrt{\frac{\lambda_2}{\lambda_1}}\,\frac{Ind}{U} \ , \ \ \widetilde{\mathcal{C}omp} \ := \ \sqrt{\frac{\lambda_2}{\lambda_1}}\,\frac{Comp}{U} \ .$$

Remark 1 That the Heron quadratic form corresponds to the sphere case amounts to a recovery of Smale's little theorem.

Ford-Smale little theorem's 9 cases, of which 7 are consistent

Case	Equation	Geometrical outcome
++	$\widetilde{\mathcal{I}nd}^2 + \widetilde{\mathcal{C}omp}^2 + \widetilde{\mathcal{C}ombi}^2 = 1$	sphere
+−	$\widetilde{\mathcal{C}ombi}^2 - \widetilde{\mathcal{I}nd}^2 - \widetilde{\mathcal{C}omp}^2 = 1$	hyperboloid of 2 sheets
−+	$\widetilde{\mathcal{I}nd}^2 + \widetilde{\mathcal{C}omp}^2 - \widetilde{\mathcal{C}ombi}^2 = 1$	hyperboloid of 1 sheet
−−	$\widetilde{\mathcal{I}nd}^2 + \widetilde{\mathcal{C}omp}^2 + \widetilde{\mathcal{C}ombi}^2 = -1$	inconsistent
+0	$\mathcal{I}nd^2 + \mathcal{C}omp^2 = \mathcal{C}ombi^2$	double cone
−0	$\mathcal{I}nd^2 + \mathcal{C}omp^2 + \mathcal{C}ombi^2 = 0$	point
0+	$\mathcal{C}ombi^2 = 1$	parallel pair of planes
0−	$\mathcal{C}ombi^2 = -1$	inconsistent
00	$\mathcal{C}ombi = 0$	plane

Key					
(pink)	Generic	(orange)	Nongeneric	(grey)	inconsistent

Figure 1:

4 Acknowledgments and Pointers

A. Ford privately communicated this to me in 2024. As the arbitrary $K = 3$ Combinatorial matrices' generalization of Sec 1's Heron rederivation of Smale's little theorem [5]. A. Ford left it to me to write this up, asking not to be a coauthor, along the lines that D.G. Kendall [8] included an unpublished theorem stated to be due to A. Casson. I contributed the sheet counts, z, and the name 'Ford–Smale little theorem'.

Pointer 1 Incidentally, this work of Kendall's extended Smale's little theorem to Kendall's little theorem. I.e. that the abovementioned arena of triangles is metric-Geometrically a sphere as well, with the standard spherical metric. [18] covers this in detail. Including its interplay with Combinatorial matrices as raised in [14, 15] and the current Article. And its rooting on Hopf's little map $H : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$. Which provides a further selection principle [18] for $\boldsymbol{F}$ (up to proportion) within the arena of Combinatorial matrices $\mathfrak{C}\mathrm{M}_3(2)$.

'Little' refers to the $\mathbb{S}^2 = \mathbb{CP}^1$ subcase of results that generalize to $\mathbb{CP}^{N-2}$. Corresponding to N-body problems in 2-d [5, 8]. Casson's own more general result is that N-simplices in dimension n are also Topologically spheres of each appropriate dimension.

Pointer 2 I subsequently supported the current work and [14] by looking into why all Combinatorial matrices share eigenspaces: [15].

Pointer 3 I further evolved the result through various further iterations during the "Linear Algebra of Quadrilaterals 2024 Summer School" at the Institute of the Theory of STEM. This is left to the next Article [17].

I also thank K. Everard for proofreading the manuscript, and the other participants at the Summer School and the Applied Combinatorics and Topology Discussion Group.

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