Eigentheory of Combinatorial Matrices

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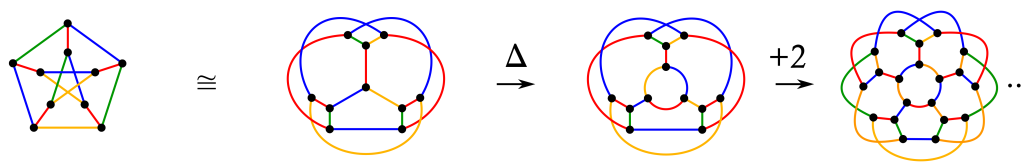
Abstract

We consider the $\mathbb{R}$-valued Combinatorial matrices. These are square, say of size K, depend on just 2 parameters and are symmetric. We classify their consequently real eigenvalues and eigenvectors. The key feature is an eigenspace of size $\geq k := K - 1$. Which for $K \geq 3$ is degenerate, in the sense of having $\geq k \geq 1$ equal eigenvalues. In the generic case, this rests in turn upon an $O(k)$ symmetry. While in the remaining isotropic case, all eigenvalues are equal, backed by the full symmetry group available: $O(K)$. Finer classification by rank and various notions of signature is also provided. Given any set of Combinatorial matrices of the same size, they can be taken to share eigenbases. Modulo a stated caveat, such share eigenspaces as well.

Dynamics' centre of mass (CoM) hierarchies return the Jacobi vectors as eigenvectors. The relative such exhibit a network ambiguity which corresponds to the unlabelled rooted binary trees. At a first glance, this appears to be an instance of Dynamics producing Combinatorial objects. These eigenvectors turn out however to arise for any Combinatorial matrix. By which a more natural perspective is that Combinatorics produces more Combinatorics, with Dynamics then just enacting a subcase of this.

Flat Geometry's triangle matrices – or more generally 2-simplex matrices – were presented in Articles 1-4. Including various Abstract Algebra properties that Ford subsequently determined to follow solely from these all being equal-K Combinatorial matrices. The current article extends Ford's erudite conclusion to further Linear Algebra properties, mostly of a Spectral nature. Finally Spectral classifications are taken further using Graph Theory and Order Theory.



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1 Introduction

1.1 Combinatorial matrices, with a first few examples

Definition 1 A *Combinatorial matrix* [18] is a $K \times K$ square matrix of the following form.

$$\boldsymbol{C} = \begin{pmatrix} x + y & x & \dots & x \\ x & & & \vdots \\ \vdots & & & x \\ x & \dots & x & x + y \end{pmatrix} . \tag{1}$$

This is symmetric. $x, y \in \mathbb{N}, \mathbb{Z}$ cover many Combinatorial uses. We however extend to $x, y \in \mathbb{R}$ to encompass Dynamics and Linear Algebra use as well.

Naming and Notational Remark 1 Let us term A. Ford's shorthand notation [71]

$$\boldsymbol{C} = (x + y , x)_K \tag{2}$$

Ford's symbol of the zeroth kind. This is the notation selected for use in the current Article.

Remark 1 We next start to build up a repertoire of specific examples by taking simple cases within the framework of our incipient choice of parameters x, y . Further examples shall appear as we develop our main theme: Eigentheory alias Spectral Theory.

Example $\mathbb{0}$) Setting

$$x = 0 = y , \tag{3}$$

we find the *zero matrices* of all sizes K ,

$$\mathbb{0} = (0, 0)_K . \tag{4}$$

Example $\mathbb{I}$) Setting

$$x = 0 , \tag{5}$$

$$y = 1 , \tag{6}$$

we arrive at the *identity matrices*

$$\mathbb{I} = (1, 0)_K . \tag{7}$$

While relaxing to

$$x = 0 , \quad y \neq 0 \tag{8}$$

yields the *matrices proportional to each identity.*

Example $\mathbb{1}$) Setting instead

$$x = 1 , \quad y = 0 , \tag{9}$$

we encounter the matrices of 1s,

$$\mathbb{1} := (1, 1)_K . \tag{10}$$

Each of which we shall refer to as a *units matrix* .

Now relaxing to

$$x \neq 1 , \quad y = 0 \tag{11}$$

produces the *matrices proportional to each units.*

Example $\mathbb{T}$) Next setting

$$t \;:=\; x \;+\; y \;=\; 0\;, \tag{12}$$

$$y \;=\; 1\;, \tag{13}$$

we obtain the *tracefree version of the units*,

$$\mathbb{T} \;:=\; (\,0,\,1\,)_K\;. \tag{14}$$

t is here the *trace per unit size*: an intensive variable in Physics parlance. The trace itself is the extensive variable counterpart,

$$T \;=\; K\,t \;=\; K\,(\,x \;+\; y\,)\;. \tag{15}$$

Finally relaxing to

$$t \;=\; 0\;,\;\; y \;\neq\; 0 \tag{16}$$

returns the *matrices proportional to the tracefree units.*

1.2 Explaining some notation

Notational Remark 2 Among these, $\mathbb{I}$ and $\mathbb{T}$ are distinguished as the pieces of $\mathbb{1}$ that are irreducible: Representation-Theoretically [21, 43] significant.[1] We pick Ford's symbol of the zeroth kind as the current Article's notation since this corresponds to expanding each $\boldsymbol{C}$ with respect to the linear basis (LB) consisting of these irreducibles. So e.g. a coordinate-free rendition of (1) is

$$\boldsymbol{C} \;=\; t\,\mathbb{I} \;+\; x\,\mathbb{T} \;=:\; (\,t,\,x\,)_K\;. \tag{17}$$

As an incipient foil, *Ford's symbol of the first kind* [71]

$$\boldsymbol{C} \;=\; y\,\mathbb{I} \;+\; x\,\mathbb{1} \;=:\; [\,y\;,\;\; x]_K\;. \tag{18}$$

I.e. now in terms of the LB consisting of the identity and the units matrix: $\mathbb{I}$ and $\mathbb{1}$. Indeed, this bracket being square alludes to using the units matrices. Whereas the roundness of the zeroth symbol's bracket amounts to reserving the most commonly encountered bracket for our most commonly used symbol.

Example $\mathbb{V}$) Having encountered trace via tracefree Combinatorial matrices dropping out of the above incipient-parameter analysis, the following adapted-parameter condition is natural. Setting

$$t \;=\; -\,1\;, \tag{19}$$

$$x \;=\; 1 \tag{20}$$

introduces the *trace-reversed units*

$$\mathbb{V} \;=\; (\,-1,\,1\,)_K\;. \tag{21}$$

Relaxing to

$$t \;=\; -\,q \;\neq\; 0 \tag{22}$$

leaves us with the *matrices proportional to the trace-reversed units.*

[1] Matrices proportional to the identity also have particular Representation-Theoretic significance through entering Schur's Lemma.

Notational Remark 3 A second foil notation is then espied. *Ford's symbol of the minus-oneth kind* [71]

$$\boldsymbol{C} \;=\; (\,y \;+\; 2\,x\,)\,\mathbb{I} \;+\; x\,\mathbf{V} \;=: \;\langle\, y \;+\; 2\,x,\, x\,\rangle_K \tag{23}$$

corresponds to the LB consisting of the identity and the trace-reversed units matrix: $\mathbb{I}$ and $\mathbf{V}$. This is the minus-oneth symbol in the sense of being the reflection about the zeroth symbol of the first symbol. As an 'image' of a type of piecewise-linear bracket – square – it is then denoted by another type of piecewise-linear bracket: the chevron.

Naming Remark 2 *Irreducible*, *units* and *trace-reversed* cases of *Combinatorial matrix symbols* are respectively truer names for the above three symbols.

1.3 Outline of the rest of this Article

We introduce the notion of arenas [62, 54, 63] in Sec 2, with Combinatorial matrix [71] and widely-used Linear Algebra [10, 47, 51] examples. We classify Combinatorial matrices' eigenvalues and eigenspectra in Sec 3. And their eigenvectors and eigenspaces in Sec 4. Including the paradigm shift from N-body problem terminology and conceptualization to the general Combinatorial matrix setting. Sec 4 includes pointing out that any Combinatorial matrices of the same size can be taken to share eigenbases. Modulo a caveat in Secs 4 and 5, they share eigenspaces as well.

In the process, further Algebraic and Geometric examples of specific Combinatorial matrix are pointed out. These include the 3 triangle matrices (more generally 2-simplex matrices) in Article 1. I pointed out a number of Abstract Algebra and Linear Algebra properties of these in Articles 2 to 4. In Article 5, A. Ford demonstrated that these Abstract Algebra properties follow purely from their being Combinatorial matrices of the same size, rather than in any way being Geometrically specific. The current Article reaches the same conclusion for their Linear Algebra – more specifically Spectral Theory – results.

Appendix A serves to condense most of Sec 3 and 4's results. The remainder – the paradigm shift – enters instead the Conclusion (Sec 5)'s comparison table. Finally Appendix B takes the classification of eigenspectra further, using Order Theory [62, 33, 40] and its underpinning Graph Theory [38].

2 Arenas

Structure 0 Given a type of Mathematical object, the corresponding *arena* is the space formed by the totality of Mathematical objects of this type. What topologies are natural to each arena is then always a good question. So via arenas, Modern Applied Topology [62, 54, 63] becomes adjacent to every other STEM subject, or indeed to every other subject with at least some objects sharply-defined enough that we can contemplate what their arenas are.

2.1 Arenas of Combinatorial matrices

Definition 0 Focusing this modern Applied Topology line of thought on our main subject matter, $\mathbb{R}$-valued Combinatorial matrices form the arenas

$$\mathfrak{CM}_{\mathbb{R}}(K)$$

for each fixed K . The cumulative arenas

$$\mathfrak{CM}_{\mathbb{R}}[K] \;:=\; \coprod_{P\,=\,0}^{K} \mathfrak{CM}_{\mathbb{R}}(P)$$

up to whichever fixed K . And the fully cumulative arena

$$\mathfrak{CM}_{\mathbb{R}} \;:=\; \mathfrak{CM}_{\mathbb{R}}[\infty] \; .$$

2.2 Vector spaces

Remark 1 Both for the Readers' convenience and as regards developing our subject matter, various examples of arenas that entered widespread use long before the advent of modern Applied Topology are pointed out in the current Article.

Example 1 *Vector spaces* [10, 44, 55, 79] $\mathfrak{V}$ over $\mathbb{R}$ are our first such: the arenas of all $\mathbb{R}$-valued vectors with a given number of components. Where relevant, we shall index these by their dimension, $\mathfrak{V}^d \;=\; \mathbb{R}^d$. Vector spaces are more generally the arenas of whichever objects that are meaningfully represented by vectors. So e.g. polynomials, matrices under addition and multiplication by a scalar, and functions [48] are also covered.

Structure 1 We have the good fortune that [71] the $\mathfrak{CM}_{\mathbb{R}}(K)$ are vector spaces, and, additionally, for $K \;\geq\; 2$, are copies of the same vector space. This occurs via each Combinatorial matrix being described by just 2 parameters – our incipient choices for which are x and y , both of which are active for $K \;\geq\; 2$. These can each take arbitrary values in $\mathbb{R}$. Yielding the common vector space

$$\mathfrak{CM}_{\mathbb{R}}(\,K \;\geq\; 2\,) \;=\; \mathbb{R}^2 \; . \tag{24}$$

This accounts for why the 3 LBs mentioned above each have 2 elements.

For $K \;=\; 1$, Combinatorial matrices collapse to just numbers. With only 1 active parameter: the one that becomes the intensive version of the trace, $t \;=\; x \,+\, y$. Yielding the 1-d vector space $\mathbb{R}^1$ over $\mathbb{R}$, i.e. just $\mathbb{R}$ itself:

$$\mathfrak{CM}_{\mathbb{R}}(1) \;=\; \mathbb{R}^1 \;=\; \mathbb{R} \; . \tag{25}$$

Here all 3 symbols' bases collapse to just $\mathbb{I} \;=\; 1$. Finally $K \;=\; 0$ is exceptional in that the sole object supported here is the *unmatrix*, which in many ways is not a matrix at all [79, 76]. While the unmatrix contains just an empty set's amount of information, the set of unmatrices itself constitutes a point. So identifying this as the zero point, it is possible to view

$$\mathfrak{CM}_{\mathbb{R}}(0) \;=\; \mathbb{R}^0 \;=\; \{0\} \;: \tag{26}$$

also a vector space. Here all 3 symbols' bases collapse to the empty set.

Remark 2 Thus also the $\mathfrak{CM}_{\mathbb{R}}[K]$ are disjoint sums of vector spaces. With $\leq\; 3$ types of vector space present, due to the swift onset of the persistent $\mathbb{R}^2$ vector space.

Hence

$$\begin{array}{rclcl} \mathfrak{CM}_{\mathbb{R}}[0] & = & \mathbb{R}^0 & = & \{0\} \; , \\ \mathfrak{CM}_{\mathbb{R}}[1] & = & \mathbb{R}^0 \;\amalg\; \mathbb{R}^1 & = & \{0\} \;\amalg\; \mathbb{R} \; , \\ \mathfrak{CM}_{\mathbb{R}}[K] & = & \mathbb{R}^0 \;\amalg\; \mathbb{R}^1 \;\amalg\; \coprod_{p\,=\,1}^{k} \mathbb{R}^2 & = & \{0\} \;\amalg\; \mathbb{R} \;\amalg\; \coprod_{p\,=\,1}^{k} \mathbb{R}^2 \; . \end{array} \tag{27}$$

Where the last result holds for each $K \geq 2$, with

$$k := K - 1 . \tag{28}$$

This immediately extends to describing $\mathfrak{CM}_{\mathbb{R}}$ as well:

$$\mathfrak{CM}_{\mathbb{R}} = \mathbb{R}^0 \amalg \mathbb{R}^1 \amalg \coprod_{p=1}^{\infty} \mathbb{R}^2 = \{0\} \amalg \mathbb{R} \amalg \coprod_{p=1}^{\infty} \mathbb{R}^2 \tag{29}$$

2.3 Eigenspectra, multiplicities, and eigenspaces

Definition 1 Let λ_e denote some eigenvalue [44, 10, 55, 36, 70, 47, 45, 79, 12, 37, 51] of a size-K square matrix $\boldsymbol{M}$. The *Algebraic multiplicity* [10, 20, 51, 36, 47, 79] α_e of λ_e is the number of times that this occurs as a root of the matrix's characteristic polynomial.

Structure 2 The totality of eigenvalues for our matrix form its *eigenspectrum*,

$$\mathfrak{e}\text{spec}(\boldsymbol{M}) = \{ \lambda_e \,|\, e = 1 \text{ to } E \} = \lambda_1 \quad \dots \quad \lambda_E . \tag{30}$$

Let us call the multi-set version of this with Algebraic multiplicities included the *multi-eigenspectrum*

$$\mathfrak{me}\text{spec}(\boldsymbol{M}) = \begin{matrix} \lambda_1 & \dots & \lambda_E \\ \alpha_1 & \dots & \alpha_E \end{matrix} . \tag{31}$$

Eigenspectra are also arenas, albeit, for finite matrices, they are rather structurally simple ones.

Structure 3 Let us denote eigenvectors corresponding to λ_e by $\boldsymbol{v}_{e_i}$. So as to form a linearly independent (LI) set that is as large as possible. I.e. a LB for the *eigenspace* $\mathfrak{E}\text{ig}_e(\boldsymbol{M})$ corresponding to λ_e . In some cases, this is of dimension α_e , while in others, not as many LI eigenvectors as this can be found. This deficit is measured by the following further multiplicity.

Definition 3 An eigenvalue λ_e's *geometrical multiplicity* [10, 20, 55, 45, 36, 47, 79] γ_e is the dimension of its corresponding eigenspace,

$$\gamma_e := \dim (\mathfrak{E}\text{ig}_e(\boldsymbol{M})) . \tag{32}$$

Structure 4 There is also a larger notion of eigenspace:

$$\mathfrak{E}\text{ig}(\boldsymbol{M}) := \bigoplus_{e \in \mathfrak{e}\text{spec}(\boldsymbol{M})} \mathfrak{E}\text{ig}_e(\boldsymbol{M}) . \tag{33}$$

These smaller and larger notions of eigenspace are somewhat more structured simple and widely-used examples of arena. Their structure lies well within basic Linear Algebra, consisting of vector subspaces and direct sums thereof respectively.

3 Combinatorial matrices' eigenspectra

3.1 Eigenvalue degeneracy due to symmetry

Naming Remark 3 This involves one of two unrelated uses of 'degeneracy' used in literature on Spectral Theory and its applications. Namely, the one that is often found in Quantum Mechanics (QM) [34, 17, 16, 52] and Mathematical Physics [14]. Its Linear-Algebraic diagnostic is that Algebraic multiplicity $\alpha_e > 1$ for some eigenvalue λ_e .

Classification Theorem 1 for Combinatorial matrices)

G) A Combinatorial matrix's eigenspectrum consists of the following.

$$z := K\,x + y \text{ with } \alpha_z = 1 , \tag{34}$$

$$y \text{ with } \alpha_y = k , \tag{35}$$

Unless the Combinatorial matrix is of one of the following exceptional kinds.

I)

$$x = 0 , \tag{36}$$

for which the sole eigenvalue is

$$y \text{ with } \alpha_y = K . \tag{37}$$

U)

$$K = 0 , \tag{38}$$

for which there are no eigenvalues at all.

Notational Remark 4 Let us henceforth index these eigenvalues by their Algebraic multiplicities! Let us also denote the $\boldsymbol{C}$ of type G) by $\boldsymbol{G}$ and those of type I) by $\boldsymbol{I}$.

Remark 1 These exceptional cases arise from the following argument. For $K \geq 2$, $k \geq 1$, so both eigenvalues are realized (not necessarily distinctly). The linear equation for equal eigenvalues is then

$$K\,x + y = z = y . \tag{39}$$

Which cancels down to

$$K\,x = 0 . \tag{40}$$

And which of course admits 2 solutions: $x = 0$ and $K = 0$. The first is meaningful: isotropy. While the second is spurious, for our linear system was only defined for $K \geq 2$.

Finally, one needs to append the 2 cases excluded by the argument.

For $K = 1$, $k = 0$. Since this was to be the Algebraic multiplicity of the eigenvalue y, here this eigenvalue does not occur at all. This gives a rather trivial realization of isotropy: since there is only 1 direction, every direction must be the same!

While $K = 0$ – the unmatrix – has no room for any eigenvalues. So the unmatrix realizes the *uneigenspectrum*: an incarnation of the empty set $\emptyset$ consisting of no eigenvalues... Also it comes to pass that the above spurious solution coincides with a non-spurious appended case.

The unmatrix even manages to be isotropic in the even more trivial sense that *all* directions are the same whenever there are *no* directions. For all that this realization merits the qualifier *unisotropic* [so long as this is not confused with the much more widely used *anisotropic*...]

The generic case G) of the theorem is covered in e.g. [61], without mention however of the exceptional cases I) and U).

Remark 2 In the generic case G), $K \geq 2$ exhibits a $k \,|\, 1$ partition of the underlying vector space, as labelled by eigenvalues, into the corresponding eigenspaces.

$K = 3$ is minimum for this partition to be into larger and smaller pieces: $2 \,|\, 1$ (Subfig c). And thus is also minimum to have a dimensionally-nontrivial eigenspace: dimension ≥ 2. These features clearly persist for all subsequent K.

In contrast, case I)'s the eigenspace is unsplit: a maximally coarse-grained 1-piece partition K. Finally for case U), the eigenspace is the unpartition of the empty set!

* * *

Remark 3 The current subsection's notion of degeneracy is furthermore indeed underpinned by symmetry. This point is particularly emphasized in the QM literature [34, 17, 16, 52] for the ground-state energy eigenvalue of the Hamiltonian operator. Each realization of this notion can then be indexed by the corresponding symmetry group.

In case I), the corresponding total isotropy group is realized; indeed I) is intended to denote total isotropy. All Combinatorial matrices are symmetric, and ours are real-valued. Thus we enjoy the *real-symmetric* combination. For which the total isotropy group is the *orthogonal* group $O(K)$.

While in case G), this is restricted to the *next-largest partial* isotropy group: $O(k)$. Which manifests a continuum amount of degeneracy for $K \geq 3$ and thus $k \geq 2$. G) denotes that among real Combinatorial matrices, this constitutes the generic case.

We have already in effect pointed out that exceptionally for $K \leq 1$, it is isotropy that is generic.

U) is sufficiently distinct to merit its own category: it is still totally isotropic, in its own way, but is not even a Linear Algebra object!

Remark 4 In the above two ways, real Combinatorial matrices are highly non-generic within the space of all (or even all real-symmetric) square matrices...

Remark 5 The isotropic condition $x = 0$ replays a condition used in Sec 1. It corresponds to 2 examples there: $\mathbb{0}$ and $q\,\mathbb{I}$. The current Subsection does not have the means to distinguish between these two cases, while the next does.

Remark 6 Let us also introduce the 0-*symmetry-degeneracy fractions*

$$\mathcal{A}_e := \frac{\alpha_e}{K} .$$

Clearly for whichever set of partitioning fraction variables[2] the total sum of fractions is unity:

$$\sum_{\mathfrak{e}\text{spec}(\boldsymbol{M})} \mathcal{A}_e \ = \ 1 \ . \tag{41}$$

In case G), this reads

$$\mathcal{A}_1 \ + \ \mathcal{A}_k \ = \ 1 \ . \tag{42}$$

While in case I), this just becomes the identity $1 \ = \ 1$, via $\alpha_K \ = \ K$.

3.2 Non-full rank

Remark 7 We consider here a size-K matrix's counts of zero and nonzero eigenvalues by K_0 and K_* respectively. These exhaustively obey

$$K_0 \ + \ K_* \ = \ K \ . \tag{43}$$

K_0 constitutes a further notion of degeneracy. By the above equation, one could just as well conceptualize that K_* constitutes a such. K_0 furthermore meets the definition of *nullity*, and K_* of *rank*; in effect, these provide Spectral reformulations of these basic Linear Algebra notions. The above equation then amounts to a simple Spectral proof of the *rank–nullity theorem* [10, 36, 44, 55, 54, 79]. The current subsection's notion of degeneracy is can then be characterized as non-full rank or, equivalently, non-empty kernel: in the form of the zero-eigenvalue count.

Let us also introduce the 0-*eigenvalue degeneracy fraction*

$$\mathcal{K}_0 \ := \ \frac{K_0}{K} \ .$$

And the 0-*eigenvalue nondegeneracy fraction*

$$\mathcal{K}_* \ := \ \frac{K_*}{K} \ .$$

Which of course obey

$$\mathcal{K}_0 \ + \ \mathcal{K}_* \ = \ 1 \ . \tag{44}$$

Thus casting the rank–nullity theorem into 'mass fraction' form.

Classification Theorem 2 for Combinatorial matrices

The isotropic case I) now splits into 2 subcases as follows.

Rank 0)

$$y \ = \ 0 \ , \tag{45}$$

yielding the zero matrix

$$\mathbb{0} \ = \ (0, 0)_K \ .$$

For which the eigenvalues are all 0 . I.e.

$$0 \ \text{ with } \ \alpha_K \ = \ K \ : \ K_* \ = \ 0 \ \text{ and } \ K_0 \ = \ K \ . \tag{46}$$

And rank K)

$$y \ \neq \ 0 \ , \tag{47}$$

[2]Compare for instance mass fractions in Physics and Dynamics, and partial-pressure fractions in Physics and Chemistry.

for which we have a matrix proportional to the identity,

$$q \mathbb{I} \ , \quad q \neq 0 \ . \tag{48}$$

Here the eigenvalues are

$$q \text{ with } \alpha_K = K : \ K_* = K \text{ and } K_0 = 0 \ . \tag{49}$$

The generic case G) also contains 2 subcases that manifest zeros.

Rank 1)

$$y = 0 \ , \quad K x \neq 0 \ . \tag{50}$$

These are the matrices proportional to the units matrix[3]

$$\mathbb{1} := (1, 1)_K \ .$$

So now

$$q \mathbb{1} \ , \quad q \neq 0 \ . \tag{51}$$

Its eigenvalues are

$$0 \text{ with } \alpha_k = k : \ K_0 = k \ . \tag{52}$$

$$K x \text{ with } \alpha_1 = 1 : \ K_0 = 1 \ . \tag{53}$$

Rank k) Now

$$z = K x + y = 0 \ , \quad y \neq 0 \ . \tag{54}$$

Returing the matrices proportional to

$$\boldsymbol{P} = K^{-1} (k, -1)_K \ . \tag{55}$$

For which the eigenvalues are

$$0 \text{ with } \alpha_1 = 1 : \ K_0 = 1 \ . \tag{56}$$

And

$$y \text{ with } \alpha_k = k \ . \tag{57}$$

G) contains furthermore a zero-free case: rank K . Here

$$K_* = K \text{ and } K_0 = 0 \ .$$

Remark 8 For $K = 0$, the unmatrix has no eigenvalues and thus no capacity to exhibit zero eigenvalues.

Remark 9 The current Subsection partners the linear equation

$$x = 0$$

– shared with Secs 1 and 3.1 – with the following new linear equation.

$$z = K x + y = 0 \ . \tag{58}$$

These are also the 2 ways in which $\det \boldsymbol{C} = 0$ determinant and thus that $\boldsymbol{C}$ can be singular.

Compare (58) with Sec 1's zero-trace equation (12); both determinant and trace are invariants. In fact, for $K = 1$, $t = 0$ and $z = 0$ coincide. This reflects that $K = 1$ are just the numbers, which do not support distinct trace and determinant...

[3]This matrix occurs in Graph Theory and also in the role of identity for the element-wise product of matrices. See Subsec 3.3 for a further special Combinatorial matrix proportional to this one.

3.3 Combinatorial projectors

Structure 5 At the level of eigenvalues, a *proper projector* [36, 45, 47, 29, 79] has eigenvalues 0 and 1. Since this is a specialization of having 2 distinct eigenvalues, it is compatible with class G). And more specifically with zero-count degenerate such.

Remark 10 There are 2 orders in which a Combinatorial matrix can implement such eigenvalues at the level of a linear system of equations. Firstly,

$$z = K x + y = 0 , \quad y = 1 . \tag{59}$$

Which is solved by

$$\boldsymbol{P}(K) = K^{-1} (k, -1)_K . \tag{60}$$

Secondly,

$$z = K x + y = 1 , \quad y = 0 . \tag{61}$$

Which is solved by

$$\boldsymbol{P}_{\perp}(K) = K^{-1} (1, 1)_K = K^{-1} \mathbb{1} . \tag{62}$$

Remark 11 We already encountered $\boldsymbol{P}(K)$ in the previous subsection; we have now established that this is a projector. While $\boldsymbol{P}_{\perp}(K)$ is the special Combinatorial matrix alluded to in footnote 3. Since Combinatorial matrices are symmetric, and we have taken them to be $\mathbb{R}$-valued, projectors in this context are automatically orthogonal [36, 45, 47, 29, 79]. Finally, $\boldsymbol{P}_{\perp}(K)$ is the orthogonal complement of $\boldsymbol{P}(K)$.

3.4 Combinatorial involutors

Structure 6 At the level of eigenvalues, a nontrivial involutor [13, 76] has eigenvalues ± 1. Again, this is a specialization of having 2 distinct eigenvalues, which is thus compatible with symmetry-generic Combinatorial matrices. Now more specifically with the zero-count nondegenerate case.

Remark 12 There are 2 orders in which a Combinatorial matrix can implement such eigenvalues at the level of a linear system of equations. These can now be jointly posed and solved as follows.

$$z = K x + y = \pm 1 , \quad y = \mp 1 . \tag{63}$$

Which are solved by

$$\pm \boldsymbol{J}(K) = \pm K^{-1} (2 - K, 2)_K . \tag{64}$$

* * *

3.5 N-body problem subcase of Combinatorial matrix eigenvalues

Notational Remark 5 For this let us use the notation N in place of K.

Structure 7 Here one has a constellation of N points-or-particles in $\mathbb{R}^d$ space. Given a possibly transient absolute origin, each point-or particle has a position vector relative to this. The space of all possible constellations is the configuration space [11, 23, 57] *constellation space*,

$$\mathfrak{q}(d; N) = \mathbb{R}^{d N} .$$

Each point-or-particle can also be allotted a label. The space of all possible LCs of point-or-particle labels – position labels – is *constellation label space*,

$$\mathfrak{Lq}(N) \;=\; \mathbb{R}^N \;.$$

One then passes to separation vectors between points-or-particles. Absolute origin dependence cancels out of these.

Structure 8 The space of all linearly independent (LI) separation vectors is the configuration space *relative space*

$$\mathfrak{Rel}(\,d,\,N\,) \;=\; \mathbb{R}^{d\,N} \;.$$

Naming Remark 4 This name is used in e.g. [57, 76], with reference to an LI set of relative separations or of relative differences.

Remark 13 Each separation in an LI set can be allotted another label, now with values running from 1 to

$$N \;:=\; n \;-\; 1 \;. \tag{65}$$

The totality of LCs of which form in turn *relative label space*,

$$\mathfrak{LRel}(n) \;=\; \mathbb{R}^n \;.$$

[57, 76] explain how the above $\mathfrak{L}$ versions are active factors in the corresponding multi-index tensor product $\mathfrak{L}$ -less versions. Passing to $\mathfrak{Rel}(n)$ and $\mathfrak{LRel}(n)$ amounts to quotienting out translations, $Tr(d)$. Various further simple quotienting procedures [32, 35, 57, 73, 76] permit handling dilations and 2-d rotations.

Remark 14

$$\boldsymbol{P}(N) \;:=\; N^{-1}\,(\,n,\,\text{-}1\,)_N \;. \tag{66}$$

shall be interpreted as a projector in Sec 4.2. Our first of many names for this is *positions-to-relative separations matrix* at the level of the internal labels [57, 76]. A second is *Lagrange matrix* [4, 57, 76]. For all that $\boldsymbol{P}$ is numerically, and yet not Physical-dimensionally, equal to the equal-masses case of this [65, 76]. For $N \;=\; 3$, the above specializes to

$$\boldsymbol{P} \;:=\; \frac{1}{3}\,(\,2,\,\text{-}1\,)\;. \tag{67}$$

3.6 N-vertex and n-simplex cases, with triangle or 2-simplex examples

Remark 15 These refer to two ways of describing the Geometrical counterpart of the N-body problem. Within the translation and rotation quotiented setting, the following are natural for $N \;=\; 3$.

A) **Apollonius' theorem** [1, 28, 53, 76] for expressing a triangle's median lengths in terms of its side lengths. The Euler 3-cycle – over sides – of this can be expressed as a matrix equation [58, 64, 76]. Whose matrix turns out to be proportional to an involutor: the *Apollonius involutor*,

$$\boldsymbol{J} \;:=\; \frac{1}{3}\,(\,\text{-}1,\,2\,)\;. \tag{68}$$

A functional alias for this is *sides-medians length-exchange involutor*. Medians, and $\boldsymbol{J}$, can be defined to transcend to [76] arbitrary dimension – whether spatial or configuration-occupying

– So $\boldsymbol{J}$ is more generally a 2-simplex matrix. We finally recognize this as the $\boldsymbol{J}(3)$ subcase of (64).

B) **Heron's formula** [2, 13, 46, 56, 76]. The square of this, when viewed as a quadratic form, is built out of the following matrix.

$$\boldsymbol{F} \ := \ (\text{-}1, 1) \ . \qquad (69)$$

This was first phrased in linear system form by Euler [3] and first explicitly written down as a matrix by Buchholz [26]. At which level the names *Heron matrix* or, more functionally, *sides-data triangle area formula matrix*, are suitable.

Article 2 however subsequently pointed out that this also occurs in the Euler 3-cycle of cosine rules, and even of triangle inequalities. By which the name *fundamental triangle matrix* [64] and the notation $\boldsymbol{F}$ are more appropriate. Its fundamentality [58, 65] is further warranted by its ties to Hopf's little map [8, 41, 42, 77]. It furthermore transcends to [76] arbitrary dimension – whether spatial or configuration-occupying – sealing our final name for it: *fundamental* 2-*simplex matrix*. We also recognize this as the $\mathbb{V}(3)$ subcase of the trace-reversed Combinatorial matrix (21).

Structure 9 (67, 68, 69) are the 3 triangle matrices, or more generally by transcending arbitrary dimension, the 3 2-simplex matrices. In each of their equations, we have dropped the K subscripts since they are all 3 's).

Remark 16 All of $\mathbb{T}$, $\boldsymbol{J}$, $\boldsymbol{F}$ and $\boldsymbol{P}$ for $N = K \geq 2$ are generic in sense G). $\boldsymbol{P}$ is of the rank-k subclass, while the other 3 are of the rank-K subclass.

Remark 17 Among the $N = K = 3$ matrices, $\boldsymbol{P}$ is the only projector of rank 2 .

$\boldsymbol{J}$ is one of the two signs of nontrivial involutor.

$\boldsymbol{F}$ is the trace-reversed matrix.

Thus our purely Algebraic considerations manage to find all of these. For all that these considerations do not single them out among various other involutors, projectors and Representation-Theoretically privileged matrices.

Remark 18 Every projector $\boldsymbol{B}$ can be associated with an involutor $\boldsymbol{K}$ according to [10]

$$\boldsymbol{K} \ = \ 2\,\boldsymbol{B} \ - \ \mathbb{I} \ . \qquad (70)$$

For $\boldsymbol{P}(N)$, the corresponding involutor is $-\boldsymbol{J}(N)$ This includes the 2-simplex CoM label-removing projector $\boldsymbol{P}$ pairing with minus the Apollonius involutor, $-\boldsymbol{J}$. So

$$\boldsymbol{J} \ + \ 2\,\boldsymbol{P} \ = \ \mathbb{I} \ . \qquad (71)$$

* * *

3.7 Real-symmetric matrices's eigenvalues

Notational Remark 6 Let us denote the arbitrary real symmetric matrix by $\boldsymbol{S}$.

Remark 18 It is well-known that all eigenvalues of $\boldsymbol{S}$ are real.

Remark 19 It is thus meaningful to allot a sign to each. Whether $+$ or $-$ for each nonzero eigenvalue. Or $+$, $-$ or 0 for every eigenvalue: the 3-*valued* notion of sign.

3.8 Physicists' signatures

Notational Remark 7 Let $K_\pm$ denote the counts of positive and negative eigenvalues respectively. In each case with Algebraic multiplicity included. Let

$$\mathcal{K}_\pm \ := \ \frac{K_\pm}{K} \ .$$

So that

$$\mathcal{K}_0 \ + \ \mathcal{K}_+ \ + \ \mathcal{K}_- \ = \ 1 \ .$$

Also for matrices of nonzero rank – $K_* \neq 0$ – define the *nondegenerate sign fractions*

$$\mathcal{ND}_\pm \ := \ \frac{K_\pm}{K_*} \ .$$

So that

$$\mathcal{K}_0 \ + \ \mathcal{K}_* \ (\ \mathcal{ND}_+ \ + \ \mathcal{ND}_- \ ) \ = \ 1 \ .$$

The $K_\pm$ are furthermore reflectively-symmetrically defined. To the extent that which is $+$ and which is $-$ is often taken to be a convention.[4]

Thus

$$\Delta\, K \ := \ K_+ \ - \ K_-$$

is in some ways a more meaningful difference than

$$\delta\, K \ := \ K_0 \ - \ K_* \ .$$

Hence the difference in notation. Δ is moreover not only reflectively-symmetric but also a proto-index.[5]

$\Delta\, K$ is furthermore (one sign convention choice of) the Physicists' signature in summary. The signature in detail exhibits how many $+$'s, $-$'s and 0 's are present. E.g. $- + + +$ for one sign convention for Minkowski spacetime. Or $+ + + 0$ for the 4-body problem's CoM label removing projector $\boldsymbol{P}$ ... This notation is used in Fig 2, with a truncated version of it in Figs 4 and 6. More efficiently especially for much larger examples, it is Körner's [50] triple of signs

$$(\, K_+, \, K_-, \, K_0 \,) \ .$$

Though S. Sánchez' presentation [60]

$$(\, K_0, \, K_*, \, \Delta\, K \,)$$

[4]So in Special Relativity, spacetime is modelled with $-$ for time and $+$ for space or vice versa!

[5]This is in the sense of index theorems; compare the Poincaré index formula [24]. The rank–nullity [79], Gauss–Bonnet [80], Riemann-Roch [27] and Atiyah–Singer [25] index theorems. And quite a few basic Combinatorics examples in [62] and basic Geometry examples in [76]; see footnote 6 for some examples of each.

is a more elegant sign-space LB choice. Picked so as to manifest the signature-in-summary proto-index among its LB elements... This kind of parametrization also permits exhibition of beloved cases in which $K_{\pm}$ are infinite and yet ΔK manages to remain finite.

Classification Theorem 3 for Combinatorial Matrices In the generic case G) away from zeros – rank K – there are for $K \geq 2$ 4 nontrivial cases for signs of eigenvalues.

$+$+)

$$y > 0 , \quad z = K x + y > 0 . \tag{72}$$

is *positive-definite*: all $\lambda_e > 0$,

$$K_+ = K .$$

$-$+)

$$y < 0 , \quad z = K x + y < 0 \tag{73}$$

is *negative-definite*: all $\lambda_e > 0$:

$$K_- = K .$$

$+$-)

$$y > 0 , \quad z = K x + y < 0 \tag{74}$$

is minimumly indefinite with sign convention

$$K_- = 1 , \quad K_+ = k .$$

$-$-) is also minimumly indefinite

$$y < 0 , \quad K x + y > 0 , \tag{75}$$

albeit with the opposite sign convention:

$$K_- = k , \quad K_+ = 1 .$$

With zeros, rank k supports just 2 cases: its single nonzero eigenvalue can be $-$ or $+$, giving $\pm$ subcases. I.e. respectively

$$y > 0 , \quad z = 0 . \tag{76}$$

And

$$y < 0 , \quad z = 0 . \tag{77}$$

Rank 1 also supports just 2 cases: its nonzero eigenspace can be positive- or negative-definite: $\pm$ subcases. I.e. respectively

$$y = 0 , \quad z > 0 . \tag{78}$$

And

$$y = 0 , \quad z < 0 . \tag{79}$$

Case I) with no 0's – rank k – supports just 2 sign choices: positive- or negative-definite: subcases $\pm$. I.e. respectively

$$y > 0 , \quad x = 0 . \tag{80}$$

And

$$y < 0 , \quad x = 0 . \tag{81}$$

Remark 20 While previous subsections have used linear (systems of) equations, we have now passed to linear (systems of) inequalities.

Remark 21 $K = 2$ has both $+$ -) and $-$ +) collapse to the *balanced* [70, 76] situation:

$$K_+ = K_- .$$

And indeed are *minimumly nontrivially balanced*:

$$K_+ = 1 = K_- .$$

Balanced entails $+ \leftrightarrow -$ symmetry. In the $+-$ case, this has well-documented consequences [15]. For $K \geq 3$, however, there is a larger $+$ or $-$ eigenspace. I.e.

$$K_+ > K_- \text{ or } K_+ < K_- ,$$

corresponding to the presence of a symmetry-degenerate eigenspace.

Remark 22 The *Physicists' signature in summary* provides the following further interpretation. Balanced is the corresponding *null proto-index condition* [6]

$$\Delta K = 0 .$$

While the quantifier of departure from balance,

$$Imbalance(\boldsymbol{M}) = \Delta K ,$$

is the corresponding nontrivial proto-index.

Also for $K = 2$ with one zero, both rank k and rank 1 subcases conflate to

$$K_+ = 1 = K_0 , \quad K_- = 0 .$$

Or

$$K_- = 1 = K_0 , \quad K_+ = 0 .$$

In contrast, for $K \geq 3$, the rank 1 subcase has

$$K_+ > K_0 , \quad K_- = 0 .$$

Or

$$K_- > K_0 , \quad K_+ = 0 .$$

We gather up all of the current Subsection's cases into Fig 2's end-table.

Remark 23 All of $\mathbb{T}$, $\boldsymbol{J}$ and $\boldsymbol{F}$ are of type G with rank K and signature in detail -+ .

* * *

Definition 1 [15] A real symmetric matrix with no zero eigenvalues is *elliptic* if all of its eigenvalues are of the same sign. *Hyperbolic* if but 1 has opposing sign. And ultrahyperbolic if it has ≥ 2 copies of each sign. Let us use the same adjectives to describe the nondegenerate sectors of degenerate symmetric matrices.

Remark 24 Ultrahyperbolicity is rather harder and much less well understood [15]. They are however banished forever from our $\mathfrak{CM}_{\mathbb{R}}$ arena by i) of the following.

[6]This conceptual type covers the Euler characteristic on the circle and the tori, and thus Gauss–Bonnet type theorems thereupon. Grinberg's theorem in planar Hamiltonian Graph Theory [60, 62], Whose index is inside-outside triangulation strength imbalance. And which theorem we thus renamed ZIPHoN: 'zero-index planar Hamiltonian Necessity'. Varignon's theorem and Euler's 3-simplex theorem in Flat Geometry, along with 'smaller' infinite families of generalizations [59, 76]. Whose common index is the left-right child imbalance in unlabelled rooted at-most binary trees that first enters this area of study as per Sec 4.

Proposition 1 i) Neither Combinatorial matrices nor their zeros-nondegenerate sectors, can be ultrahyperbolic.

ii) Combinatorial matrices with non-empty zeros-degenerate sector must have an elliptic or empty nondegenerate sector.

<u>Proof</u> Such matrices of size K must have a symmetry-degenerate eigenspace of size $\geq k$.

i) Thus there is no room to partition signs into $2m$ pieces of size ≥ 2 . .

ii) The zeros-degenerate sector uses up 1 sign of eigenvalue. So only 1 sign is left for what nondegenerate sector there may be. Thus it is either empty. Or nonempty but left with insufficient sign types to support 2 distinct signs of its own, forcing it to be elliptic. □

3.9 Multiplicity equalities

Definition 1 An eigenvalue is called *semisimple* [51] if its Algebraic multiplicity coincides with its geometrical multiplicity.

Remark 25 A matrix is diagonalizable iff all of its eigenvalues are semisimple.

Proposition 2 All the $\boldsymbol{C} \in \mathfrak{CM}_{\mathbb{R}}$

i) have

$$\alpha_e = \gamma_e \text{ for each } \lambda_e . \tag{82}$$

ii) Are diagonalizable.

iii) Enjoy the arena equation

$$\mathfrak{E}\text{ig}(\boldsymbol{C}) = \mathfrak{V}^K = \mathbb{R}^K . \tag{83}$$

<u>Proof 1</u> All of these properties are inherited from the real symmetric matrices. □

4 Combinatorial matrices' eigenvectors

4.1 Finite Real symmetric matrices' eigenvectors

Remark 1 Finite such are well-known to be not only diagonalizable but also *diagonalizable using orthogonal matrices*. They furthermore admit *orthonormal LBs* (*ONLBs*). And their eigenbases can be taken to be eigenONLBs. These ONLBs can furthermore be taken to be *complete* [48], so we can expand in terms of these without losing any information in the process.

Remark 2 Our $\mathfrak{CM}_{\mathbb{R}}(K)$ inherit all of these features by being finite, real and symmetric. More specifically, proposition 2.iii) can be viewed as a completeness relation. I.e. the *Spectral completeness relation* that the eigenvectors of $\boldsymbol{C}$ form a LB for the whole K-dimensional vector space that $\boldsymbol{C}$ naturally acts upon.

4.2 N-body subcase of Combinatorial matrices' eigenvectors

Remark 3 In this context, the generic G)'s lone eigenvector corresponds to the CoM position vector's label R . Corresponding to the normalized version of the vector of 1 's. And whose linear span (LS) forms the eigenspace

$$\mathfrak{E}\mathrm{ig}_1(\boldsymbol{G}) = \mathfrak{CoM}(1) = \mathbb{R} .$$

Remark 4 Also in this context, the generic case's symmetry-degenerate eigenspace is relative label space:

$$\mathfrak{E}\mathrm{ig}_n(\boldsymbol{G}) = \mathfrak{LRel}(n) = \mathbb{R}^n . \tag{84}$$

This can be studied by considering a LB of separations between points-or-particles.

Remark 5 These eigenspaces fit together to form

$$\mathbb{R}^N = \mathfrak{E}\mathrm{ig}(\boldsymbol{G}) = \mathfrak{E}\mathrm{ig}_1(\boldsymbol{G}) \oplus \mathfrak{E}\mathrm{ig}_n(\boldsymbol{G}) = \mathfrak{CoM}(1) \oplus \mathfrak{D}\mathrm{if}(n) . \tag{85}$$

Remark 6 In contrast, in the isotropic case I), there is a single irreducible eigenspace

$$\mathbb{R}^N = \mathfrak{E}\mathrm{ig}(\boldsymbol{I}) = \mathfrak{E}\mathrm{ig}_N(\boldsymbol{I}) . \tag{86}$$

Remark 7 $N = 2$ is minimum for the above LB to be nonempty.

$N = 3$ is minimum for this LB to not be non-diagonal. Passing to *point-or-particle clustering separations* – between subsystem CoMs – attains diagonality however. In the Dynamics context, this (non)diagonality is manifested by such as the total moment of inertia and the kinetic energy. The corresponding clustering separation vectors have hitherto been called *relative Jacobi vectors* [5, 31, 32, 57, 73, 76].

$N = 3$ is furthermore minimum for ambiguity in labelling clusterings. Here these arise from how the input points-or-particles are labelled. Corresponding to the number of ways of of leaving out a single point-or-particle. Or, equivalently, of forming a pair subsystem, whose relative separation is our current object of interest.

$N = 4$ is minimum for clusterings to possess a distinct notion of network ambiguities. The clustering structure can here be H- or K-shaped; these have often been called the Jacobi-H and -K .

Remark 8 The inertia and kinetic quadrics in relative coordinates can be modelled using the Lagrange matrix. We have already linked this to our projector onto relative label space, $\boldsymbol{P}$. Thus making contact with Combinatorial matrices and their Spectral Theory.

Remark 9 Given Remark 1, we can now further qualify on the one hand that $\boldsymbol{P}$ amounts to projecting onto relative label space. Hence the name *relative label space projector*. Along the CoM label direction, thus projecting out the CoM label. *Projecting along* [10] has subsequently also been described as *projecting out*. Hence the further name *CoM label-removing projector*. And, upon tensoring up, the names *CoM-removing projector* and *relative space projector* follow. The first of which bears some relation to the common practise in Physics of passing to the CoM frame. And furthermore explains Montgomery's [73] alias for relative space: *centred configuration space*, with reference to centering about the CoM position. This space featured in e.g. [32, 35] long before the above and Sec 4.2's references.

On the other hand, the orthogonal complement projector $\boldsymbol{P}_{\perp}$ projects onto the CoM label space.

Naming Remark 5 A truer name for relative Jacobi coordinates is *eigenclusterings* [59, 57, 76].

* * *

Remark 10 $\boldsymbol{P}$ is generic in sense G), so the full underlying symmetry is $O(n)$. In the N-body problem context, these have been termed *internal rotations*, alias *democracy transformations* [31]. These are internal in the sense that they act not on space but on the internal space of point-or-particle labels. By sending eigenONLBs for $\mathfrak{Rel}(n)$ to other eigenONLBs.

Remark 11 The above features attributed to $N = 2$ to 4 are all persistent. Both types of ambiguity are furthermore increasingly persistent. Network ambiguity is indexed by [39, 67] the *unlabelled rooted binary trees* (*URBT*) [38, 49, 62]. Whose counts are [67] the *Wedderburn–Etherington numbers* [6, 9, 22, 74]. While the labelling ambiguity goes like the sizes of the corresponding orbits of the permutation group S_N acting on the labels.

Structure 10 Each $N \geq 1$ supports a generalized-K eigenONLB of relative Jacobi vectors. The generalized-K corresponds to each N supporting an URBT which, upon defoliating once [67], is the straight path P_n [67]. Aside from P_3 corresponding to K-shaped clustering, P_2 is T-shaped: side and corresponding median. While P_1 just involves the incipient point-or-particle separation, and P_0 is just the ungraph presentation of the empty set. This said, we do not need a clustering conceptualization to understand $N = 1$ to 2. This is by replaying the above minimum cases, in the form that $N = 2$ supports no clustering other than its sole separation. And that $N = 1$ supports no separations at all!

Remark 12 Take any eigenONLB (of relative label space!) of relative Jacobi vectors and adjoin R. This forms the corresponding eigenONLB (now for all of $\mathfrak{V}$) (*absolute*) *Jacobi vectors*, alias *eigenclustering vectors with CoM position adjoined*. Hitherto, in the Dynamics literature, the Jacobi vectors have been associated with CoM hierarchies. Which can be reformulated as choices of eigenONLB for the Lagrange matrix, and thus for the relative space projector $\boldsymbol{P}$. Which exists for any N-body problem in any $\mathbb{R}^d$.

Remark 13 When the full $O(N)$ is present, it sends ONLBs of

$$\mathfrak{Eig}_N(\boldsymbol{I}) = \mathfrak{Eig}(\boldsymbol{I}) = \mathfrak{V}^N = \mathbb{R}^N .$$

to other ONLBs.

4.3 Generalization to any Combinatorial matrix

Remark 14 Let us now shift away from the above context to Combinatorial matrices in full generality. The generic G) case's lone eigenvector U corresponds to the sum of the counts acted upon. Or equivalently, given subsequent normalization, the *average of the counts* (*AoC*). Whose LS forms the eigenspace

$$\mathfrak{Eig}_1(\boldsymbol{G}) = \mathfrak{AoC}(1) = \mathbb{R} .$$

Remark 15 Relative separation labels of pairs of point-or-particle positions become differences between counts. Now forming the generic G)'s symmetry-degenerate eigenspace *difference space*

$$\mathfrak{Eig}_k(\boldsymbol{G}) = \mathfrak{Dif}(k) = \mathbb{R}^k . \tag{87}$$

This can be studied by forming a LB of differences between counts.

Remark 16 These eigenspaces fit together to form

$$\mathbb{R}^K = \mathfrak{Eig}(\boldsymbol{G}) = \mathfrak{Eig}_1(\boldsymbol{G}) \oplus \mathfrak{Eig}_k(\boldsymbol{G}) = \mathfrak{AoC}(1) \oplus \mathfrak{Dif}(\boldsymbol{G}) . \tag{88}$$

Remark 17 In contrast, in the isotropic case I), there is a single irreducible eigenspace

$$\mathbb{R}^K = \mathfrak{Eig}(\boldsymbol{I}) = \mathfrak{Eig}_K(\boldsymbol{I}) . \tag{89}$$

Remark 18 For $K \geq 3$, the above LB of differences is not diagonal. Passing to LCs of these – *count subset differences*: between 2 subsets' counts – attains diagonality however.

Naming Remark 6 Given this more general context, *combinatorial-matrix eigenvectors*, or for short *eigencombinatorial vectors*, is in turn a truer name than eigenclustering vectors.

Remark 19 $K = 3$ is minimum for ambiguity in labelling differences of counts. Here these arise from labelling the input counts. 3 labellings of differences of counts are possible. Corresponding to the number of ways of of leaving out 1 count. Or, equivalently, of picking 2 counts to form a difference out of.

Remark 20 $\boldsymbol{P}$ now amounts to projecting onto difference space, hence the name *difference space projector*. Projecting out AoC direction space, hence the further name *AoC-removing projector*. $\boldsymbol{P}_\perp$ now complementarily amounts to projecting onto AoC space, hence *AoC space projector*. Projecting out difference space, hence also *difference space-removing projector*.

* * *

Remark 21 Take any eigenbasis of eigencombinatorial vectors and adjoin U . This forms the corresponding *extended* basis of *eigencombinatorial vectors* with CoM position adjoined.

Remark 22 The eigencombinatorial vectors exhibit the same network ambiguity of URBT form as described above.

Proposition 4 Any $K \geq 1$ Combinatorial matrix can be equipped with an extended generalized-K eigencombinatorial eigenbasis.

Proof 2 A generalized K is available for all K [67] as the straight-path P_k URBT. Form the difference between a first pair of objects. Next form the difference between the sum of these and twice a third object. Apply this move recursively between the sum of the first $k - 1$ objects used and $k - 1$ times a kth object. Finally adjoin U . □

Proof of proposition 2. i) Using the K LB, $\gamma_e = \alpha_e$ for each λ_e . ii) and iii) then separately follow from i). □

Exercise 1 Prove proposition 3.

* * *

Remark 23 In the case of a general network, each step uses rather the difference between left- and right-child sums. For the N-body problem subcase, this specializes to left and right clusterings' masses.

Remark 24 The URBT ambiguity was long known to arise from CoM hierarchies: Mechanics to Combinatorics ([39] and the literature survey in an Appendix of [67]). A more natural perspective is that general Combinatorial matrices give further standard Combinatorial objects as their eigenvectors. Dynamics' CoM hierarchy then produces Combinatorial objects by enacting a subcase of this. We have thus passed to a purely Combinatorial explanation.

Remark 25 Eigencombinatorial eigenbases are but a measure-0 subset of the possible eigenONLB. This is based upon the relative sizes of the finite permutation subgroup S_K versus the infinite orthogonal group $O(k)$ corresponding to allowing all $\mathbb{R}$-LCs.

Naming Remark 7 So far as the Author is aware, the Combinatorial literature has not pinned a name on this generalized setting for what Molecular Physicists call internal rotations or democracy transformations. 'Internal' here refers to LI relative separation labels. So a natural name for the Combinatorial counterpart is *count-difference rotations.* Acting by sending eigenONLBs for $\mathfrak{Dif}(n)$ to other eigenONLBs.

Remark 26 When the full $O(K)$ is present, it sends ONLBs of

$$\mathfrak{E}\text{ig}_K(\boldsymbol{I}) = \mathfrak{E}\text{ig}(\boldsymbol{I}) = \mathfrak{V}^K = \mathbb{R}^K .$$

to other ONLBs.

4.4 Eigevector classification

Classification theorem 3 for Combinatorial matrices With reference to a cover by some of the above-defined cases, a Combinatorial matrix's normalized eigenvectors take the following corresponding forms.

G) The normalized unit vector. Alongside any normalized LB choice of LCs of count-difference vectors.

I) Any normalized LB for $\mathbb{R}^K$ will do.

U) The *uneigenbasis* consisting of an empty set's worth of eigenvectors.

Proof G). For the lone eigenvalue, the eigenvector equation is

$$(-k\,x\,,x\,)_K \cdot \boldsymbol{x} = \mathbf{0}\ . \tag{90}$$

Which is solved by

$$x_1 = \ldots = x_K = 1\ . \tag{91}$$

Finally divide by the corresponding normalization factor

$$\sqrt{\sum_{i\,=\,1}^{K} 1^2} = \sqrt{K}\ . \tag{92}$$

For the other eigenspace's eigenvalue,

$$x\,\mathbb{1} \cdot \boldsymbol{x} = \mathbf{0}\ . \tag{93}$$

Which is solved as claimed.

I) The eigenvector equation now reads

$$(\,0\,,0\,)_K \cdot \boldsymbol{x} = \mathbf{0}\ . \tag{94}$$

Which places no restrictions on what $\boldsymbol{x}$ can serve as an eigenvector.

U) Now there is no eigenvector equation, but no vectors to restrict either. The restriction of the empty set $\emptyset$ by the (empty set of equations) is of course just $\emptyset$ again. □

4.5 Sharing eigenbases and eigenspaces

Corollary 1 Any generic set of same-size Combinatorial matrices can be taken to share eigenbasis.

Proof For $K = 0$, all must be copies of the unmatrix, and thus share the same empty set of eigenspaces.

For $K \geq 1$, by theorem 3 any LB for $\mathbb{R}^K$ will do for class I). Thus pick the extended version of the generalized-K LB so as to match class G). □

Remark 27 In Fig 3, this alignment is drawn out using green for I)'s single eigenspace versus blue and yellow for G)'s pair.

Proposition 5 Suppose that we are given a set of size-K Combinatorial matrices. Then they share eigenspaces iff either of the following hold.

i) $K \leq 1$.

ii) $K \geq 2$ and they are either all generic G) or all isotropic I).

<u>Proof</u> For $K = 0$, each matrix in the set can only be a copy of the unmatrix. All of which share the same eigenspaces: no eigenspaces at all!

For $K = 1$, only 1 eigenspace can be realized and thus must be shared by all.

For $K \geq 2$, two cases work out. Firstly, a set of isotropic matrices I) with the same K shares the same K-fold eigenspace. I.e. the whole vector space acted upon. Secondly, a set of generic matrices G) with the same K share the same 1-d eigenspace in each case with the same k-fold complement. The remaining case – sets containing ≥ 1 G) and ≥ 1 I) do not work out, by concurrently realizing both the split and the unsplit eigenspaces. □

Corollary 2 Our 3 2-simplex matrices

i) possess a shared eigenbasis, which can be taken to be the extended version of T .

ii) They share eigenspaces.

<u>Proof</u> i)

$$\text{All of the 2-simplex matrices are Combinatorial and of the same size } K = N = 3 \ . \tag{95}$$

So theorem 3 gives that they share eigenbasis. And that this can be taken to be the extended generalized-K eigenbasis. Which for $\mathrm{K} = 3$ is the extended T .

ii)

$$\text{All of the 2-simplex matrices are furthermore generic G) .} \tag{96}$$

So proposition 5 $\Rightarrow$ they share eigenspaces. □

5 Conclusion

Remark 1 The current Article has thus replaced Articles 2 to 4's piecemeal Geometrical considerations of the Linear Algebra properties of the 2-simplex matrices $\boldsymbol{P}$, $\boldsymbol{J}$, $\boldsymbol{F}$. Paralleling Ford's [71] observations that the 2-simplex matrices' Abstract Algebra properties – commutators, and simple products leading into commutative monoids – follow just from their being Combinatorial matrices of the same size. Proposition 5's condition " $K \geq 2$ and they are either all generic G) or all isotropic I)" is the caveat alluded to in the Abstract and Introduction. By this, the current Article is not quite as clean as Ford's, in which any Combinatorial matrices of the same size will do. But for the 2-simplex matrices, (95) guarantees $K \geq 2$. While (96) guarantees the all generic G) subcase. So our parallel is secure.

Remark 2 Also, unique specifications of $\boldsymbol{P}$ and $\boldsymbol{J}$ as a particular Combinatorial matrix projector and involutor follow from whichever of Ford's account and the current one.

Remark 3 We form a comparison table in Fig 1 for the current Article's paradigm shift from N-body problem use of Combinatorial matrices to general use. A large part of the theory of centres of mass (CoM) is thereby reduced to purely a matter of Combinatorics. And we have a precise name for what Physicists' 'hierarchies of subsystems' CoMs' are. I.e. one very specific realization of the unlabelled rooted binary trees (URBTs); see [67] for the precise correspondence. With each CoM's 2 input subsystems being the right and left children of that CoM as viewed as a node.

Remark 4 We summarize many of the current Article's other results so far in Appendix A. And take our study of Spectral classification for Combinatorial matrices further in Appendix B. This is by use of Order Theory alongside more structurally sparse Graph Theory undepinning this.

Pointer 1 Combinatorial matrices are often taken to be $\mathbb{N}$- or $\mathbb{Z}$-valued in Combinatorics. Thus forming the arenas $\mathfrak{CM}_{\mathbb{N}}$ and $\mathfrak{CM}_{\mathbb{Z}}$. The current Article's analysis extends to $\mathfrak{CM}_{\mathbb{R}}$ for its Dynamics and Geometry significance, and Linear Algebra specifics of our workings. Let us leave the yet more general $\mathfrak{CM}_{\mathbb{C}}$ for another occasion.

Pointer 2 As regards n-simplex matrices, some remaining open questions are as follows [68, 69, 72]. Which sets of same-size quadrilateral matrices commute, form multiplicative commutative monoids, share eigenspaces and share eigenbases? For here not all of the matrices in any of these sets considered are Combinatorial...

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Wheelerian comparison table		
Notions	N-**body matrices** (Dynamics); N-**vertex matrices**, n-**simplex matrices** (Geometry)	**General Combinatorial matrices** $\boldsymbol{C}$ **of size** K (Combinatorics)
Examples	In particular the equal-masses Lagrange matrix, $\boldsymbol{L}$, which is numerically equal to CoM label-removing projector $\boldsymbol{P}$. Whose complement is the CoM label projector $\boldsymbol{P}_{\perp}$. The Apollonius involutor $\boldsymbol{J}$ and the fundamental 2-simplex matrix $\boldsymbol{F}$ also appear in Geometrical study	Two specific cases among which are the average-of-counts removing projector $\boldsymbol{P}$. And its complement the average-of-counts projector $\boldsymbol{P}_{\perp}$. We do not however need $\boldsymbol{P}$ in order to develop the theory.
Eigenspaces: lone	CoM label direction space $\mathfrak{CoM}(1)$	Average-of-counts direction space $\mathfrak{AoC}(1)$ [A 2025]
symmetry-degenerate	Relative label space $\mathfrak{LRel}(n)$ [1990s?]	Difference space $\mathfrak{Dif}(k)$ [A 2025]
Totality of eigenvectors: traditional name	Jacobi vectors [19th century]	
Degenerate space's eigenvectors' traditional name	Relative Jacobi vectors [1990s?]	
Degenerate space's eigenvectors: conceptual name	Eigenclustering vectors [SA 2018]	Eigencombination vectors [A 2025]
Totality of eigenvectors: conceptual name	Eigenclustering vectors extended by CoM vector	Eigencombination vectors extended by average-count vector [A2025]
Network ambiguity	H versus K network ambiguity for $N = 4$ [19th century] → Unlabelled rooted binary trees: Mechanics producing Combinatorics [e.g. S 2002]. Now however explained as a subcase of Combinatorics producing more Combinatorics [A2025]	Unlabelled rooted binary trees: Combinatorics producing more Combinatorics [A2025]
Labelling ambiguity	S_N orbits	S_K orbits
Full ambiguity: generic case G)	$O(n)$, called internal rotations or democracy transformations	$O(k)$ now named count-difference rotations [A 2025]
Isotropic case I)	$O(N)$	$O(K)$

Figure 1:

A Eigenvalue and eigenvector classifications

Remark 1 We condense many of Sec 3 and 4's other results into tables 2 and 3 respectively.

Classification of Combinatorial matrices'eigenvalues

Symmetry class	Eigen-values		Rank	Nullity	Notions of signature: s_{Math}	Notions of signature: s_{Phys}	Notions of signature: $s_{\text{Phys-detail}}$	Examples	Notes	
Generic combinatorial matrices *G*. **Class G):** O(k)	z 1 1	y k k	K	0	K -K k -k	K -K K -2 2- K	+ ... + – ... – + ... + –	𝕋, ***J*** ⊂ ***J***(K), ***F*** ⊂ 𝕍 or proportional	Nondegenerate (no zero eigenvalues)	elliptic hyperbolic
	0 1 1	y k k	k	1	k -k	k -k	– ... – + + ...+ 0 – ... – 0	***P*** ⊂ ***P***(K) or proportional	Degenerate (zero eigenvalues)	Nondegenerate sector is elliptic
	z 1 1	0 k k	1	k	1 -1	1 -1	+ 0 ... 0 – 0 ... 0	𝟙 or proportional, including $\boldsymbol{P}_{\perp}$		
Isotropic matrices *I*. **Class I):** O(K)		0 K K	K	0	K -K	K -K	+ ... + – ... –	𝕀 or proportional		
		0 K K	0	K	0	0	0 ... 0	𝟘 is the only example	Nondegenerate (no zero eigenvalues)	elliptic
The unmatrix *U*. **Class U):** *id*		— — —	0	0	0	0	—		Fully degenerate (all zero eigenvalues)	

Key			
Key	λ_e	Eigenvalue	
	α_e	Algebraic	multiplicities
	γ_e	geometric	

Figure 2:

Classification of Combinatorial matrices by eigenspaces, with a shared K-basis of eigenvectors				
Isotropic I)		$\frac{1}{\sqrt{K}}\begin{pmatrix}1\\ \vdots\\ \vdots\\ \vdots\\ 1\end{pmatrix}$,	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\\ 0\\ \vdots\\ 0\end{pmatrix}, \dots , \frac{1}{\sqrt{k\,K}}\begin{pmatrix}1\\ \vdots\\ \vdots\\ 1\\ -k\end{pmatrix}$	
Eigenvalues		y		
Geometric multiplicities		K		$=$ K
Eigenspaces		$\mathfrak{E}\mathrm{ig}_y\,(\boldsymbol{I})$ $\mathbb{R}^K$		$=$ $\mathfrak{E}\mathrm{ig}\,(\boldsymbol{I})$ $=$ $\mathbb{R}^K$
Generic G): requires $K \geq 2$		$\frac{1}{\sqrt{K}}\begin{pmatrix}1\\ \vdots\\ \vdots\\ \vdots\\ 1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\\ 0\\ \vdots\\ 0\end{pmatrix}, \dots , \frac{1}{\sqrt{k\,K}}\begin{pmatrix}1\\ \vdots\\ \vdots\\ 1\\ -k\end{pmatrix}$	
Eigenvalues		z	y	
Geometric multiplicities		1 $+$	k	$=$ K
Eigenspaces		$\mathfrak{E}\mathrm{ig}_z\,(\boldsymbol{G})$ $\oplus$ $\mathbb{R}$ $\oplus$	$\mathfrak{E}\mathrm{ig}_y\,(\boldsymbol{G})$ $\mathbb{R}^k$	$=$ $\mathfrak{E}\mathrm{ig}\,(\boldsymbol{G})$ $=$ $\mathbb{R}^K$
Unmatrix U)		—		
Eigenvalues		—		
Geometric multiplicities		—		
Eigenspaces		$\emptyset$		$=$ $\mathfrak{E}\mathrm{ig}\,(\boldsymbol{U})$

Figure 3:

B Boosting classifications using Order Theory

Structures 11 and 12 Classificatory-table versus classificatory-key depictions are illustrated in Fig 4 for the current Article's cumulative-K Spectral classifications. On the one hand, tables can encode some simple patterns of coarse graining. On the other hand, the key diagram can be considered to be a rooted tree [38, 62], which is a subcase of poset [33, 40]. Rooted trees are not in general preserved under quotients, but more general posets can accommodate these. In this way, key diagrams are stronger when the objects under classification are sharply enough defined to have meaningful quotients.

Remark 1 Fig 5 abstracts posets from the previous figure. Including the following illustrative quotients. b) Treating the signs as distinguishable but meaningless. c) Identifying equal

	Spectral classification of Combinatorial matrices											
	Table presentation											
Combinatorial matrices	C											
Symmetry class	G								I			U
Rank	K				k		1		K		0	
Signature (Physicists')	K	$-K$	$K-2$	$2-K$	k	$-k$	1	-1	K	$-K$	0	0
Signature (Physicists-in-detail)	$++$	$-+$	$+-$	$--$	$+$	$-$	$+$	$-$	$+$	$-$		

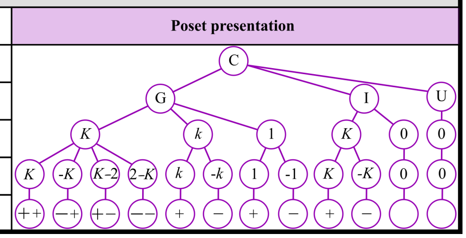

Figure 4:

signatures. d) Both at once. b) still manages to be a rooted tree, while c) and d) exhibit cycles.

* * *

Remark 2 Graphs underlying these posets are exhibited in rows 3 and 4. See [75] for an explanation of the specific style of these presentations of graphs.

Remark 3 In Fig 6, we split into Combinatorial matrices with each individual value of K .

$K = 0$ forms a disjoint chain: involving objects not present for any subsequent K .

$K = 1$ is also particularly simple, since here the matrices are just numbers, and these support just the one lone eigenspace.

$K = 2, 3$ have extra scope for identifications. $K = 2$'s third row exceptionally contains more than just rank information. By discerning between whether it is the subsequently-solo or the subsequently-degenerate eigenvalue that is 0 . Cutting this out, corresponding by rank alone, excises the paler vertices and edges in Figs 5 to 7.

$K = 3$ is minimum for the generic case to have a symmetry-degenerate eigenspace.

$K = 4$ is the minimum generic value including our quotienting considerations. With less quotienting, $K = 3$ can play this role. This role corresponds to the Combinatorial matrix arena $\mathfrak{CM}_{\mathbb{R}}$ being Spectrally truncated at, and persistently past, this value. Which is these matrices' main Spectral feature. The Author shall eventually consider small-sized square matrices' Jordan normal forms, which are not afflicted by any such truncation, for subsequent comparison.

Remark 4 Underlying graphs for these quotients are provided in Figs 7 and 8. The underlying homeomorph irreducibles are in row 4, cycle systems in row 5, and the homeomorph irreducibles of the cycle systems themselves in row 6. For the first 2 graphs in 5's homeomorph irreducibles, just remove the non-encircled 1's and 2's. All are planar bar Fig 8 column 1 rows 3 to 6, by virtue of the marked $K_{3,3}$ forbidden subgraph.

Remark 5 All the above planar graphs furthermore correspond to upper-planar [30] posets: a more stringent condition.

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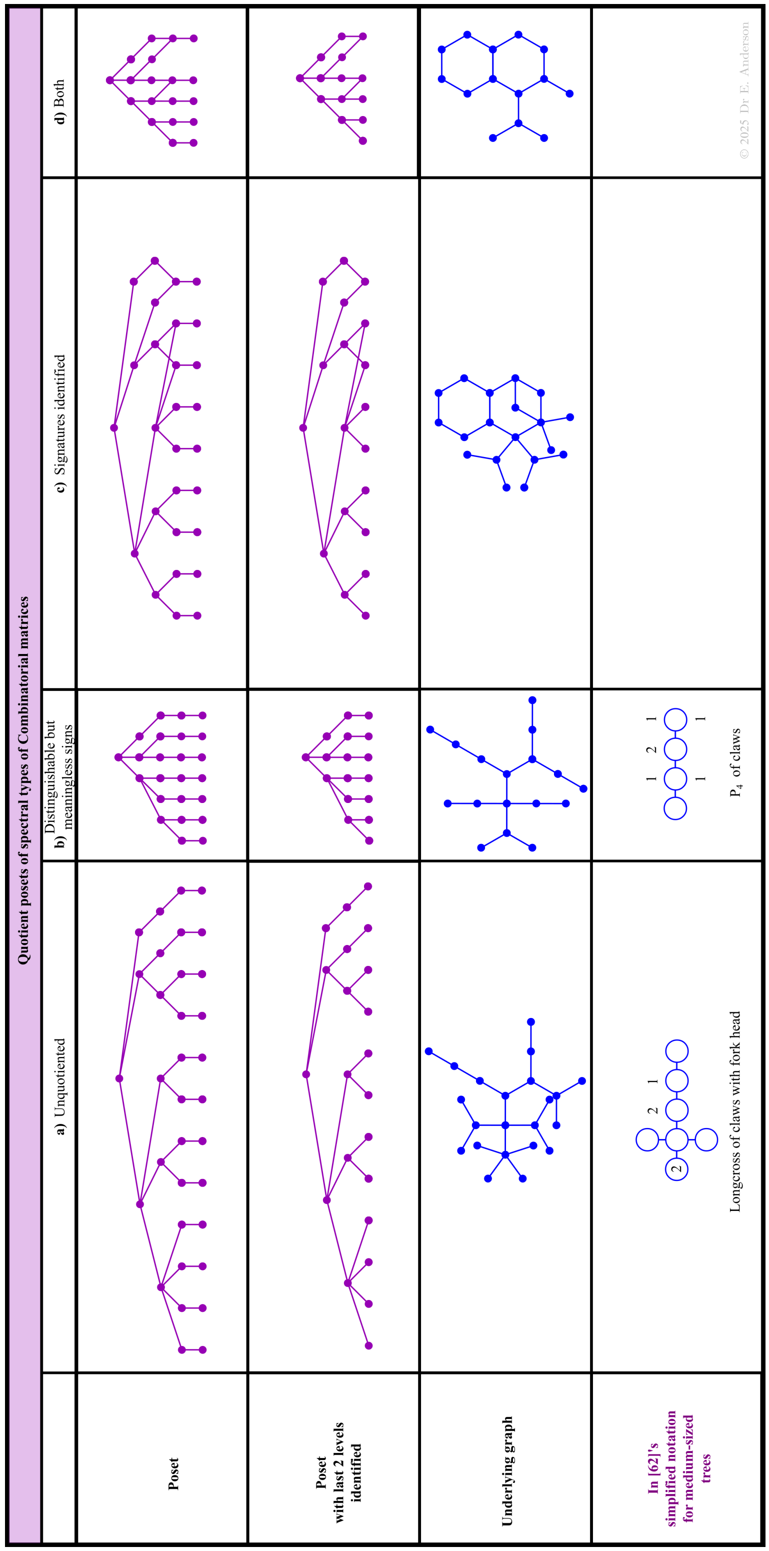
Quotient posets of spectral types of Combinatorial matrices
a) Unquotiented
b) Distinguishable but meaningless signs
c) Signatures identified
d) Both
Poset
Poset with last 2 levels identified
Underlying graph
In [62]'s simplified notation for medium-sized trees
2 2 1
Longcross of claws with fork head
1 2 1
1 1
P4 of claws
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Figure 5:

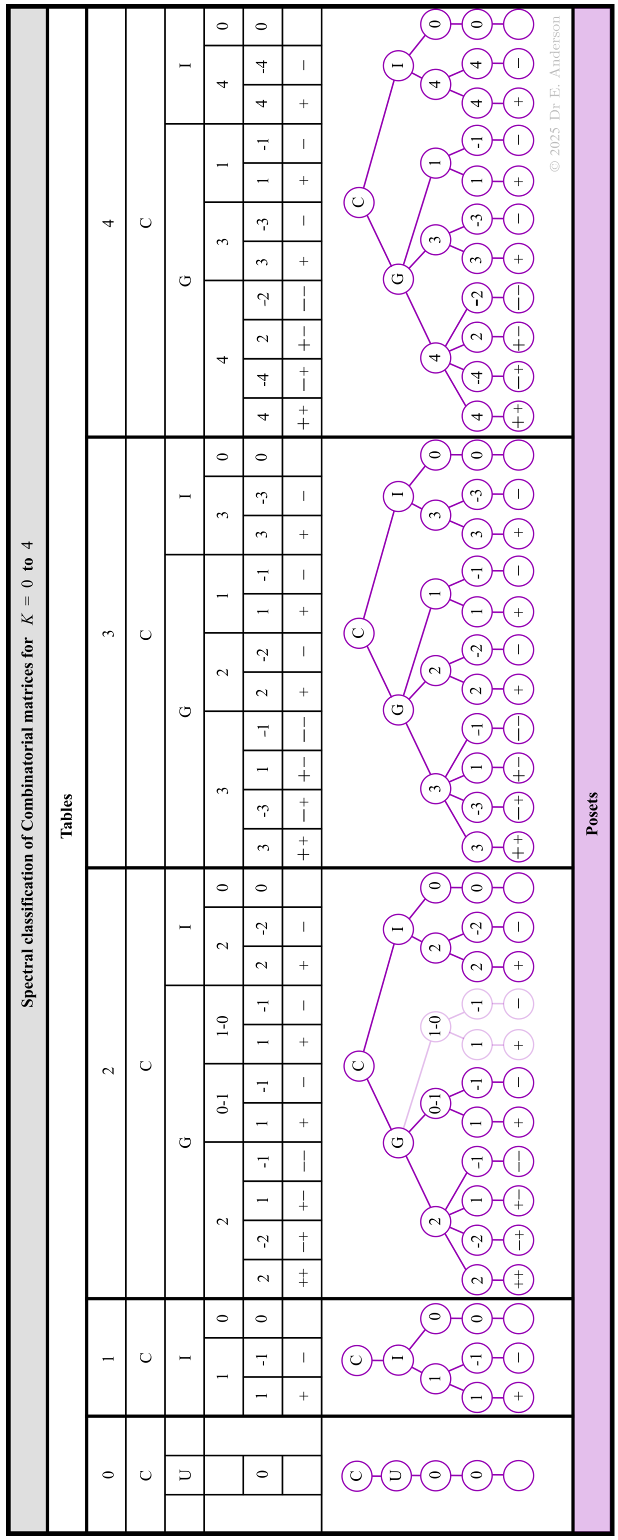
Spectral classification of Combinatorial matrices for K = 0 to 4
Tables
Posets
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Figure 6:

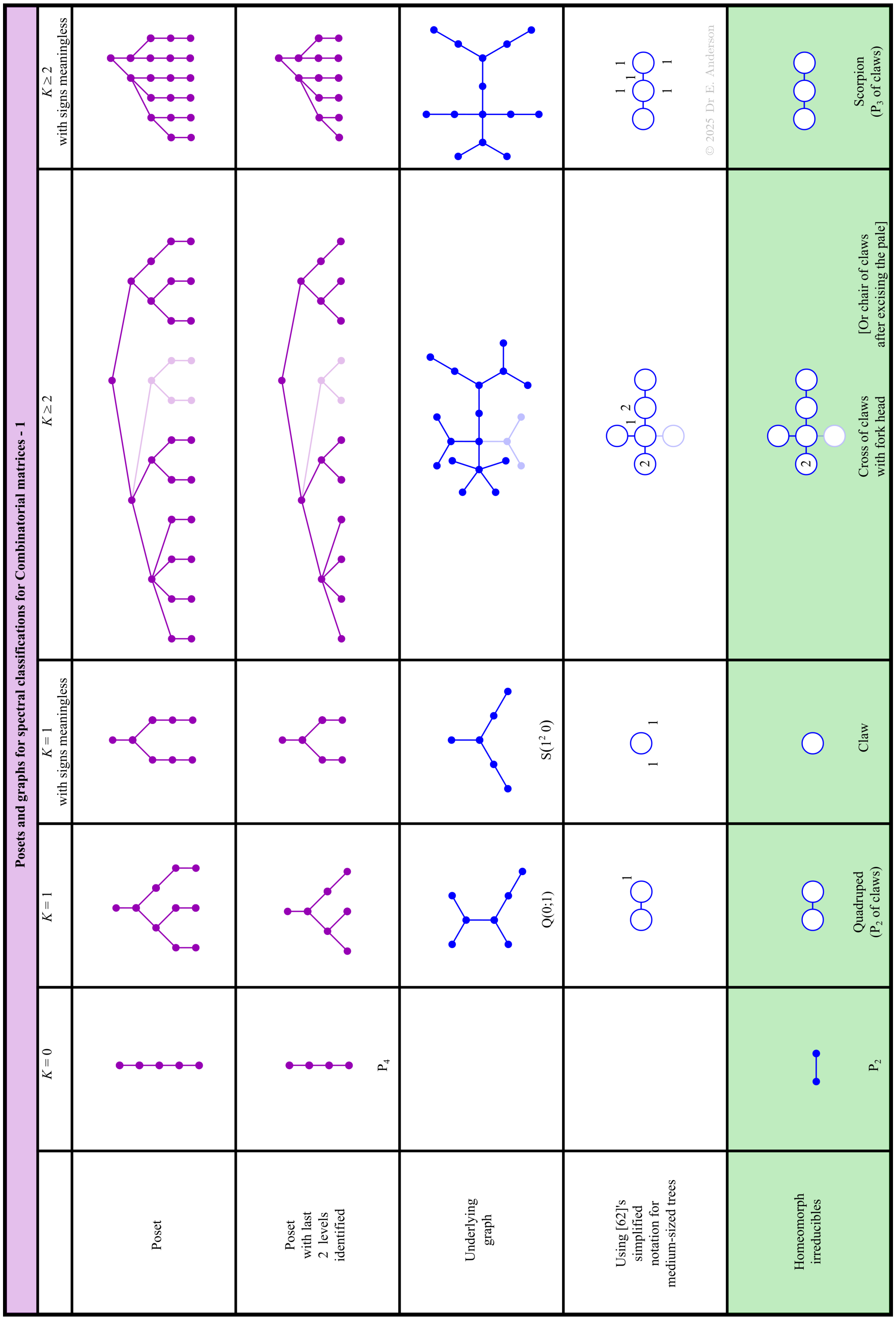
Posets and graphs for spectral classifications for Combinatorial matrices - 1
K = 0
K = 1
K = 1 with signs meaningless
K ≥ 2
K ≥ 2 with signs meaningless
Poset
Poset with last 2 levels identified
P4
Underlying graph
Q(0;1)
S(1² 0)
Using [62]'s simplified notation for medium-sized trees
Homeomorph irreducibles
P2
Quadruped (P2 of claws)
Claw
Cross of claws with fork head
[Or chair of claws after excising the pale]
Scorpion (P3 of claws)
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Figure 7:

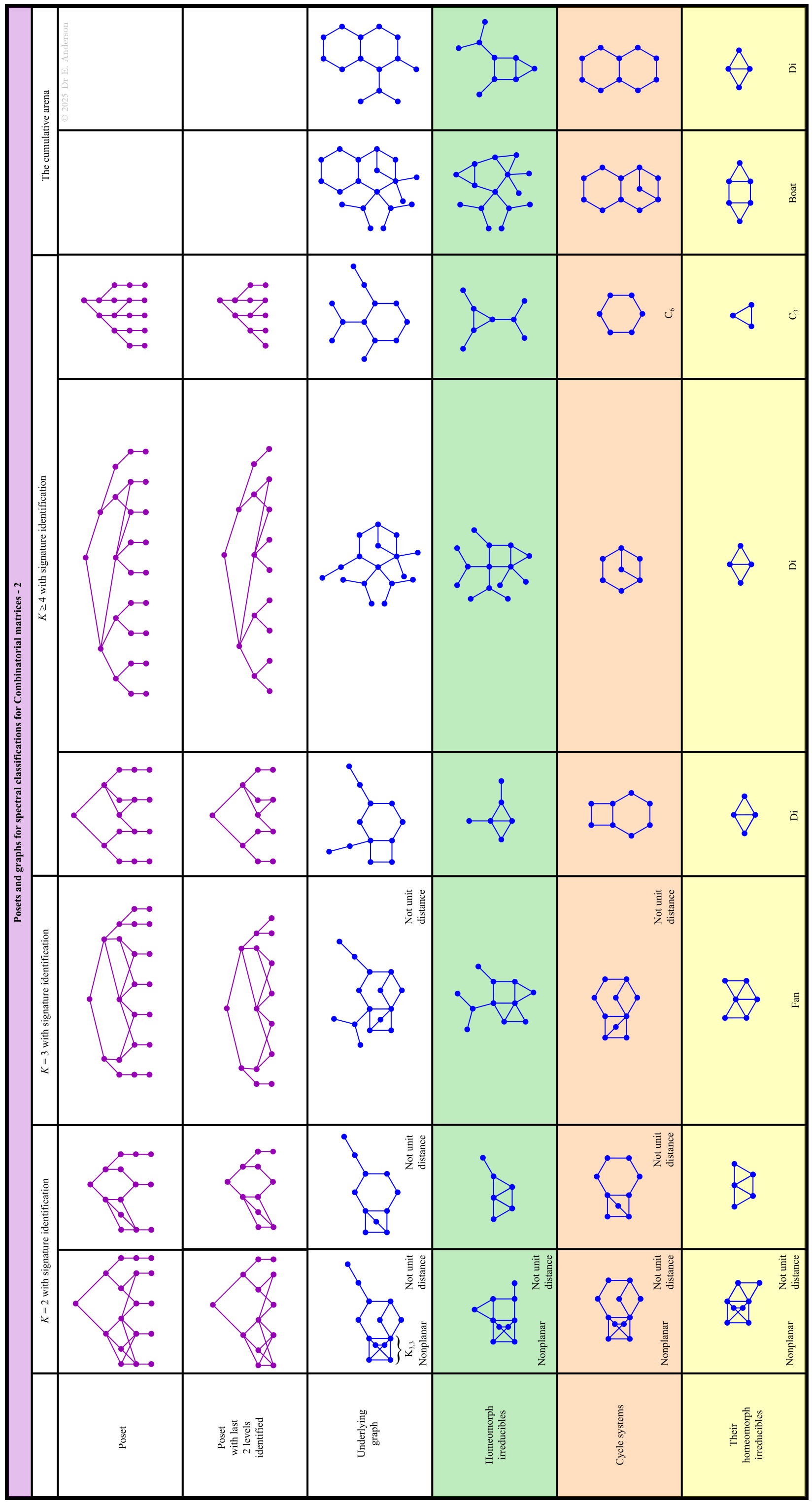
Posets and graphs for spectral classifications for Combinatorial matrices - 2
K = 2 with signature identification
K = 3 with signature identification
K ≥ 4 with signature identification
The cumulative arena
© 2025 Dr. E. Anderson
Poset
Poset with last 2 levels identified
Underlying graph
Homeomorph irreducibles
Cycle systems
Their homeomorph irreducibles
K3,3
Nonplanar
Not unit distance
Di
Fan
C6
C3
Boat

Figure 8: