Triangle, but not most Quadrilateral, Matrices are Combinatorial

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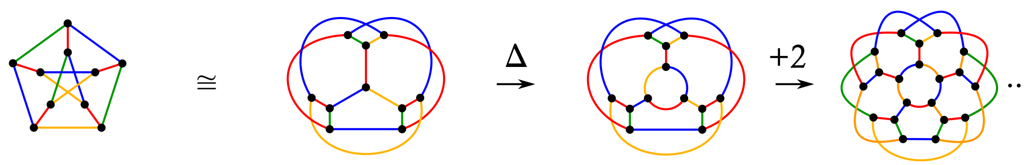
Abstract

A further reason is given for why the matrix theory of triangles is simpler and neater than that for quadrilaterals. Namely, that it is purely in terms of combinatorial matrices. That all triangle matrices commute with each other is then but a simple consequence. Also sets of invertible combinatorial matrices multiplicatively form commutative groups. Suppose that some invertability is dropped, as occurs for the triangle matrices since these include a Lagrange matrix: a type of projector. Then commutative monoid structure persists, now also just on combinatorial matrix grounds.

New combinatorial algebra characterizations are also given for the Lagrange and Apollonius matrices of the N-body problem and the triangle respectively. As well as an alternative generalization for the latter. Arenas of combinatorial matrices, with algebraically-distinguished cases demarcated, are also provided.



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1 Introduction

1.1 Motivation

Recent works [19, 25, 27, 28] have developed much of the theory of triangles from three matrices. Namely, the Lagrange [3] projector $\boldsymbol{P}$ [20, 26, 17], the Apollonius [1] involutor $\boldsymbol{J}$ [19] and the fundamental triangle matrix $\boldsymbol{F}$ [25, 27, 28], previously a.k.a. Heron [2] matrix [11]. In contrast, generalizing to quadrilaterals [20] produces many more matrices [22, 33, 35, 37] with large amounts of multiplicative incompatibility. And less results about such matrices' properties.

Various reasons for triangles' exceptionally neat linear algebra results have been given.

1) Flat geometry. That for triangles separations coincide with sides. In contrast, for quadrilaterals 2 of the 6 separations are instead diagonals.

2) Linear algebra. That for $N = 3$ the eigenclustering vectors [30, 29] (a.k.a. relative Jacobi vectors [12, 14, 38] come in the right number – 2 – to constitute a basis for space. As part of the diagonal of d-dimensional N-body problems with $N = d + 1$, Which has so far usually been called 'simplex-' or 'Casson-' [13] rather than just 'basis-diagonal' [21].

3) Topology and differential geometry. That in particular the triangle contains a 'hidden angel' [6, 9, 19, 41], namely 'Hopf's little map' [5, 8, 16]

$$\mathit{H} \; : \; \mathbb{S}^3 \longrightarrow \mathbb{S}^2 \; . \tag{1}$$

For all that the theory of quadrilaterals involves a larger Hopf map

$$H \; : \; \mathbb{S}^5 \longrightarrow \mathbb{CP}^2 \; . \tag{2}$$

There are two issues here here. (2) is both not quite as mathematically special as (1). Nor is (2) known to be tied to area formulae or matrices encoding these. In contrast, there are close ties [19, 40] between (1) and the quadratic form formulation of Heron's formula. Incidentally, the $\mathbb{S}^2$ and $\mathbb{CP}^2$ are realized as [13] the shape spaces modulo similarities of triangles and quadrilaterals respectively.

4) Flat geometry. There is a large increase in the number of area formulae upon passing from triangles to quadrilaterals [4, 23, 22].

5) Distance geometry, representation theory (...) In any case, quadrilateral area formulae do not generalize all facets of [10, 18, 20, 22, 39] the surprisingly deep Heron's formula.

The purpose of this note is to add one more reason.

6) Algebraic combinatorics. Namely, that $\boldsymbol{P}$, $\boldsymbol{J}$ and $\boldsymbol{F}$ are all *combinatorial matrices* [7, 24]. In particular, this by itself guarantees that sets of such will both brackets-commute and multiplicatively form at least a commutative monoid. So these results found to hold for [27, 28] triangle matrices in fact hold for any set of combinatorial matrices of compatible size. In contrast, none of the compatible fragments of the linear algebra of quadrilaterals enjoy the property of consisting purely of combinatorial matrices.

1.2 Contents outline

Section 2 serves to define combinatorial matrices. And to provide two new notations for these, which would rather simplify [25, 27]'s presentation. One is for the matrix of 1s and the identity. while the other uses the trace-tracefree irreducible decomposition: with representation theory benefits.

Section 3 points out that combinatorial matrices of compatible size always commute. And finds all combinatorial matrices that are projectors or involutors. [36] provides a spectral counterpart to the present article.

Section 4 considers arenas of algebraically-distinguished combinatorial matrices. In the process, there is cause to introduce a third trace-reversed notation. Arenas are configuration spaces, each comprising the totality of some given type of mathematical object. Work included on these here can be viewed as mapping the layout of algebraically-distinguished combinatorial matrices within the overall arena [24] of combinatorial matrices. This is now available as an ambient background for the matrix theory of triangles and for the matrices that [27] opens up with as well.

Section 5 finally surveys the more patchy extent to which combinatorial matrices have been found to date within the growing maelstrom of quadrilateral linear algebra.

2 Combinatorial matrices

2.1 Definition

Definition 1 A *combinatorial matrix* [7] is a square matrix of the following form.

$$\begin{pmatrix} x + y & x & \dots & x \\ x & & & \vdots \\ \vdots & & & x \\ x & \dots & x & x + y \end{pmatrix} = y\mathbb{I} + x\mathbb{1} . \quad (3)$$

Notation 1 The present article considers $x, y \in \mathbb{R}$, for all that e.g. any other field $\mathbb{F}$ would do. The second expression above is a basis-free rendition of Knuth's formulation [7]. Compare [28]'s use of coordinate-free notation for arrays. And so happens to be a coordinate-free rendition of a componentwise expression used by Also, in the standard basis, $\mathbb{I}$ is the corresponding identity matrix, and $\mathbb{1}$ the 'block' matrix whose entries are all 1's. Finally take our matrices to be of size K .

2.2 Linearity and notation

Lemma 1 The combinatorial matrices for a given K form a 2-d vector space under matrix addition and multiplication by a field-valued scalar.

Notation 2 Sincethe present article covers many combinatorial matrices and properties thereof, the following further shorthands are proposed. Let us denote (3) by

$$[\, y, \, x \,]_K \quad (4)$$

in the *block basis*. And by

$$(x + y, x)_K \tag{5}$$

in the *irreducible basis*. I.e. now expanding in terms of $\mathbb{I}$ and $\mathbb{T}$: the tracefree part of $\mathbb{1}$ in [28]'s notation. This second notation comes with representation theory benefits, and is more practical to use. In the event of dealing with a set or space of compatible matrices, we can afford to drop the K subscript.

Lemma 2 The above symbols enjoy the following properties.

i) *Homogeneous-linearity*

$$[\lambda y, \lambda x] = \lambda [y, x] , \tag{6}$$

$$(\lambda a, \lambda b) = \lambda (a, b) . \tag{7}$$

ii) *Additivity of compatible matrices*

$$[y, x] + [w, v] = [y + w, x + v] , \tag{8}$$

$$(y, x) + (w, v) = (y + w, x + v) . \tag{9}$$

2.3 Some basic algebra and representation theory examples

Example 0 In our notation, the zero matrix $\mathbb{0}$ is

$$[0, 0] = (0, 0) . \tag{10}$$

Example I The identity matrix $\mathbb{I}$ is

$$[1, 0] = (1, 0) . \tag{11}$$

Remark 1 The only matrices for which our two symbols' entries coincide are

$$[x, 0] = (x, 0) . \tag{12}$$

By homogeneous linearity, these are proportional to the identity matrix. This includes the previous examples as the $x = 0$ and 1 subcases respectively.

Example 1 The matrix of 1's a.k.a. block matrix $\mathbb{1}$ is

$$[0, 1] = (1, 1) . \tag{13}$$

Example T The tracefree part $\mathbb{T}$ of the block matrix is

$$[-1, 1] = (0, 1) . \tag{14}$$

These amount to finding each basis' sparse binary concomitant of the identity.

2.4 Some recently discussed examples from flat geometry

Example 3.P For a triangle,

$$\boldsymbol{P} \ = \ \left[1 \ , \quad - \frac{1}{3} \right] \ = \ \frac{1}{3} \, (\, 2, \, \text{-}1 \,) \ . \tag{15}$$

Example 3.J

$$\boldsymbol{J} \ = \ \left[\text{-}1 \ , \ \ \frac{2}{3} \right] \ = \ \frac{1}{3} \, (\, \text{-}1, \, 2 \,) \ . \tag{16}$$

Example 3.F

$$\boldsymbol{F} \ = \ [\, \text{-}2, \, 1 \,] \ = \ (\, \text{-}1, \, 1 \,) \ . \tag{17}$$

Example K.F [27]

$$\boldsymbol{F}(K) \ = \ [\, \text{-}2, \, 1 \,]_K \ = \ (\, \text{-}1, \, 1 \,)_K \ . \tag{18}$$

Example K.P

$$\boldsymbol{P}(K) \ = \ K^{-1} \, [\, K, \, \text{-}1 \,]_K \ = \ K^{-1} \, (\, k, \, \text{-}1 \,)_K \ . \tag{19}$$

Example K.M [27]

$$\boldsymbol{M}(K) \ = \ K^{-1} \, (\, \text{-}1, \, k \,)_K \ . \tag{20}$$

Remark 1 Above

$$k \ := \ K \ - \ 1 \ . \tag{21}$$

Whose K-body problem interpretation is as the dimension of the label space for relative space. Alternatively, it is the dimension of the simplices formed by the corresponding configurations.

2.5 A first few applications

Application 1 In particular, the 'fundamental linear dependence for triangles' [28]

$$\boldsymbol{F} \ = \ \boldsymbol{J} \ - \ \boldsymbol{P} \tag{22}$$

becomes (dropping the 3-subscripts)

$$(\, \text{-}1, \, 1 \,) \ = \ 3^{-1} \, (\, \text{-}1, \, 2 \,) \ - \ 3^{-1} \, (\, 2, \, \text{-}1 \,) \ . \tag{23}$$

And so

$$K^{-1} \ \{ \ (\, \text{-}1, \, k \,)_K \ - \ (\, k, \, \text{-}1 \,)_K \ \} \ = \ K^{-1} \, (\, \text{-}K, \, K \,)_K \ = \ K \, K^{-1} \, (\, \text{-}1, \, 1 \,)_K \ = \ (\, \text{-}1, \, 1 \,)_K \ . \tag{24}$$

By which (22) generalizes to

$$\boldsymbol{F}(K) \ = \ \boldsymbol{M}(K) \ - \ \boldsymbol{P}(K) \ . \tag{25}$$

This explains why $\boldsymbol{M}(K)$ features in [27], though an alternative shall be provided in section 3.

Lemma 3 i)

$$\text{tr}[\, y, \, x \,]_K \ = \ K \, (\, x \ + \ y \,) \ . \tag{26}$$

ii) [7, 24]

$$\det [\, y, \, x \,]_K \ = \ (\, y \ + \ K \, x \,) \, y^k \ . \tag{27}$$

Exercise 1 Prove this.

Corollary 1 For a combinatorial matrix to be tracefree,[1]

$$x + y = 0 . \tag{28}$$

The only tracefree possibilities are then the matrices $\propto \mathbb{T}$.

Corollary 2 For a combinatorial matrix to be singular, one or both of the following must hold.

$$y = 0 , \tag{29}$$

$$y = -K x . \tag{30}$$

If the first equation holds alone,

$$(x, x)_K , \tag{31}$$

of nullity k .

If the second equation holds alone,

$$x (-k, 1)_K \propto \boldsymbol{P}(K) , \tag{32}$$

of nullity 1 .

Finally if both hold at once,

$$(0, 0)_K = \mathbb{0} : \tag{33}$$

the zero matrix of nullity K .

Remark 1 So the arbitrary-N equal-masses Lagrange projectors can also be characterized as that N's unique projector onto a $\geq$ 1-d linear space. This is for nontrivial N -body problems – $K = N \geq 3$ – since no room is left below this.

3 Some algebra of combinatorial matrices

3.1 Multiplication and commutativity

Lemma 4 (Multiplication rule)

$$(x_1 + y_1, x_1)_K \; (x_2 + y_2, x_2)_K =$$

$$(K x_1 x_2 + x_1 y_2 + x_2 y_1 + y_1 y_2 , \; K x_1 x_2 + x_1 y_2 + x_2 y_1)_K . \tag{34}$$

Proof Expand and then regroup each of the following.

$$\prod_{i = 1}^{2} (x_i + y_i) + k x_1 x_2 . \tag{35}$$

$$\sum_{i = 1}^{2} (x_i + y_i) x_j + (k - 1) x_1 x_2 , \;\; j \neq i \; . \square \tag{36}$$

[1]This is subject to the assumption that the underlying field $\mathbb{F} = \mathbb{R}$. Characteristic-k fields offer other possibilities at this juncture.

Corollary 3 (Squares)

$$(x + y, x)_K{}^2 = (x (K x + 2 y) + y^2 , \; x (K x + 2 y))_K \; . \qquad (37)$$

<u>Proof</u> Set $1 = \text{blank} = 2$ in Lemma 4. □

Corollary 4 All compatible combinatorial matrices commute with each other.

<u>Proof</u> Lemma 4's expression is $1 \leftrightarrow 2$ invariant. □

Remark 1 Thus that the three triangle matrices commute can be explained simply from their being of the conceptual type 'combinatorial matrix'. Without any reference to geometry.

Remark 2 Matrix multiplication is associative, Lemma 4 gives that combinatorial matrices close, and Corollary 4 that they commute. Example I provides the multiplicative identity combinatorial matrix. Given a set of combinatorial matrices, if any singular such are present, then the inverse property fails. One is thus left with [24] a commutative monoid [15]. Remark 1's explanation then plays out once more. If none are singular, the inverses are themselves combinatorial (**Exercise 2**). In such cases, the set of Combinatorial matrices forms a commutative group.

3.2 Combinatorial projectors

Definition 1 A matrix $\boldsymbol{P}$ is a *projector* if it obeys

$$\boldsymbol{P}^2 = \boldsymbol{P} \; . \qquad (38)$$

A $\mathbb{R}$-valued such is furthermore an *orthogonal projector* if also

$$\boldsymbol{P}^{\mathrm{T}} = \boldsymbol{P} \; . \qquad (39)$$

Remark 1 $\boldsymbol{P}$ is a *nontrivial projector* if both its image and its kernel are of dimension ≥ 1. For if its kernel is of dimension 0, nothing is projected out and one has the identity matrix $\mathbb{I}$. While if its image is of dimension 0, everything is projected out. And one has the zero matrix $\mathbb{0}$ acting as a total annihilator. By the rank–nullity Theorem, both of the above can only simultaneously occur if the overall dimension is zero.

Proposition 1 i) The only possible combinatorial projectors are

$$(0, 0)_K \; , \;\; K^{-1} (1, 1)_K \; , \;\; K^{-1} (k, \text{-}1)_K \; , \;\; (1, 0)_K \; : \;\; \text{all of which are orthogonal} \; . \qquad (40)$$

ii) Among which, the nontrivial cases are the second and third, provided that $K \geq 2$.

<u>Proof</u> i) Equating non-diagonal and diagonal elements in turn, combinatorial projectors obey the following pair of simultaneous quadratic equations.

$$x (K x + 2 y) = x \; , \qquad (41)$$

$$x (K x + 2 y) + y^2 = x + y \; . \qquad (42)$$

Replace (42) with (42) − (41):

$$y^2 = y \;\Rightarrow\; y(y - 1) = 0 . \tag{43}$$

So

$$y = 0 \text{ or } 1 . \tag{44}$$

Thus

$$x(Kx \mp 1) = 0 . \tag{45}$$

So either

$$x = 0 , \tag{46}$$

giving our first and last cases.

Or

$$x = \pm K^{-1} . \tag{47}$$

Which returns our second and third cases. Using factorization in both cases, and use of the definition of k in the latter.

ii) The first and fourth are the zero and identity matrices. For $K = 1$, the second is 1 : of trivial kernel. While the third is 0 : of trivial image. Such triviality cannot however occur for $K \geq 2$. □

Remark 2 For $K = 2$, the two nontrivial projectors have images of equal dimension 1 .

Remark 3 However for $K \geq 3$, corresponding to the nontrivial N-body problems, the following new combinatorial-algebraic characterization applies.

Principle 1 For $K \geq 3$, *The Lagrange matrix* $\boldsymbol{P}$ *is the sole nontrivial combinatorial projector of largest image.*

Remark 4 Aside from the trivial identity, projectors are singular. The projector property fixes the constant of proportionality previously found in characterizing the singular combinatorial matrices.

3.3 Combinatorial involutors

Definition 1 A matrix $\boldsymbol{J}$ is an *involutor* if it obeys

$$\boldsymbol{J}^2 = \mathbb{I} . \tag{48}$$

Remark 1 It is a *nontrivial involutor* if it is not the identity. So that 2 is the minimum power returning the identity.

Proposition 2 i) The only possible combinatorial involutors are

$$\pm(1, 0)_K , \quad \pm 2K^{-1}\left(1 - K2^{-1}, 1\right)_K . \tag{49}$$

ii) Among which, the last three are the nontrivial cases.

Proof i) Equating non-diagonal and diagonal elements in turn, combinatorial projectors obey the following pair of simultaneous quadratic equations.

$$x\,(\,K\,x\;+\;2\,y\,)\;=\;0\;, \tag{50}$$

$$x\,(\,K\,x\;+\;2\,y\,)\;+\;y^2\;=\;1\;. \tag{51}$$

Replace (51) with (51) - (50):

$$y^2\;=\;1\;\;\Rightarrow\;\;(\,y\;+\;1\,)(\,y\;-\;1\,)\;=\;0\;. \tag{52}$$

So

$$y\;=\;\pm 1\;. \tag{53}$$

In either case,

$$x\;=\;0 \tag{54}$$

is possible. Factorizing, this returns the first two cases.

If not,

$$x\;=\;\mp 2\,K^{-1}\;. \tag{55}$$

Factorizing, this gives the last two cases.

ii) The first case is the identity, and is thus discarded. □

Remark 1 For $K = 1$, the above four cases collapse to just two: ± 1 .

Remark 2 For $K = 2$, they are distinct but the last 2 become

$$\pm\mathcal{T}\;. \tag{56}$$

I.e. the mininum *transpositor* (transposition matrix) up to sign.

Principle 2 For $K \geq 2$, *the matrix* $\boldsymbol{J}(N)$ *is, up to sign, the sole nontrivial combinatorial involutor.*

Remark 3 The $K = 2$ case, as the transposition, admits a more basic permutation interpretation.

The $K = 3$ case is consequently the first for which some other characterization is desirable. Flat geometry obliges: this smallest nontrivial combinatorial involutor encodes Apollonius' sides-medians length exchange Theorem.

For $K \geq 4$, an alternative to [27]'s $\boldsymbol{M}(K)$ matrices has by now been motivated. The $\boldsymbol{M}(K)$ are such that the difference (25) holds. But our $\boldsymbol{J}(K)$ possess the involution property (48) itself. At the partial cost that now

$$\boldsymbol{J}(K)\;-\;\boldsymbol{P}(K)\;=\;\text{-}K^{-1}\,(\,1,\,1\,)_K\;=\;\text{-}K^{-1}\mathbb{1}\;\neq\;\boldsymbol{F}(\boldsymbol{K})\;. \tag{57}$$

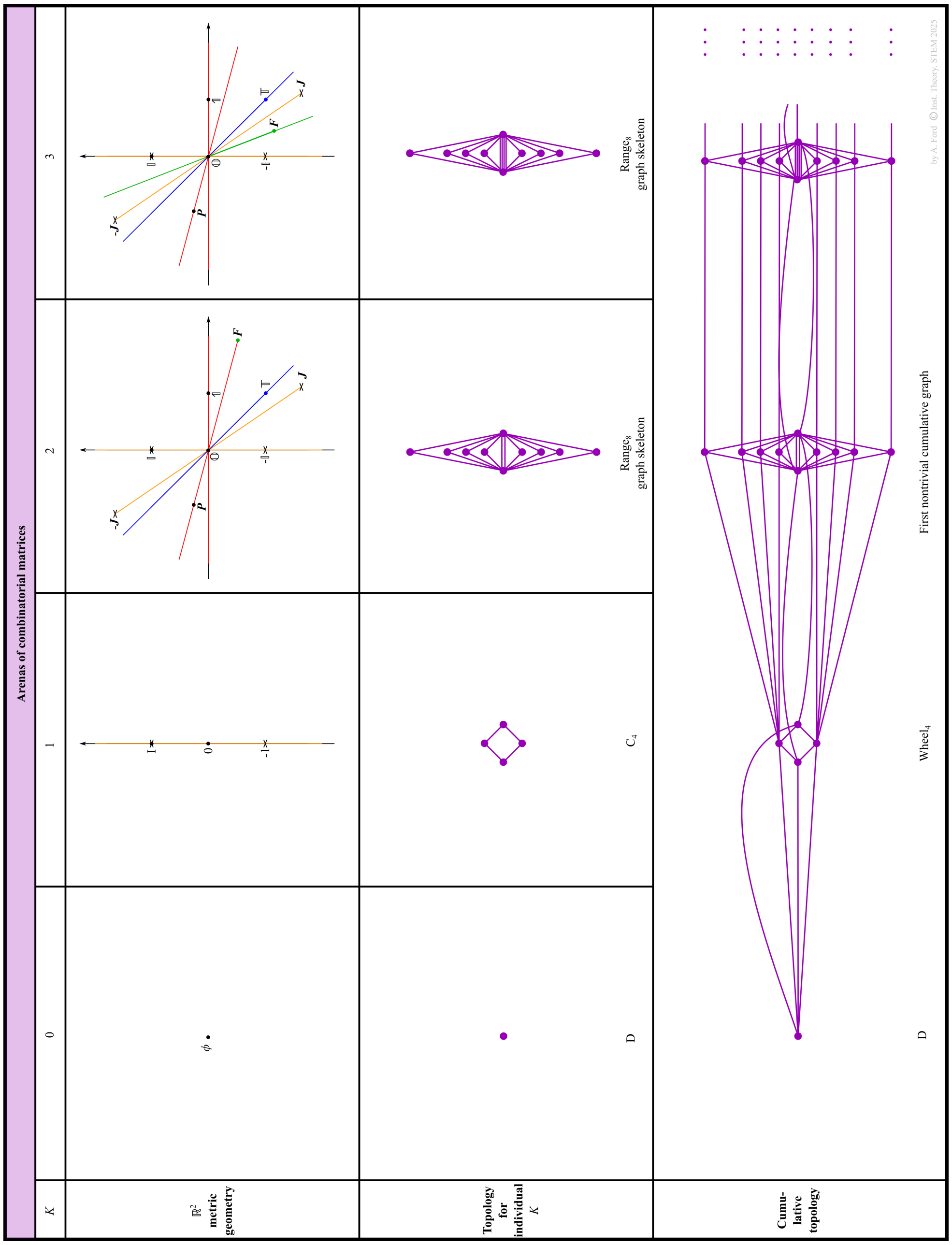

Figure 1:

4 Arenas of combinatorial matrices

4.1 Algebraically special subarenas

Remark 1 These CM(K) are plotted in the first row of Fig 1. Including the intersecting pair of lines of singularity, and that of proportionality to an involution. Viewing the first of these pairs as a determinant condition gives an invariant reason to include the zero-trace line as well. The $\boldsymbol{F}(K)$ are also algebraically distinguished, as *trace-reversed* matrices [36] and the difference of the two irreducibles in play. The general combinatorial matrices that enjoy this property are

$$q\,(\,-1,\,1\,)_K \;=\; q\,\boldsymbol{F}(K)\ . \tag{58}$$

These form yet another line.

All six of these lines share a common intersection point: $\mathbb{0}$. The next most special point is the identity $\mathbb{I}$, since this is both a trivial involutor and a trivial projector. There are additionally three other projectors and three other involutors, all six of which are realized as further distinct points for $K \geq 2$. Plus the irreducible $\mathbb{T}$ point. $\boldsymbol{F}$ is itself the final special point marked.

Notation 1 Expanding in terms of $\mathbb{I}$ and $\boldsymbol{F}(K)$,

$$\langle\,2\,x \;+\; y,\,x\,\rangle_K \tag{59}$$

in the *trace-reversed basis.* Squared brackets are nicest – irreducibles – while round and angled brackets are next-nicest reflections thereabout.

Exercise 3 Show that Lemma 2 extends to this, and express all of the matrices in subsections 2.3-4 in terms of this basis.

Remark 2 The figure's first three columns of present non-generic simplifications realized by $K < 3$. $K = 2$ has 2 coincident lines. $K = 1$ is overall $\mathbb{R}$ rather than $\mathbb{R}^2$. And $K = 0$ is just a point representing the empty set. Also for $K = 3$, $\boldsymbol{F}$ exceptionally additionally enjoys the difference property (22); this is a metric-level property.

Remark 3 In formulating topological arenas $\text{TopCM}(K)$ for combinatorial matrices' algebraically-distinguished subcases, the choice of including the point at infinity has been made. For $K \geq 2$, this corresponds to viewing the metric-level arenas' parameter spaces as $\mathbb{C}$ and then taking the Riemann sphere model for this. $K = 1$ then involves the corresponding compactification to a circle. Floorplan edges encode path-connectedness to stairwells' matrix padding. Such padding is a well-defined operation for combinatorial matrices. For it corresponds to adding a single row-and-column border of x 's. Excepting the placement of an $x + y$ in the corner.

4.2 Graph and order theory analysis

Remark 1 The overall order structure formed is not a poset since it contains triangles.

Remark 2 The padding operation by itself produces posets which are additionally trees. With specifics presented in Fig 2. These stabilize to homeomorphs adding 1 vertex everywhere.

Remark 3 The $\text{TopCM}(K)$ are all planar. And stably settle down to a fixed form for $K \geq 3$. $K = 2$ suffices however for the cumulative arena $\text{TopCM}[K]$ to be nonplanar (Fig 3).

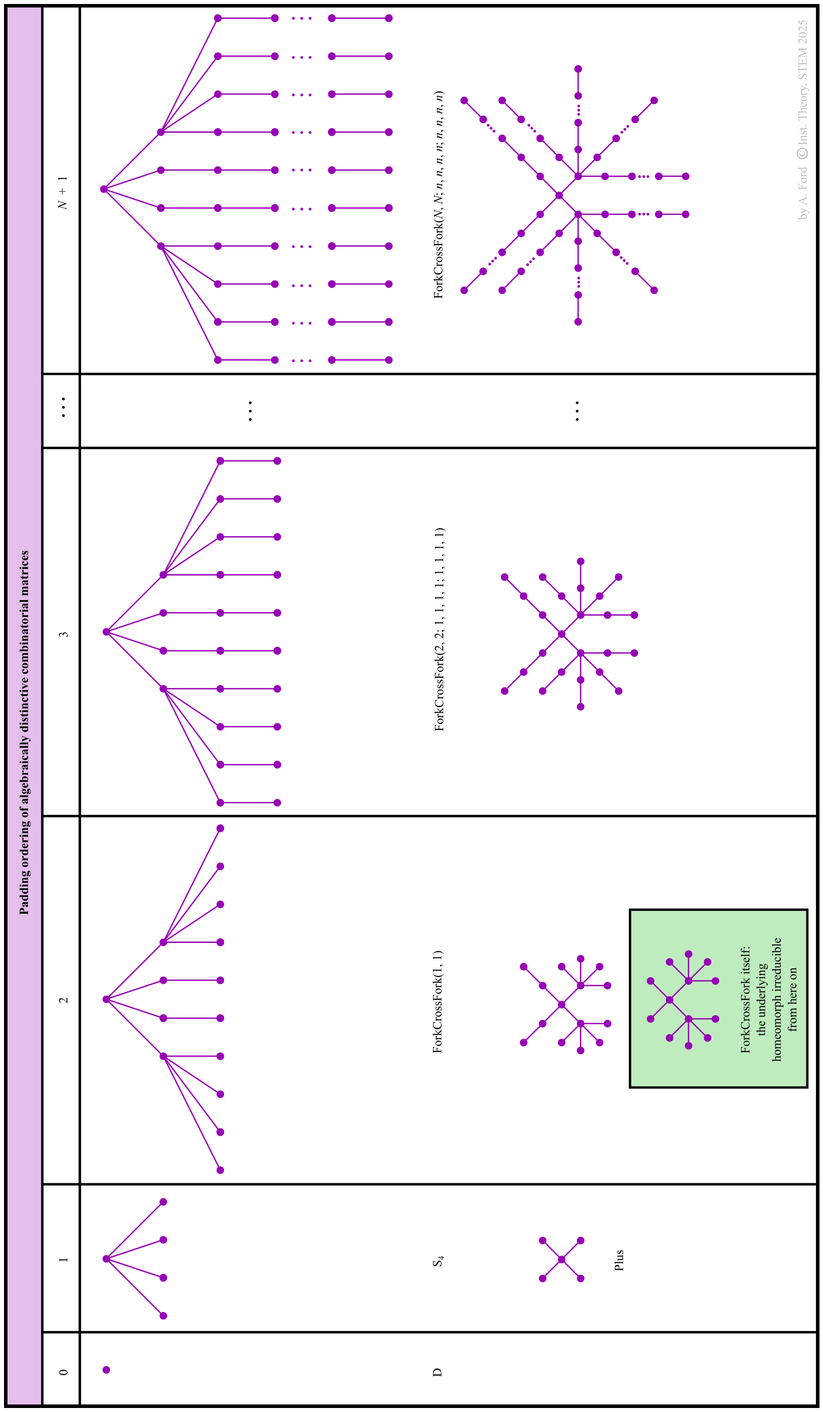
Padding ordering of algebraically distinctive combinatorial matrices
0
1
2
3
N + 1
D
S4
Plus
ForkCrossFork(1, 1)
ForkCrossFork(2, 2; 1, 1, 1, 1; 1, 1, 1, 1)
ForkCrossFork(N, N; n, n, n, n; n, n, n, n)
ForkCrossFork itself:
the underlying
homeomorph irreducible
from here on
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Figure 2:

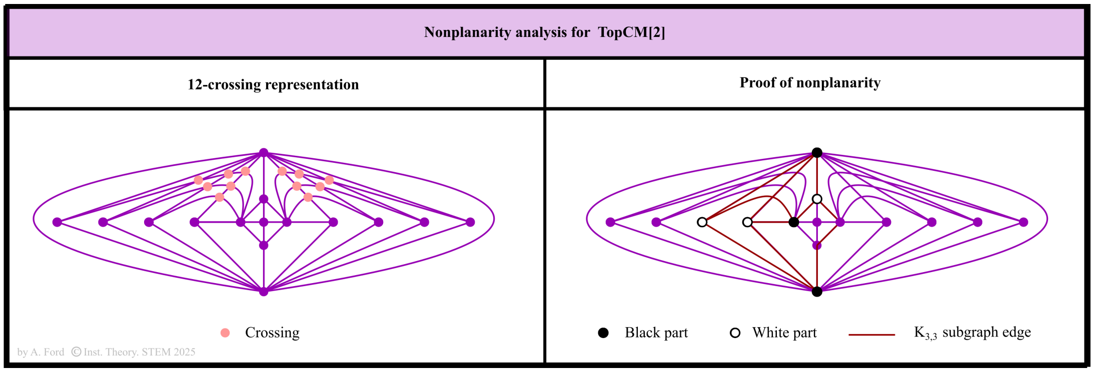

Figure 3:

5 Contrast with quadrilateral matrices

In contrast, in the matrix theory of cyclic quadrilaterals, the Lagrange projector $\boldsymbol{P}$ is compatible with [33] a 3-cycle of Ptolemy matrices. These are not individually combinatorial. Though their sum is, returning the irreducible matrix $\mathbb{T}$. Nor are any of the accompanying Brahmagupta matrices [35] combinatorial. At the level of separations, rather, the corresponding Ptolemy involutor [32] is not combinatorial either.

In the matrix theory of convex quadrilaterals, the Brahmagupta matrices are supplanted by the Bretschneider involutors [37]. These are however also not combinatorial.

While the Ptolemy matrices are involutors, they are not the most direct analogues of the Apollonius involutor $\boldsymbol{J}$. These are instead eigenclustering length-exchange matrices [30, 31]. For $K = 4$, however, these carry 3-indices, 6-indices or both, and so are not compatible with sides matrices. Among these, $\boldsymbol{F}$ [34] and $\frac{1}{2}\boldsymbol{F}(6)$ [31] are combinatorial. Furthermore, these form a poset rather than a chain series [29].

In the theory of quadrilaterals, combinatorial matrices provide but an incomplete cover. Zero commutators are then not guaranteed, and appear to be in shorter supply.

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