Stewart's Theorem generalized.
III. Solved form for any eigenstroke-length for any N-Vertex Configuration.

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Abstract

In the current Series, we use a moments method to derive 2 families of N-vertex generalizations of Stewart's Cevian-Length Theorem for triangles. Firstly, the Arbitrary-Mass Eigenclustering Length-Exchange Theorems (AMELETs). Whose EMELET (equal masses) subcases generalize Apollonius' Median-Length Theorem. While the smallest examples involve a single eigenclustering length, all larger examples produce a linear system for multiple such. The current Article's Theorem gives the general solution for any eigenclustering magnitude for any N in any dimension, in brief Combinatorial form. We term these Solved form for AMELTs (Eigenstroke-Length Theorems). We also provide their Solved form for EMELT subcases as a Corollary.

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1 The Theorem

Remark 1 The below Theorem generalizes Stewart's Cevian-length Theorem [2, 8, 12, 18] to an explicit formula for the following. The length of any eigenstroke for any N-vertex configuration in whichever dimension d . Where one can also interpret N-vertex as N-body or n-simplex depending on context. This account makes use of the notions and notation laid out in the Appendix.

Theorem 1: Solved form for AMELT [2019] Any size-N (sub)system's last nontrivial eigenstroke-Length

$$T \ = \ \sum\!\!\!\!\!\!\sum \sigma(\mathrm{I}\,,\mathrm{J}\,)\,\mathcal{M}_{\mathrm{I}}\,\mathcal{M}_{\mathrm{J}}\,R^{\mathrm{I}\,\mathrm{J}} \ . \tag{1}$$

Remark 2 The expanded version – in terms of separated-out left- and right-child quantities – is

$$T \ = \ \sum_{\mathrm{I}_- \,=\, 1}^{N_-}\sum_{\mathrm{I}_+ \,=\, 1}^{N_+} \mathcal{M}_{\mathrm{I}_-}\,\mathcal{M}_{\mathrm{I}_+}\,R^{\mathrm{I}_-\,\mathrm{I}+} \ - \ \sum_{\mathrm{H} \,=\, \mp}\ \sum_{\substack{\mathrm{I}_\mathrm{H},\ \mathrm{J}_\mathrm{H} \ = \ 1 \\ \mathrm{I}_\mathrm{H} \ < \ \mathrm{J}_\mathrm{H}}}^{N_\mathrm{H}} \mathcal{M}_{\mathrm{I}_\mathrm{H}}\,\mathcal{M}_{\mathrm{I}_\mathrm{H}}\,R^{\mathrm{I}_\mathrm{H}\,\mathrm{J}_\mathrm{H}} \ . \tag{2}$$

A schematic form for which is

$$T \ = \ \left(\sum_{\text{mutual}} - \sum_{\text{selves}} \right) \mathcal{M}_{\mathrm{I}}\,\mathcal{M}_{\mathrm{J}}\,R^{\mathrm{I}\,\mathrm{J}} \ . \tag{3}$$

In our nomenclature for Stewart's Theorem and generalizations, this is form $4'$.m). Standing for solved form ($'$) in redundant-ratio variables (4) which are 'masses' (m).

Remark 3 The above is in terms of the Appendix's total 'mass' fractions $\mathcal{M}_{\mathrm{I}}$. To make contact with the directed side-lengths formulation most widely found in the literature on Stewart's Theorem, we apply the following three trivial rearrangements. Apply the Balance Law of First Moments to convert the $\mathcal{M}_{\mathrm{I}}$ into the sides' directed-length fractions ξ_{I} . Remove denominators by multiplying through and pass to unsquared variables so as to arrive at a sides' directed-lengths version. Now arbitrary masses becomes the freedom for each eigenstroke to emanate from arbitrary points on its left- and right-childrens' eigenclustering strokes (if needs be extended to lines). Which generalizes how in Stewart's Theorem the Cevian runs from a vertex to the opposite side (extended) at an arbitrary cutting point.

2 Examples

Corollary 1: Solved form for EMELT [2019] For equal masses, the schematic form simplifies to the '3-coefficient formula':

$$T \ = \ \frac{1}{N_- \, N_+} \sum_{\text{mutual}} R^m \ - \ \frac{1}{N_-^2} \sum_{\text{self}_-} R^{s_-} \ - \ \frac{1}{N_+^2} \sum_{\text{self}_+} R^{s_+} \ . \tag{4}$$

In full,

$$T \ = \ \frac{1}{N_- \, N_+} \sum_{\mathrm{I}_- \, = \, 1}^{N_-} \sum_{\mathrm{I}_+ \, = \, 1}^{N_+} R^{\mathrm{I}_- \, \mathrm{I}_+} \ - \ \sum_{\mathrm{H} \, = \, \mp} \frac{1}{N_{\mathrm{H}}^2} \sum_{\substack{\mathrm{I}_{\mathrm{H}}, \, \mathrm{J}_{\mathrm{H}} \ = \ 1 \\ \mathrm{I}_{\mathrm{H}} \ < \ \mathrm{J}_{\mathrm{H}}}}^{N_{\mathrm{H}}} R^{\mathrm{I}_{\mathrm{H}} \, \mathrm{I}_{\mathrm{H}}} \ . \tag{5}$$

Remark 3 The above Corollary generalizes Apollonius's Median-length Theorem [1, 12, 18] to an explicit formula for the following. The length of any eigenstroke for any equal-masses N-vertex configuration.

<u>Proof</u> of Corollary 1.

For equal masses,

$$\mathcal{M}_{\mathrm{I}_\pm} \ = \ \frac{1}{\sum_{\mathrm{I}_\pm \, = \, 1}^{N_\pm}} \ = \ \frac{1}{N_\pm} \ .$$

Substitute into the expanded version of the Theorem. In each of its 3 sums, pull out a constant factor as indicated. Also the mutual sum collapses to the above simple double sum. $\Box$

Example 1 We need $N \ \geq \ 3$ for there to be any nontrivial eigenclustering magnitude to solve for.

Example 2 For Stewart's Theorem and its generalization along the $\mathrm{K}(N)$ family of eigenclusterings [23, 27], the w.l.o.g. right child is just a vertex R . So it supports no self separations. Thus one self factor drops out, and the mutual factor becomes linear. Indexing the left child by $i \ = \ 1$ to n , we are left with the following.

$$K_{n \, - \, 1} \ = \ \sum_{i \, = \, 1}^{n} \mathcal{M}_i \, R^{\mathrm{A}_i \, \mathrm{R}} \ - \ \sum_{\substack{i, \, j \ = \ 1 \\ i \ < \ j}}^{n} \mathcal{M}_i \, \mathcal{M}_j \, R^{\mathrm{A}_i \, \mathrm{A}_j} \ . \tag{6}$$

Schematically,

$$K_{n \, - \, 1} \ = \ \sum_{\text{mutual}} \mathcal{M}_i \, R^{\mathrm{A}_i \, \mathrm{R}} \ - \ \sum_{\text{self}} \mathcal{M}_i \, \mathcal{M}_j \, R^{\mathrm{A}_i \, \mathrm{A}_j} \ . \tag{7}$$

I.e. now with just a single child contributing self terms.

Applying the Balance Law of first moments, the $\mathcal{M}_i$ can be reinterpreted as the side fractions ξ_i . We thus recover Article II's result.[1].

Pointer 1 Since Article II already further collapsed this down to $\mathrm{K}(N)$'s Solved form for EMELT [25] generalization of Apollonius' Theorem, we shall not do so again here.

Example 3 $N \ = \ 5$ suffices to see that there are less new eigenstrokes to solve for at each N than there are eigenclusterings. For the last stroke of $\mathrm{K}(5)$ coincides with that of $\mathrm{H}(5)$.

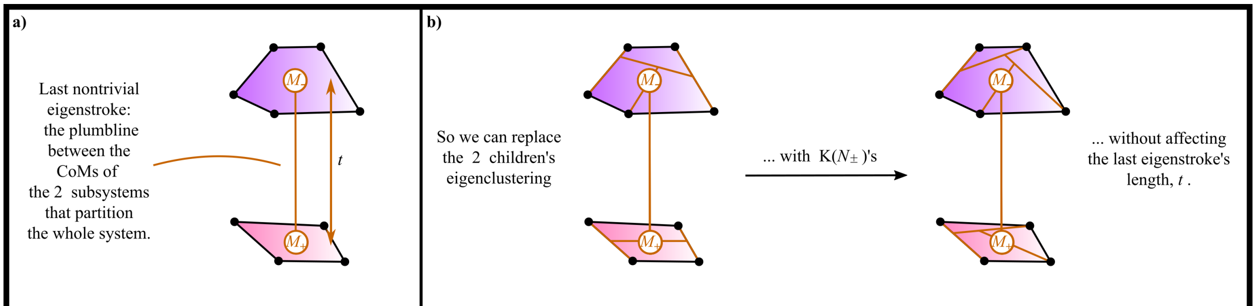
Pointer 2 This prompts our assessment of how eigenclustering magnitudes, and thus solutions for these, live on a smaller arena than the eigenclustering networks' and the ELETs'; see Article IV.

Pointer 3 While we have plenty of further interesting examples, we elect to leave these to subsequent Articles [30].

[1] Or at least a version of this that the next draft shall contain.

3 Proof of the Theorem

Trick 1. The last nontrivial eigenstroke t of an eigenclustering is a plumbline between its two childrens' centres of mass (CoM) (Fig 1.a). This is independent of how its corresponding right and left children are themselves eigenclustered. Thus we can w.l.o.g. take these to be $\mathrm{K}(N_\pm)$ (Subfig b). So as to take advantage of how we have already worked out everything for these in closed form in [23, 26].



c)

$$\frac{\mathrm{M}_+\mathrm{M}_-}{\mathrm{M}} T + \sum_{\mathrm{H}=\pm} \sum_{j_\mathrm{H}=1}^{n_-} \frac{\mathrm{M}_{j_\mathrm{H}}}{\mathrm{M}_{j_\mathrm{H}+1}} \mathrm{A}_{j_\mathrm{H}+1} O_{j_\mathrm{H}-1} = \underbrace{\frac{1}{\mathrm{M}} \sum_{\substack{\mathrm{I},\,\mathrm{J}=1 \\ \mathrm{I}<\mathrm{J}}}^{N} \mathrm{I}\,\mathrm{J}\, R^{\mathrm{I}\,\mathrm{J}}}_{\text{Split into:}}$$

Trick 2. Make 2 uses of Solved form of K-AMELT [25].

$$\sum_{\mathrm{H}=\mp} \frac{1}{\mathrm{M}_\mathrm{H}} \sum_{\substack{\mathrm{I}_\mathrm{H},\,\mathrm{J}_\mathrm{H}=1 \\ \mathrm{I}_\mathrm{H}<\mathrm{J}_\mathrm{H}}}^{N_\mathrm{H}} \mathrm{I}_\mathrm{H}\,\mathrm{J}_\mathrm{H}\, R^{\mathrm{I}_\mathrm{H}\,\mathrm{J}_\mathrm{H}} \qquad \overbrace{\frac{1}{\mathrm{M}} \sum_{\mathrm{H}=\mp} \sum_{\substack{\mathrm{I}_\mathrm{H},\,\mathrm{J}_\mathrm{H}=1 \\ \mathrm{I}_\mathrm{H}<\mathrm{J}_\mathrm{H}}}^{N_\mathrm{H}} \mathrm{I}_\mathrm{H}\,\mathrm{J}_\mathrm{H}\, R^{\mathrm{I}_\mathrm{H}\,\mathrm{J}_\mathrm{H}}}^{\text{the 2 self parts}} + \overbrace{\frac{1}{\mathrm{M}} \sum_{\mathrm{I}_-=1}^{N_-} \sum_{\mathrm{I}_+=1}^{N_+} \mathrm{I}_-\,\mathrm{J}_+\, R^{\mathrm{I}_-\,\mathrm{J}_+}}^{\text{and the mutual part.}}$$

d)

$$\frac{\mathrm{M}_+\mathrm{M}_-}{\cancel{\mathrm{M}}} T = -\frac{1}{\cancel{\mathrm{M}}} \sum_{\mathrm{H}=\mp} \frac{\mathrm{M}_{\overline{\mathrm{H}}}}{\mathrm{M}_\mathrm{H}} \sum_{\substack{\mathrm{I}_\mathrm{H},\,\mathrm{J}_\mathrm{H}=1 \\ \mathrm{I}_\mathrm{H}<\mathrm{J}_\mathrm{H}}}^{N_\mathrm{H}} \mathrm{I}_\mathrm{H}\,\mathrm{J}_\mathrm{H}\, R^{\mathrm{I}_\mathrm{H}\,\mathrm{J}_\mathrm{H}} + \frac{1}{\cancel{\mathrm{M}}} \sum_{\mathrm{I}_-=1}^{N_-} \sum_{\mathrm{I}_+=1}^{N_+} \mathrm{I}_-\,\mathrm{J}_+\, R^{\mathrm{I}_-\,\mathrm{J}_+}$$

Figure 1:

Principle 1. Sánchez' Useful Lemma [19, 26, 29] is

$$\iota_{\mathrm{Eig}} = \iota_{\mathrm{sep}} \; .$$

For our arbitrary Nth stroke, this returns the top equation in Subfig c).

We next apply two of Article II's pedestrian steps.

1)

$$\frac{1}{\mathrm{M}} - \frac{1}{\mathrm{M}_\mathrm{H}} = \frac{\mathrm{M}_\mathrm{H} - \mathrm{M}}{\mathrm{M}\,\mathrm{M}_\mathrm{H}} = -\frac{1}{\mathrm{M}} \frac{\mathrm{M}_{\overline{\mathrm{H}}}}{\mathrm{M}_\mathrm{H}} \; .$$

Where $\overline{\mathrm{H}}$ is the opposite sign to H . By which we arrive at the equation in Subfig d).

2) Divide by $\mathrm{M}_-\,\mathrm{M}_+$ to obtain the following.

$$T = \sum_{\mathrm{I}_-=1}^{N_-} \sum_{\mathrm{J}_+=1}^{N_+} \mathcal{M}_{I_-} \mathcal{M}_{J_+} R^{I_-\,J_+} - \sum_{\mathrm{H}=\mp} \sum_{\substack{\mathrm{I}_\mathrm{H},\,\mathrm{J}_\mathrm{H}=1 \\ \mathrm{I}_\mathrm{H}<\mathrm{J}_\mathrm{H}}}^{N_\mathrm{H}} \mathcal{M}_\mathrm{H}\, \mathcal{M}_\mathrm{H}\, R^{\mathrm{I}_\mathrm{H}\,\mathrm{J}_\mathrm{H}} \; .$$

Trick 3. Apply Sánchez' exclusion-sign function (17) to package this in the stated form. □

4 Discussion

Naming Remark 1 A truer name for Apollonius' Theorem is *Median-Length Theorem.* This Theorem extends to however in any dimension [22, 29], as does the notion of median. By which yet truer names are 3-*vertex Median-Length Theorem* (Geometry). 2-*simplex Median-Length Theorem* (Combinatorics, more advanced Geometry and Topology). 3-*body Median-Length Theorem* (a mixture of Physics-and-Dynamics with Geometry). Or, using the mass reformulation, 3-*body Equal-'masses' Plumbline-Length Theorem*, which is purely Physical.

Naming Remark 2 From Projective Geometry, a truer name for Cevian [5, 6, 10, 18] is *triangle cotransversal* [26]. Giving Stewart's Theorem the yet truer name *Triangle Cotransversal-Length Theorem.*

This Theorem also holds in any dimension [26, 29], however, as does the notion of cotransversal. By which yet truer names are, matching with the above, 3-*vertex Cotransversal-Length Theorem.* 2-*simplex Cotransversal-Length Theorem.* 3-*body Cotransversal-Length Theorem*, though this is now a more egregious mixture of Physics-and-Dynamics with Projective Geometry. Using the mass reformulation, however, 3-*body Arbitrary-'masses' Plumbline-Length Theorem* entails a fully Physical conceptualization. This mattered less for Apollonius' Theorem, since, firstly, medians can be viewed as but a basic Geometry notion (for all that they are implicitly Affine-Geometric [9, 29]). Secondly, Physicists have been using medians in CoM calculations all the way back to Archimedes...

Naming Remark 3 3 classical results, by Apollonius [1, 12, 18], Stewart [2, 8, 12, 18] and Euler (4-Body Theorem) [3, 14, 15, 16, 21] are generalized by the above Theorem in a manner opened up by Jacobi [4, 11, 13, 17]. So an alternative honorific name for the current Article's Theorem is *SAJE*, after these four sages. The four scribes who discovered it and then wrote it up are however Sánchez, and the current three Authors, ascribing a further name, SAFE.

Starting the truer names name with *Solved form of...* follows our conceptual nomenclature [26, 29] for variants of Stewart's Theorem. Two complications with ending the name are as follows. Firstly, ending with AMELET will not do, since this was designed for how K's top equation gives a linear combination of Lengths. Which feature subsequently persists for all ELETs. Secondly, given that most ELETs return coupled linear systems, there are in fact multiple notions to distinguish, among which we highlight in particular the following four.

1) The pth stroke-length for a given eigenclustering.

2) All the stroke-lengths for a given eigenclustering, with is an extended set of results like 1).

3) A stroke-length of some eigenclustering, without reference to that eigenclustering, since some of these are shared between eigenclusterings.

4) All the eigenstroke-lengths supported by a given N-vertex, n-simplex or N-body configuration.

So while 2) builds up 1)'s stated eigeclustering's set of stokes, 3) is instead a freeing from any particular eigenclustering. And 4) establishes a distinct totality. While this could be done by forming the union of all eigenclusterings supported, this would not be a disjoint union, by which it is less useful.

Truer names are thus as follows.

1) *Solved form for an Arbitrary-Mass Eigenclustering's pth Stroke-Length Theorem.*

2) *Solved form for an Arbitrary-Mass Eigenclustering's Stroke-Lengths Theorem.*

3) *Solved form for some Arbitrary-Mass Eigenstroke-Length Theorem.*

4) *Solved form for all Arbitrary-Mass Eigenstroke-Lengths Theorem.*

The closest to the spirit of the current Article is 4): the less-structured totality. So our choice of name is *Solved form for AMELT*, where L stands for total-plural 'Lengths'.

Eigenclustering, stroke and eigenstroke come from considering the joint eigentheory of vertex sets, simplices and bodies. Being well within the basic understanding of each of the abovementioned subject areasthere is then no need to have multiple naming streams as above. The sole competing truer name left is *Solved form for AMPLT*, with reference to *all of an Arbitrary-Masses' Plumbline-Lengths*. Which Physicists might conceivably prefer. In Geometry, the E in AMELT would stand furthermore for eigen[transversal]. Where [transversal] is projectively-dual portmanteau notation for transversal or cotransversal: a truer name for 'stroke'.

Naming Remark 4 Finally, the consequently truer name for our Corollary arbitrarily generalizing Apollonius' Theorem is then *Solved form for EMELT* (or *EMPLT*). This context serves furthermore to provide the simplest examples illustrating the necessity of covering both cotransversals: medians in Apollonius' Theorem and transversals: bimedians in Euler's Quadrilateral Theorem.

Pointer 4 Article IV shall also finish off Articles I and II's Multi-linear Algebra considerations in the light of the current Article' developments.

End-Remark Our Corollary is a close analogue of the *Democratic-Separations RoG Lemma*. Where RoG stands for radius of gyration, the square of which is related to the total moment of inertia by

$$\iota \;=\; \mathrm{M}\, R \;.$$

The Lemma is then that

$$R \;=\; N^{-2} \sum R^{\mathrm{I\,J}} \;=\; \frac{1}{N^2} \sum_{\substack{\mathrm{I,\,J}\;=\;1 \\ \mathrm{I}\;<\;\mathrm{J}}}^{N} R^{\mathrm{I\,J}} \;.$$

Furthermore, our Theorem is a slightly less close analogue of the *Mass-weighted Democratic-Separations RoG Lemma*,

$$R \;=\; \sum \mathcal{M}_{\mathrm{I}}\, \mathcal{M}_{\mathrm{J}}\, R^{\mathrm{I\,J}} \;=\; \sum_{\substack{\mathrm{I,\,J}\;=\;1 \\ \mathrm{I}\;<\;\mathrm{J}}}^{N} \mathcal{M}_{\mathrm{I}}\, \mathcal{M}_{\mathrm{J}}\, R^{\mathrm{I\,J}} \;.$$

Ab initio, this is the LHS of Sánchez' Useful Lemma. The end form of the Theorem results from subtracting off a number of pieces of the RHS. Due to eigenclustering expansions of ι containing their own side-Length contributions, which are sequentially cancelled off.

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A Notions and Notation used

At the level of sets

Definition 1 Let $\mathfrak{S}$ be a set, $\mathfrak{W} \subseteq \mathfrak{S}$ and $s \in \mathfrak{S}$. Then the corresponding Naïve Set-Theoretic *characteristic function*, alias *indicator function*, is as follows.

$$\chi(s) \;\stackrel{\text{shorthand}}{:=}\; \chi(\,\mathfrak{W};\,\mathfrak{S}\,)(s) \;:=\; \begin{cases} 1 & \text{if } s \in \mathfrak{W} \\ 0 & \text{otherwise} \quad . \end{cases} \;. \tag{8}$$

Definition 2 Let $\mathfrak{W}^{\mathrm{c}}$ denote the complement of $\mathfrak{W}$ in $\mathfrak{S}$. *Sánchez' exclusion-sign function* [19] is

$$\sigma(s) \;\stackrel{\text{shorthand}}{:=}\; \sigma(\,\mathfrak{W};\,\mathfrak{S}\,)(s) \;:=\; \begin{cases} 1 & \text{if } s \in \mathfrak{W}^{\mathrm{c}} \\ \text{-1} & \text{if } s \in \mathfrak{W} \;. \end{cases} \tag{9}$$

Lemma 1 [19]

$$\sigma(s) \;=\; (-)^{\chi(s)} \;. \tag{10}$$

<u>Proof</u> Re-express

$$\chi(s) \;=\; \begin{cases} 1 & \text{if } \; s \;\in\; \mathfrak{W} \\ 0 & \text{if } \; s \;\in\; \mathfrak{W}^{\mathrm{c}} \;. \end{cases} \;. \tag{11}$$

Then

$$(-)^{\chi(s)} \;=\; \begin{cases} (-)^1 & \text{if } \; s \;\in\; \mathfrak{W} \\ (-)^0 & \text{if } \; s \;\in\; \mathfrak{W}^{\mathrm{c}} \end{cases} \;=\; \begin{cases} \text{-1} & \text{if } \; s \;\in\; \mathfrak{W} \\ 1 & \text{if } \; s \;\in\; \mathfrak{W}^{\mathrm{c}} \;. \end{cases}$$

Finally the order of assignment of piecewise functions' pieces is immaterial. □

Naming Remark 1 This is termed 'exclusion-sign' since multiplying it in *preserves the excluded objects while flipping the sign of the included objects.*

Definition 3 Let $\mathfrak{T}$ be the set $\{\, 1, \,...\, , \, N \,\}$ with arbitrary elements $I, \, J, \, ...$. Form the binary-product set $\mathfrak{T} \times \mathfrak{T}$ with arbitrary elements such as $(\, I, \, J \,)$. And take $\mathfrak{S} \;=\; \mathfrak{S}\mathrm{ym}(\, \mathfrak{T} \times \mathfrak{T} \,)$: the symmetric part of our binary-product set. For which the arbitrary element is as above but additionally subject to $I \;<\; J$. Consider the partition $\mathfrak{T} \;=\; \mathfrak{U} \;\amalg\; \mathfrak{U}^{\mathrm{c}}$. Then take

$$\mathfrak{W} \;:=\; (\, \mathfrak{U}, \mathfrak{U} \,) \;\amalg\; (\, \mathfrak{U}^{\mathrm{c}}, \mathfrak{U}^{\mathrm{c}} \,) \;\;=\;\; \{\, \mathrm{self} \,\} \;, \tag{12}$$

$$\mathfrak{W} \;:=\; (\, \mathfrak{U}, \mathfrak{U}^{\mathrm{c}} \,) \;\;=\;\; \{\, \mathrm{mutual} \,\} \;. \tag{13}$$

I.e. are respectively the induced *self* and *mutual* partitions of our symmetric $\mathfrak{S}$.

At the level of sets of separations for N vertices

Structure 1 Now interpret the above $\mathfrak{T}$ as the N-vertex label set $\mathfrak{L}$. Denote the $\binom{N}{2}$ corresponding separation vectors by $\boldsymbol{r}^{\mathrm{I\,J}}$, and their magnitudes by $r^{\mathrm{I\,J}}$. Only $n \;:=\; N \;-\; 1$ of the $\boldsymbol{r}^{\mathrm{I\,J}}$ are linearly independent. Introduce furthermore[2] the 'weak-Conway separation Lengths'

$$R^{\mathrm{I\,J}} \;=\; \left(r^{\mathrm{I\,J}} \right)^2 \;.$$

These have many uses in Geometry; in the context of Stewart's Theorem and all of its generalizations in the current Article, they render the equation (system) linear. For a given configuration, all of the previous paragraph's variables are in bijective correspondence with the (I, J) that form $\mathfrak{S}\mathrm{ym}(\, \mathfrak{L} \;\times\; \mathfrak{L} \,)$. I.e. a subcase of how the above $(I, \, J)$ constitute $\mathfrak{S}\mathrm{ym}(\, \mathfrak{T} \;\times\; \mathfrak{T} \,)$.

Definition 4 The *symmetric double sum* over our N vertices is

$$\mathbb{\Sigma} \;\;=\;\; \sum_{\substack{\mathrm{I,\,J \;=\; 1} \\ \mathrm{I \;<\; J}}}^{N} \;. \tag{14}$$

At the level of eigenclustering

♠ ◇ ♣

Definition 5 An N-vertex configuration's *eigenclustering vectors* are linear combinations of its relative separation vectors that diagonalize its inertia quadric, alias total moment of inertia, ι . *Eigenclustering lengths* are then the corresponding magnitudes. Eigenclustering vector is a truer name for what are more widely termed *relative Jacobi vectors* [4, 11, 13, 17]; the alias *relative Jacobi magnitude* is more occasionally used.

Definition 6 An eigenclustering length is *nontrivial* if it is not just a side-length.

Definition 7 An *eigenclustering network* is a basis choice of eigenclustering vectors, with reference to how these fit together but without reference to whether or how the vertices are labelled. An *eigenstroke* is our term for the length of an eigenclustering vector without reference to any particular eigenclustering network it belongs to. This is a relevant distinction because some eigenclustering vectors belong to multiple distinct eigenclustering networks. It also serves to pin the "eigen" descriptor on names referring to an eigenclustering

[2] More generally, weak Conway notation uses capitalized letters to denote the squares of uncapitalized-letter symbols, and we even use 'Length' to mean $(\,\mathrm{length}\,)^2$.

vector free from reference to any eigenclustering network. For which using “eigenclustering" would either be ambiguous or require clunking up names with further words.

Structure 2 The eigenclustering networks are in $1 : 1$ correspondence with [17, 24] the unlabelled rooted binary trees.

♠ ◇ ♣

Structure 3 Let us now partition the vertex set $\mathfrak{L}$ into 2 nonzero pieces as indicated in Fig 2. Where left and right child is then standard binary-tree nomenclature [7]. And ‘mass’ signifies a $\mathbb{R}$-valued generalization of physical mass ($\mathbb{R}_0$-valued), so as to cover external cases as well as internal and corner cases.

	Total		Left-child		Right-child			Total / Left-child / Right-child
Vertex-label sets	$\mathcal{L}$	=	$\mathcal{L}_-$	$\amalg$	$\underbrace{\mathcal{L}_+}_{\mathcal{L}_-{}^{c}}$		Ordering of label sets	$\mathcal{L} = \underbrace{(1, \ \ldots \qquad , N)}$ $\underbrace{(1, \ \ldots \ N_-,}\ \underbrace{N_- + 1, \ \ldots \ , N)}$ $\underbrace{(1, \ \ldots \ N_-)}\ \underbrace{(N_- + 1, \ \ldots \ , N)}$ $\mathcal{L} = \quad \mathcal{L}_- \quad \amalg \quad \mathcal{L}_+$
Of size	$\underbrace{\lvert\mathcal{L}\rvert}_{N}$	=	$\underbrace{\lvert\mathcal{L}_-\rvert}_{N_-}$	+	$\underbrace{\lvert\mathcal{L}_+\rvert}_{N_+}$			
Vertices indexed by	I	=	I_-	+	I_+	Vertex `masses' indexed likewise		
	M	=	M_-	+	M_+	Overall `masses'		

Figure 2:

Definition 8 The subject of the curent Article is the nontrivial eigenstroke-Length $T = t^2$ between the two children’s CoMs.

Structure 4 Specializing the first Subappendix’s developments,

$$\chi(\mathrm{I},\mathrm{J}) \overset{\text{shorthand}}{:=} \chi(\,\mathfrak{S}\mathrm{ym}(\mathfrak{L} \times \mathfrak{L})\,,\ \mathfrak{W}(\mathfrak{L}_-)\,)(\mathrm{I},\mathrm{J}) = \tag{15}$$

$$\begin{cases} 1 & \text{if } (\mathrm{I},\mathrm{J}) \in (\mathfrak{L}_-, \mathfrak{L}_-) \amalg (\mathfrak{L}_+, \mathfrak{L}_+) \quad = \quad \{\text{ self-signed }\} \\ 0 & \text{if } (\mathrm{I},\mathrm{J}) \in (\mathfrak{L}_-, \mathfrak{L}_+) \quad = \quad \{\text{ mutual-signed }\}\ . \end{cases} \tag{16}$$

$$\sigma(\mathrm{I},\mathrm{J}) \overset{\text{shorthand}}{:=} \sigma(\,\mathfrak{S}\mathrm{ym}(\mathfrak{L} \times \mathfrak{L})\,,\ \mathfrak{W}(\mathfrak{L}_+)\,)(\mathrm{I},\mathrm{J}) = \begin{cases} 1 & \text{if } (\mathrm{I},\mathrm{J}) \in \{\text{ mutual-signed }\} \\ \text{-}1 & \text{if } (\mathrm{I},\mathrm{J}) \in \{\text{ self-signed }\} \end{cases} . \tag{17}$$

Definition 9 self_- refers to the vertex pairs within the left child, self_+ within the right, and $\mathrm{selves} = \mathrm{self}_- \amalg \mathrm{self}_+$. While mutual comprises the vertex pairs bridging between the 2 children.

Definition 10 For use in our Theorem, the *left-* and *right-child ‘mass’ fractions* are

$$\mathcal{M}_{I_\pm} = \frac{\mathrm{I}_\pm}{\mathrm{M}_\pm}\ . \tag{18}$$

Where each ratio’s $\pm$ take matching values. Finally, for use in our comparison with the Democratic-Separations RoG Lemmas, *the total ‘mass’ fractions* are

$$\mathcal{M}_I = \frac{\mathrm{I}}{\mathrm{M}}\ . \tag{19}$$

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