Stewart's Theorem generalized to
1 Theorem per Eigenclustering Network per N-Simplex.
I. $N = 4$ alias P_3

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Abstract

We use a 'physical' moments method to derive a family of N-simplex generalizations of Stewart's Cevian-Length Theorem for triangles. Our family has a number of different Theorems that increases with N. The smallest case past Stewart – $N = 4$ – that we concentrate on in this Article has 2. One of which is the mass-weighted generalization of Euler's 4-Body Theorem. In fact, our family members are indexed by the eigenclustering network types supported by each N. Which are unlabelled rooted binary tree valued for $N \geq 3$, so their counts are Wedderburn–Etherington numbers.

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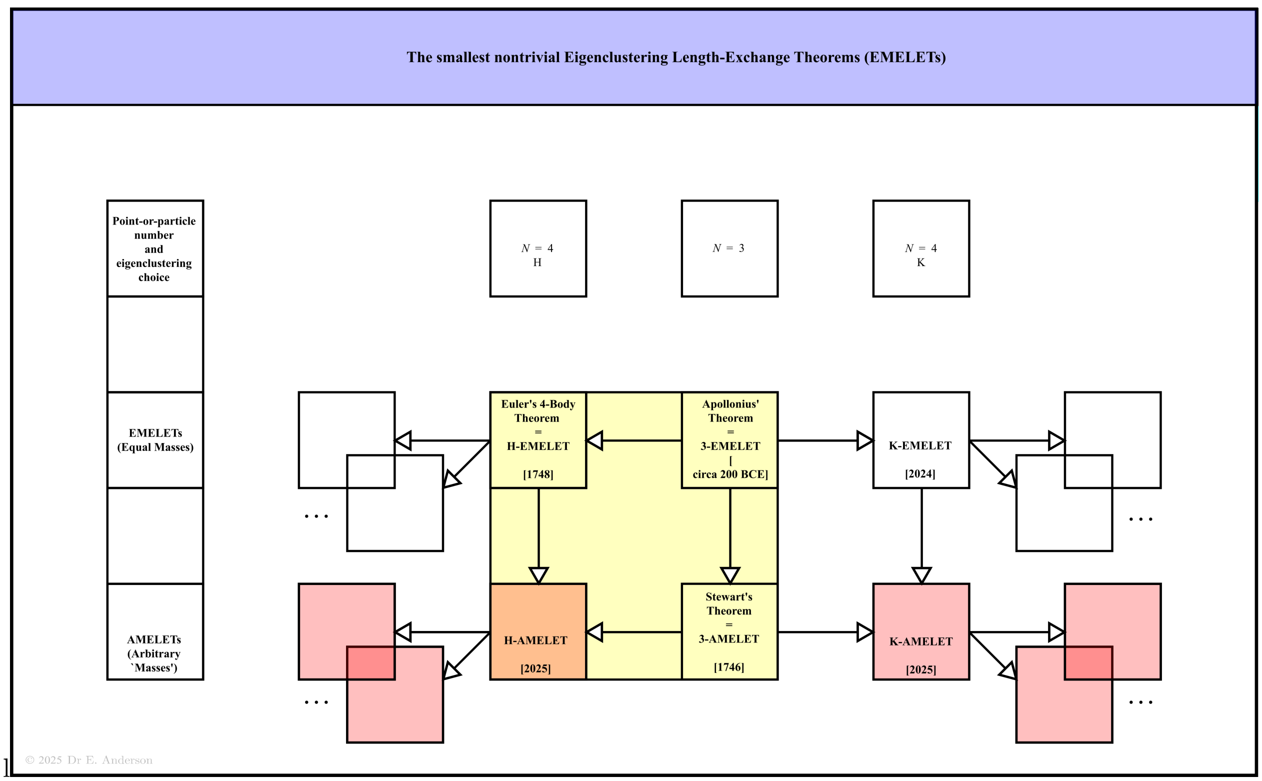

Figure 1:

1 Introduction

One of us recently showed that [55, 58, 66] new proofs of Apollonius' Medians-Length Theorem [1, 28, 47] and Euler's 4-Body Theorem [7, 36, 37, 39, 43, 58, 52] for the Newton length [5, 10, 13, 15, 43, 49] drop out as 2 of the smallest cases of a physical moments method. Two of us then reviewed how [61] this approach generates one structure per unlabelled rooted binary tree (URBT). The corresponding Theorems are EMELETs (Equal-Masses Eigenclustering Length-Exchange Theorems). For such to be nontrivial, enough room is required to support an exchange of eigenclustering lengths – alias it relative Jacobi magnitudes – [12, 27, 38]. Which slightly truncates us down to requiring $N \geq 3$ bodies. Apollonius then corresponds to the only eigenclustering network for $N = 3$. While Euler corresponds to the H-eigenclustering, alias Jacobi-H.

Hitherto, all workings made public were for equal masses (see also [59, 60, 63]). But today we generalize to arbitrary masses: AMELETs. Unveiling the first 2 members of a likewise URBT-valued 'AMELET' family, now generalizing Stewart's Cevian-Length Theorem [6, 21, 28, 44, 47, 66].[1] One of our new Theorems is the mass-weighted (mw) generalization of Euler's 4-Body Problem Theorem. Thus completing the Apollonius–Stewart–Euler square of Theorems (shaded yellow in Fig 1).

We begin in Sec 2 by recollecting Stewart's Theorem, since the current Series (see also [64, 65]) provides a large generalization of this useful result. In Sec 3, we explain the moments method that we use. We then use this to give a new proof of Stewart's Theorem in Sec 4. A generalization of Euler's 4-Body Theorem in Sec 5, and of its K-eigenclustering counterpart [59] in Sec 6. We state the scope of general result – the *AMELET* (*General-Masses Eigenclustering Length-Exchange Theorem*) in Sec 7.

In Sec 8, we interpret our three AMELETs by carefully identifying which mass ratios are relevant. In particular, the a priori under-determined K-AMELET then splits into 2 equations. Returning Stewart's Theorem for the picked-out 3-subsystem and a larger close relative for the last stroke of the K. In Sec 9, we consider setting up well-determined linear systems of ELETs a priori. We start to provide a multi-linear interpretation for AMELETs in Sec 10. This is continued in Article II [64]'s treatment of the arbitrary N's analogue of the K, and further unified after Article III's further examples.

See Appendix A for the triangle and tetrahaedron notation that we use, including the corresponding eigenclusterings. And Appendix B for the previously considered equal-masses (EM) counterparts: Apollonius = 3-EMELET, Euler = H-EMELET, and the K-EMELET.

[1] As explained in Appendix B, *triangle cotransversal* is a truer name for Cevian [3, 4, 18, 20, 34, 47].

2 Stewart's Theorem

Remark 1 In this Section, we use the triangle notation in Figs 3.a) and c).

Stewart's Theorem [Stewart 1746] [6] 0) Traditional form.

$$\overline{\mathrm{LB}}\,(\,\mathrm{AC}\,)^2 \;+\; \overline{\mathrm{CL}}\,(\,\mathrm{AB}\,)^2 \;\;=\;\; \overline{\mathrm{CB}}\;\left[\;(\,\mathrm{AL}\,)^2 \;+\; \overline{\mathrm{CL}}\,\overline{\mathrm{LB}}\;\right] \;. \tag{1}$$

Where the overline denotes directed distance.

Or, making AL the subject, form 0′)

$$\mathrm{AL} \;=\; \sqrt{\frac{\overline{\mathrm{CL}}\,(\,\mathrm{AB}\,)^2 \;+\; \overline{\mathrm{LB}}\,(\,\mathrm{AC}\,)^2}{\overline{\mathrm{CB}}} \;-\; \overline{\mathrm{CL}}\,\overline{\mathrm{LB}}} \;. \tag{2}$$

Next pass to side-length and Cevian-length variables. Alongside introducing the shorthands

$$x \;:=\; \overline{\mathrm{CL}}\;, \quad y \;:=\; \overline{\mathrm{LB}}\;, \quad d \;:=\; \overline{\mathrm{CB}}\;. \tag{3}$$

So as to obtain the following.

a) Polynomial form (homogeneous-cubic in lengths).

$$d\,\left(\,{v_a}^2 \;+\; x\,y\,\right) \;\;=\;\; x\,c^2 \;+\; y\,b^2\;. \tag{4}$$

a′) Cotransversal-length computation form.

$$v_a \;=\; \sqrt{\frac{x\,c^2 \;+\; y\,b^2}{d} \;-\; x\,y} \;. \tag{5}$$

b) Squared-length repackaging of a):

$$d\,(\,V_A \;+\; x\,y\,) \;\;=\;\; x\,C \;+\; y\,B\;. \tag{6}$$

b′) And of a′):

$$V_A \;=\; \frac{x\,C \;+\; y\,B}{d} \;-\; x\,y\;. \tag{7}$$

c) Squares-or-ratios form [56, 66].

$$V_A \;=\; \xi\,C \;+\; (\,1 \;-\; \xi\,)\,B \;+\; \xi\,(\,\xi \;-\; 1\,)\,A\;. \tag{8}$$

There is now no unprimed-primed distinction, since the factorization's distinguishing factor has been absorbed by defining the ratios.

d) Redundant-variable counterpart of c)

$$V_A \;=\; \xi_1\,C \;+\; \xi_2\,B \;-\; \xi_1\,\xi_2\,A\;. \tag{9}$$

Where

$$\xi_1 \;=\; \xi\;, \quad \xi_2 \;=\; 1 \;-\; \xi \quad \text{whence the redundancy} \quad \xi_1 \;+\; \xi_2 \;\;=\;\; \xi \;+\; 1 \;-\; \xi \;=\; 1\;. \tag{10}$$

We can furthermore write ξ and ξ^{c} for ξ_1 and ξ_2 . Where the c-superscript stands for 'complement'.

Remark 2 Forms c) and d) have homogenized to the extent that A features in the same way as B and C . It is these forms in particular that the current Article makes contact with.

Naming Remark 1 This is the Cevian Length Theorem. Alias Triangle-Cotransversal Length Theorem, as per Appendix B.

Remark 3 Stewart's Theorem [21, 28, 44, 47, 66] can be proven for instance by the cosine rule, or by Pythagoras, or using vectors. See Sec 4 for a further proof of Stewart's Theorem in its aspect as the 3-AMELET. And Sec 10, Article II, Article III and [66] for yet further formulations of Stewart's Theorem.

3 Underlying second moments method

Remark 1 The method developed by one of us with Sánchez [51] is to equate 2 different expansions for the inertia quadric. For N-body problem configurations [12, 27, 38], comprising N points-or-particles [54, 50, 66] in the space $\mathbb{R}^d$ as equipped with the Euclidean metric.

Structure 1 On the one hand, the 'Lagrange' expansion [29, 54, 50, 66]

$$\iota_{\mathrm{Lag}} = \frac{1}{\mathrm{M}} \sum_{\substack{\mathrm{I},\,\mathrm{J} = 1 \\ \mathrm{I} < \mathrm{J}}}^{N} \mathrm{I}\,\mathrm{J}\, r^{\mathrm{IJ}\,2} . \tag{11}$$

Where the I and J are point-or-particle labels. And we also shorten the notation for the corresponding masses m_{I} to just I .
And use

$$\mathrm{M} := \sum_{\mathrm{I} = 1}^{N} \mathrm{I} \tag{12}$$

for the total mass of the system. Also the corresponding position vectors are $\boldsymbol{q}^{\mathrm{I}}$. The

$$r^{\mathrm{IJ}} = ||\boldsymbol{r}^{\mathrm{IJ}}|| = ||\,\boldsymbol{q}^{\mathrm{J}} - \boldsymbol{q}^{\mathrm{I}}\,|| \tag{13}$$

are then the corresponding inter-point-or-particle separations.

Naming Remark 2 A truer name for this expansion is thus 'democratic-separations', since it makes democratic use of all of the N-body problem's separations. By which we pass to the notation ι_{Sep} .

Structure 2 On the other hand, the 'Jacobi' version

$$\iota_{\mathrm{Jac}} = \sum_{i = 1}^{n} \mu_i^{(\mathrm{E})} {R^i_{(\mathrm{E})}}^2 = \sum_{i = 1}^{n} {\rho^i_{(\mathrm{E})}}^2 . \tag{14}$$

Where the i correspond to a linear basis (LB) of

$$n := N - 1 \tag{15}$$

relative Jacobi separation vectors. Arising from diagonalizing the inertia quadric, which for $N \geq 3$ partly takes us away from having just inter-point-or-particle separation vectors. To instead having some separation vectors between subsystem CoMs (centres of mass). Such a LB's vectors' magnitudes are the $R^i_{(\mathrm{E})}$. And whose corresponding reduced masses are the $\mu_i^{(\mathrm{E})}$. The $\rho^i_{(\mathrm{E})}$ are the corresponding mw relative Jacobi magnitudes,

$$\rho^i_{(\mathrm{E})} := \sqrt{\mu_i^{(\mathrm{E})}}\, R^i_{(\mathrm{E})} . \tag{16}$$

The E-superscripts correspond to an ambiguity in specifying such a basis. In fact 2 qualitatively distinct kinds of ambiguity are supported: point-or-particle labelling ambiguities and network ambiguities. Regarding the labelling as meaningless for many purposes, we do not include this in our superscripts, which are thus pure-network. See Fig 2 for how $N = 3$ has just the 1 network shape: T . Whereas $N = 4$ has 2 : H and K .

Naming Remark 3 A truer name for relative Jacobi vectors is *eigenclustering vectors*. Our E thus encode eigenclustering network ambiguities, which growingly persist for all subsequent N . By which we pass to the subscripting ι_{Eig} .

Remark 2 The underlying reason that this expansion works is that the mw version encodes the Euclidean norm of the $n\,d$-dimensional relative space in Cartesian coordinates.

Remark 3 Our underlying equation is thus

$$\iota_{\mathrm{Sep}} = \iota_{\mathrm{Eig}} \tag{17}$$

for whichever choice E of eigenclustering network. This was conceived of as a 'Statics counterpart' of 'counting in 2 different ways' in Combinatorics: a well-known technique for obtaining nontrivial Theorems.

Remark 4 An alternative equivalent overall mass-free formulation can be set up using

$$\iota \; = \; \mathrm{M} \, R \, . \tag{18}$$

For R the square of the *radius of gyration* (*RoG*). Then (11) becomes the *separations-democratic RoG formula* [66]. I.e.

$$R_{\mathrm{Sep}} \; = \; \sum_{\substack{\mathrm{I}, \, \mathrm{J} \, = \, 1 \\ \mathrm{I} \, < \, \mathrm{J}}}^{N} \mathcal{M}_{\mathrm{I}} \mathcal{M}_{\mathrm{J}} \, r^{\mathrm{IJ} \, 2} \, . \tag{19}$$

For *mass fractions*

$$\mathcal{M}_{\mathrm{I}} \; := \; \frac{\mathrm{I}}{\mathrm{M}} \, . \tag{20}$$

Which can furthermore be exchanged for length ratios by the balance of first moments. This provides a glimpse of how form d) of Stewart's Theorem is related to the current Section's method.

One can then also rewrite (14) as follows.

$$R_{\mathrm{Eig}} \; = \; \sum_{i \, = \, 1}^{n} \mathcal{M}_i R^{i \, 2} \, . \tag{21}$$

Now for eigenclustering-reduced-mass to total-mass ratios

$$\mathcal{M}_{\mathrm{I}} \; := \; \frac{\mu_i}{\mathrm{M}} \, . \tag{22}$$

And then finally equate

$$R_{\mathrm{Sep}} \; = \; R_{\mathrm{Eig}} \, . \tag{23}$$

4 Recovery of Stewart's Theorem as the 3-AMELET

Remark 1 In this Section, we use the triangle notation in Fig 3.a) and Subfig b)'s 3- alias T-eigenclustering notation.

Proof of Stewart's Theorem by our moments method [51, 66]. For the 3-body problem, inserting row 1 of Fig 4's reduced masses into (17) returns the following.

$$\frac{\mathrm{A\,B}\,C \;+\; \mathrm{C\,A}\,B \;+\; \mathrm{B\,C}\,A}{\mathrm{M}} \;=\; \frac{\mathrm{B\,C}}{\mathrm{B} \;+\; \mathrm{C}}\,A \;+\; \frac{(\mathrm{B} \;+\; \mathrm{C})\,\mathrm{A}}{\mathrm{M}}\,V_A \;.$$

Where V_A is the $(\,\mathrm{length}\,)^2$ of the cotransversal line-interval v_a , corresponding to the side $a(\;=\; \mathrm{BC}\;)$, itself of $(\,\mathrm{length}\,)^2 \quad A$.

So making V_A the subject,

$$V_A \;=\; \frac{\mathrm{M}}{(\mathrm{B} \;+\; \mathrm{C})\,\mathrm{A}} \; \frac{\mathrm{B\,C}[\mathrm{B} \;+\; \mathrm{C} \;-\; (\mathrm{A} \;+\; \mathrm{B} \;+\; \mathrm{C})]\,A \;+\; \mathrm{A\,B}\,C \;+\; \mathrm{C\,A}\,B}{\mathrm{M}}$$

$$= \frac{\mathrm{A}}{\mathrm{A}} \, \frac{\mathrm{C}\,B \;+\; \mathrm{B}\,C \;-\; \mathrm{B\,C}\,A}{\mathrm{B} \;+\; \mathrm{C}} \;.$$

Which readily returns form c) or d) of Stewart's Theorem (8). Provided that we set

$$\xi \;:=\; \frac{\mathrm{B}}{\mathrm{B} \;+\; \mathrm{C}} \;=\; \frac{1}{1 \;+\; \theta} \;, \quad \theta \;:=\; \frac{\mathrm{B}}{\mathrm{C}} \;: \quad . \tag{24}$$

a single mass ratio. □

Remark 2 For the interior case of Stewart's Theorem (with L lying between C and B), directed lengths collapse to just lengths. The corresponding picture in terms of masses has all masses positive. The exterior case's signs require however extending to having some 'negative masses'. Which is mathematically acceptable for all that these 'masses' then bear less relation to physical masses.

Remark 3 For a median, set equal masses in the 'base' side:

$$\mathrm{B} \;=\; \mathrm{C} \;=\; 1 \;. \tag{25}$$

Then

$$\xi \;=\; \frac{1}{2} \;.$$

And so

$$M_A \;=\; \frac{C}{2} \;+\; \left(1 \;-\; \frac{1}{2}\right) B \;+\; \frac{1}{2}\left(\frac{1}{2} \;-\; 1\right) A \;=\; \frac{C}{2} \;+\; \frac{B}{2} \;-\; \frac{A}{4} \;.$$

Thus recovering the form in column 2 of Apollonius' Theorem in Fig 5.

5 The H-AMELET generalization of Euler's 4-Body Theorem

Remark 1 In this Section, we use the tetrahaedron notation in Fig 3.d) and Fig 3.f)'s H-eigenclustering notation.

H-AMELET (H-Arbitrary-Masses-Eigenclustering Length-Exchange Theorem) a) Solving for the generalized Newton length:

$$L = \frac{1}{(\mathrm{A}+\mathrm{B})^2(\mathrm{C}+\mathrm{D})^2}\left[\begin{array}{c}(\mathrm{A}+\mathrm{B})(\mathrm{C}+\mathrm{D})(\mathrm{BC}B+\mathrm{AD}D+\mathrm{AC}E+\mathrm{BD}F) \\ -\mathrm{AB}(\mathrm{C}+\mathrm{D})^2A-\mathrm{CD}(\mathrm{A}+\mathrm{B})^2C\end{array}\right].$$

b) Polynomial form [homogeneous quartic in the masses while linear in the $(\text{lengths})^2$]:

$$(\mathrm{A}+\mathrm{B})^2(\mathrm{C}+\mathrm{D})^2L+(\mathrm{C}+\mathrm{D})^2\mathrm{AB}\,A+(\mathrm{A}+\mathrm{B})^2\mathrm{CD}\,C$$

$$=(\mathrm{A}+\mathrm{B})(\mathrm{C}+\mathrm{D})(\mathrm{BC}B+\mathrm{AD}D+\mathrm{AC}E+\mathrm{BD}F).$$

<u>Proof</u> For the 4-body problem in H-eigenclustering inserting row 2 of Fig 4's reduced masses into (17) returns the following.

$$\frac{\mathrm{AB}A+\mathrm{BC}B+\mathrm{CD}C+\mathrm{AD}D+\mathrm{AC}E+\mathrm{BD}F}{\mathrm{M}}$$

$$=\frac{\mathrm{AB}}{\mathrm{A}+\mathrm{B}}A+\frac{\mathrm{CD}}{\mathrm{C}+\mathrm{D}}C+\frac{(\mathrm{A}+\mathrm{B})(\mathrm{C}+\mathrm{D})}{\mathrm{M}}L.$$

Where N is the $(\text{length})^2$ of the (doubly-generalized) Newton transversal line-interval l, between side $\mathrm{AB}=a$ and side $\mathrm{CD}=c$. Which we are to make into the subject:

$$L=\frac{1}{(\mathrm{A}+\mathrm{B})(\mathrm{C}+\mathrm{D})}\frac{\mathrm{M}}{\mathrm{M}}$$

$$\times\left[\frac{\mathrm{AB}(\mathrm{A}+\mathrm{B}-\mathrm{M})}{\mathrm{A}+\mathrm{B}}A+\frac{\mathrm{CD}(\mathrm{C}+\mathrm{D}-\mathrm{M})}{\mathrm{C}+\mathrm{D}}C+\mathrm{BC}B+\mathrm{AD}D+\mathrm{AC}E+\mathrm{BD}F\right]$$

$$=\frac{1}{(\mathrm{A}+\mathrm{B})(\mathrm{C}+\mathrm{D})}\left[\mathrm{BC}B+\mathrm{AD}D+\mathrm{AC}E+\mathrm{BD}F-\frac{\mathrm{AB}(\mathrm{C}+\mathrm{D})}{\mathrm{A}+\mathrm{B}}A-\frac{\mathrm{CD}(\mathrm{A}+\mathrm{B})}{\mathrm{C}+\mathrm{D}}C\right].$$

Then placing this on a common denominator returns form a). While multiplying through by this denominator and taking the 2 negative terms to the LHS returns form b). □

Remark 2 For the equal-masses case,

$$\mathrm{A}=\mathrm{B}=\mathrm{C}=\mathrm{D}=1, \tag{26}$$

b) collapses to the following.

$$16L+4A+4C=4(B+D+E+F)$$

$$\Rightarrow\ 4L=B+D+E+F-A-C.$$

Which is one of Euler's 3-cycles of form 2 of Euler's 4-Body Theorem in Fig 5.

6 The K-AMELET

Remark 1 In this Section, we use the tetrahaedron notation in Fig 3.d) with instead Fig 3.g)'s H-eigenclustering notation.

K-AMELET (K-General-Masses Eigenclustering Length-Exchange Theorem) a) Solving for the spike-handle mw LC:

$$C(A + B)MP + D(A + B + C)^2 H$$
$$= \frac{A + B + C}{A + B}\,[\,(A + B)(BCB + CDC + ADD + ACE + BDF) - (C + D)ABA\,]\,. \tag{27}$$

b) Polynomial form [again homogeneous quartic in the masses while linear in the $(\text{lengths})^2$]:

$$(A + B)\left[\,C(A + B)MP + D(A + B + C)^2 H\,\right] + (A + B + C)(C + D)ABA$$
$$= (A + B + C)(A + B)(BCB + CDC + ADD + ACE + BDF)\,.$$

Proof For the 4-body problem in H-eigenclustering inserting instead row 3 of Fig 4's reduced masses into (17) returns the following.

$$\frac{1}{M}(\,ABA + BCB + CDC + ADD + ACE + BDF\,)$$
$$= \frac{AB}{A + B}A + \frac{(A + B)C}{A + B + C}P + \frac{(A + B + C)D}{M}H\,.$$

Where P is the $(\text{length})^2$ of the spike cotransversal line-interval p, between the CoM of side $AB = a$ and vertex C . And H is the $(\text{length})^2$ of the handle cotransversal line-interval p , between the CoM of the triple subsystem $A, B, C = a$ and the remaining vertex D .

The specific mw LC of which we are to make into the subject is

$$\frac{(A + B)C}{A + B + C}P + \frac{(A + B + C)D}{M}H$$
$$= \frac{1}{M}\left[\frac{AB(A + B - M)}{A + B}A + BCB + CDC + ADD + ACE + BDF\right]$$
$$= \frac{1}{M}\left[BCB + CDC + ADD + ACE + BDF - \frac{AB(C + D)}{A + B}A\right].$$

Then isolating the spike-handle mw LC returns form a). While multiplying through by a)'s denominator and taking the sole negative terms to the LHS returns form b). □

Remark 2 For

$$A = B = C = D = 1\,, \tag{28}$$

b) collapses to the following.

$$2(8P + 9H) = 2 \times 3(\,B + C + D + E + F - A\,)$$
$$\Rightarrow\ 8P + 9H = 3(\,B + C + D + E + F - A\,)\,.$$

Which is form 2 of the K-EMELET in Fig 5.

7 AMELETs in general

AMELET Theorem [Sánchez–Anderson–Ford–Everard (SAFE) 2018] Throughout, we consider $N \geq 3$ points-or-particles in $\mathbb{R}^d$ for d arbitrary, and with arbitrary 'masses' $\in \mathbb{R}$. Thereupon, for every eigenclustering network, some combination of (nontrivial-eigenclustering lengths $)^2$ can be exchanged for (side length $)^2$ data. Providing an URBT-valued family of Theorems, which are nontrivial for $N \geq 3$. The counts of which for each N return the corresponding *Wedderburn–Etherington* numbers.

Remark 1 This is the first public statement of the far more powerful arbitrary-masses version of the Theorem.

Remark 2 See [14, 16, 23, 30] for the Wedderburn–Etherington numbers. [35, 42, 53, 61] for the URBT. And e.g. [38, 61] for their application to N-body problems.

8 (Ir)relevant ratio variables

Remark 1 N equidimensional quantities support $n = N - 1$ LI dimensionless ratios [19].

Example 1 In the $N = 3$ case, the x, y split determines the cotransversal. Equivalently by the first moments balance law, the mass ratio

$$\theta := \frac{\mathrm{B}}{\mathrm{C}} \tag{29}$$

does, by which it is a relevant mass ratio variable.

In contrast, the total CoM position does not enter the specification of the cotransversal. So e.g.

$$\phi := \frac{\mathrm{A}}{\mathrm{B}} \tag{30}$$

is an irrelevant ratio variable that completes the LB of all of the mass ratio variables. This is by the position of the CoM being determined by

$$\frac{\mathrm{A}}{\text{base mass}} = \frac{\mathrm{A}}{\mathrm{B} + \mathrm{C}} = \frac{\frac{\mathrm{A}}{\mathrm{B}}}{1 + \frac{\mathrm{C}}{\mathrm{B}}} .$$

But the top fraction is already a relevant ratio in our basis, leaving the fraction in the denominator as the carrier of the irrelevant information. This argument transcends to (with other functional dependencies) if θ was defined the other way up and A/C was chosen to play the role of ϕ.

The 3-AMELET – Stewart's Theorem – manages to depend on just the 1 ratio variable. This is furthermore a function of just the relevant one, as per (24). Thus consistency is attained.

Example 2 The H case plays out similarly. Now

$$\lambda = \frac{\mathrm{A}}{\mathrm{B}} , \quad \mu = \frac{\mathrm{C}}{\mathrm{D}} \tag{31}$$

can be interchanged for how the CoM splits each post, and so are relevant ratio variables. But the Geometrical H-framework is unaffected by where along the crossbar the total CoM is. Thus the remaining independent ratio,

$$\delta := \frac{\mathrm{C}}{\mathrm{A}} , \tag{32}$$

say, is an irrelevant ratio variable for this case.

The H-AMELET is consistent with this as follows. L can be reformulated purely in terms of the relevant ratios λ, μ, as per below. Take form a) and pull out a factor of $\mathrm{B}^2\mathrm{D}^2$ in both the numerator and the denominator. Apply also the definitions of our ratios so as to obtain the following.

Form c) of H-AMELET.

$$L \;=\; \frac{1}{(1+\lambda)^2(1+\mu)^2}$$
$$\times \;\Big[\,(1+\lambda)(1+\mu)(\,F+\lambda D+\mu B+\lambda\mu E\,) - \lambda(1+\mu)^2 A - \mu(1+\lambda)^2 C\,\Big]\;. \tag{33}$$

Example 3 In the K case,

$$\sigma \;:=\; \frac{B}{A}\;,\quad \tau \;:=\; \frac{C}{A}$$

are an LB of relevant ratio variables. Completed to an overall LB by the irrelevant ratio variable

$$\delta \;=\; \frac{D}{A}\;.$$

Dividing both sides of b) by A^4 and applying the definitions, we arrive at the following.

Form c) of K-AMELET

$$(1+\sigma)\,\Big[\,\tau(1+\sigma)(1+\sigma+\tau+\delta)P + (1+\sigma+\tau)^2\delta H\,\Big] + \sigma(1+\sigma+\tau)(\tau+\delta)A$$
$$=\;(1+\sigma)(1+\sigma+\tau)(\,\tau E+\delta D+\sigma\tau B+\tau\delta C+\sigma\delta F\,) \tag{34}$$

So this case's irrelevant ratio does not cancel out. Instead, the K-AMELET is an inhomogeneous-linear equation in this. But its output cannot depend on δ . Thus the K-AMELET is actually 2 equations: the coefficient of δ , and the coefficient of δ-free must both separately be 0 .

Provided that $\tau \neq 0$, these equations are as follows.

Form d) of K-AMELET

$$P \;=\; \frac{E+\sigma B}{1+\sigma} - \frac{\sigma A}{(1+\sigma)^2} \tag{35}$$

$$H \;=\; \frac{D+\sigma F+\tau C}{1+\sigma+\tau} - \frac{\sigma A+\tau E+\sigma\tau B}{(1+\sigma+\tau)^2} \tag{36}$$

Form e) of K-AMELET Back in mass variables,

$$P \;=\; \frac{\mathrm{A}E+\mathrm{B}B}{\mathrm{A}+\mathrm{B}} - \frac{\mathrm{A}\,\mathrm{B}\,A}{\mathrm{A}+\mathrm{B}}\;, \tag{37}$$

$$H \;=\; \frac{\mathrm{A}D+\mathrm{C}C+\mathrm{B}F}{\mathrm{A}+\mathrm{B}+\mathrm{C}} - \frac{\mathrm{B}A+\mathrm{C}C(\mathrm{A}E+\mathrm{B}B)}{(\mathrm{A}+\mathrm{B}+\mathrm{C})^2}\;. \tag{38}$$

For equal masses, (37) collapses down to Apollonius' Theorem for the picked out triple subsystem $\triangle$ABE . While (36) collapses down to

$$H \;=\; \frac{3(\,D+C+F\,) - (\,A+B+E\,)}{9}\;. \tag{39}$$

This bears a striking resemblance to Apollonius' Theorem. Which [63] confirms by establishing the general formula for all of the K(N)-AMELETs' stroke lengths.

The full (37) is recognizable as form c) of Stewart's Theorem for the picked out triple subsystem $\triangle$ABE . I.e. now with

$$\xi \;=\; \frac{\mathrm{A}}{\mathrm{A}+\mathrm{B}} \;=\; \frac{1}{1+\lambda}\;.$$

While the full (36) is a 3 side-fraction variables analogue of form d) of Stewart's Theorem, as follows.

Form f) of K-AMELET. The new last stroke is

$$H \;=\; \xi_1 D + \xi_2 F + \xi_3 C - (\,\xi_1\xi_2 A + \xi_3\xi_1 E + \xi_2\xi_3 B\,)\;. \tag{40}$$

Where

$$\xi_1 \ := \ \frac{1}{1 \ + \ \sigma \ + \ \tau} \ , \ \ \xi_2 \ := \ \frac{\sigma}{1 \ + \ \sigma \ + \ \tau} \ , \ \ \xi_3 \ := \ \frac{\tau}{1 \ + \ \sigma \ + \ \tau} \ . \tag{41}$$

Which bears a striking resemblance to form f) of Stewart's Theorem.

Remark 2 We then turn back to Example 2 and rephrase it as follows.

Form f) of H-AMELET

$$L \ = \ \xi_2\chi_1 \, B \ + \ \xi_1\chi_2 \, D \ + \ \xi_1\chi_1 \, E \ + \ \xi_2\chi_2 \, F \ - \ (\ \xi_1\xi_2 \, A \ + \ \chi_1\chi_2 \, C \ ) \ . \tag{42}$$

Where ξ_1, ξ_2 are a permutation of the redundant variables used in formulating Stewart's Theorem, now pertaining to how one 'post' in the H is split. And χ_1, χ_2 are defined in precisely the same manner for the other 'post'.

9 Well-determined linear systems of subsystem ELETs

Remark 1 Each of the 3- and H-AMELETs return 1 linear equation in 1 stroke-length unknown, which is thus well-determined. The K-AMELET returns however 1 linear equation in 2 stroke-length unknowns, which is a priori under-determined. And yet a careful analysis of which mass ratio variables are irrelevant, and what functional dependence on such irrelevant variables is trapped within our linear equation, splits it into 2 linear equations. Which is the right number of equations for well-determinedness. Each of the above cases' linear system is also nonsingular, and thus solvable and uniquely so, and also indeed has been solved!

Pointer 1 We address the extent to which the splitting phenomenon exhibited by the K-AMELET generalizes to other AMELETs in Articles II and III.

Remark 2 We do know that no other single AMELET equations are well-determined. This is since all eigenclusterings contain [61], using the floor function,

$$s \ = \ \left\lfloor \frac{n}{2} \right\rfloor \ \text{ to } \ n \ - \ 1 \ \text{ nontrivial lengths } . \tag{43}$$

So for $N \ \geq \ 5$, all possible eigenclusterings have $\geq \ 2$ nontrivial lengths.

Remark 3 Even if splitting does not produce a well-determined system of linear equations, we can however conceive of ELET problems in a different way. Namely that we prescribe by hand an ELET for each subsequent nontrivial subsystem picked out by our eigenclustering. These are the ones adding a stroke that is a nontrivial eigenclustering rather than a side. Such equations are thus s in total as well.

Example 1 The K-AMELET is rendered significant as the *minimum example* nontrivially requiring such a system of equations (obtained in whichever way). Which feature subsequently persists for all larger ELETs. This second approach could have a broader scope, if the split phenomenon turns out not to be persistent.

Lemma 1 By always adding in a new stroke with each equation, the second approach's guaranteed well-determined linear systems must furthermore be LI. And thus are nonsingular and thus solvable, and uniquely so.

Remark 4 Also observe that the first method above cannot work for fixed mass ratios. For these still have the same amount of under-determination, but now have no *free-variable* mass ratio to play the role of split-inducing irrelevant variable. This applies in particular to EMELETs. Among which [63] covers the infinite series of K(N)-EMELETs. But no other examples have yet been publicly revealed.

Remark 5 Open Question 1 and Remark 4 provide some further reasons to further study other small ELETS. And simple infinite series of ELETs that admit joint treatment, among which the K(N)-ELETs have a number of maximally simplifying features.

10 A split multi-linear formulation

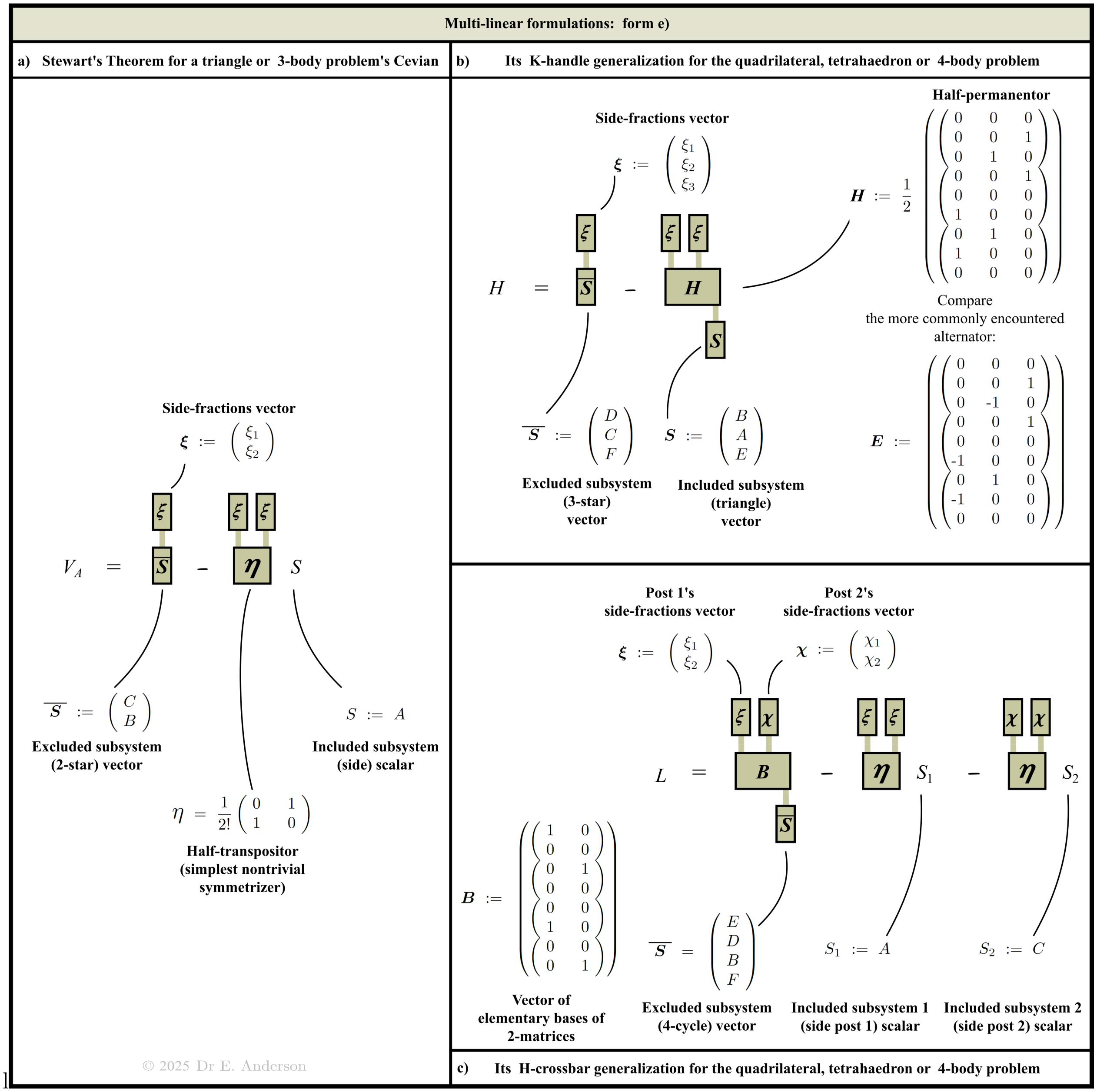

Figure 2:

Remark 1 Our final form e) for today for the the H- and K-AMELETs is presented in Fig 2.b-c). Each of these follows on from the corresponding form d). And constitutes a multi-linear repackaging. The particular notation that we use to display these here is a simple Combinatorial one: the vertical variant of [67, 66] Penrose birdtracks [22, 40]. To be clear, the split in question is unrelated to the previous sections', now referring to that into linear and quadratic pieces in the ratio variables.

Naming Remark 4 The *transpositor* is the sole transposition supported by the permutations on 2 objects. In its natural representation by a 2×2 matrix.

Naming Remark 5 The *permanent* $\boldsymbol{P}$ is the analogue of the determinant with all plus signs. Our name 'permanentor' follows from how the permanent is built out of this in the same way that the determinant is built out of the alternator alias alternating tensor alias epsilon tensor $\boldsymbol{E}$. Indeed, we would encourage the further alias *determinantor* for this aspect of the alternator.

The *symmetrizer* is then the further scaling

$$\boldsymbol{Z} := \frac{1}{K!}\boldsymbol{P} . \tag{44}$$

To the *antisymmetrizer* being

$$\boldsymbol{A} := \frac{1}{K!}\boldsymbol{E} . \tag{45}$$

These act so as to respectively carve out the totally-symmetric and totally-antisymmetric parts of an object. It so happens that the symmetrizer is much more widely used (and named, and thus quickly recognized) than the permanentor. Even though the alternator is much more widely used (and named in whichever of the first three above ways) than the antisymmetrizer.

Our half-permanentor can thus also be termed the k!-*symmetrizer* for $k := K - 1$. All of this Naming Remark applies to rank-K tensors in $\mathbb{R}^K$. The particular case that we need is for $K = 3$. Arising from the coincidence that triangles have 3 sides, while the K-eigenclustering involves 3 mass (or side) fractions. So our half-permanentor $\boldsymbol{H}$ can also be viewed as the tri-symmetrizer $3\,\boldsymbol{Z}$.

Remark 3 This development also points to a further form e) for Stewart's Theorem as per Subfig a).

Pointer 1 Articles II and III provide further, and eventually more unified, multi-linear formulations.

11 Some further context for our results

11.1 Features of our results

Remark 1 This Article's moments method is an *at-most second* moments method. In contrast, first moments methods are more widespread in Geometry. Medians concurrency at the centre of mass and barycentric coordinates thereabouts [45, 62] for triangles, and Varignon's Theorem [20, 17, 41, 52, 66] for quadrilaterals, are examples which can be viewed in terms of first moments.

Remark 2 The proof that we provided of Stewart's Theorem is furthermore dimension-independent. Be this the dimension of the space or of the 3-body configuration therein. Similar applies to vectorial and cosine-rule-based proofs [66]. And yet not to the often encountered [21] Pythagoras-based proofs.

11.2 Other generalizations of Stewart's Theorem

Remark 3 Some previously reported generalizations of Stewart's Theorem are as follows.

Generalization 1 In 1986, Branzei, Anita and Cocea [26] interrelated the following for a convex quadrilateral. The 4 distances to the vertices from an arbitrary point. And the 2-determinant combination of lengths from the intersection of the diagonals to the vertices.

Generalization 2 While in 1980 Bottema [25] equated the following, in the context of a tetrahaedron. 3 separation lengths emanating from an apex vertex. The circumradius of the triangle formed by these 3 vertices. And the distance from the apex to a point in the plane defined by the other 3 vertices. Which is a very natural generalization of a triangle's Cevian.

Remark 4 In both cases with (subsystem) areas as coefficients.

Generalization 2′ The above work of Bottema is a more detailed specialization of his N-simplex result from 1979 [24] (for N squared lengths). In which context his area coefficients are replaced by hypervolumes, and thus are highest-non-top-form-valued.

11.3 How our generalization is different

Difference 1 We interrelate, on the one hand, N-choose-2 separation lengths: the full democratic amount. Which is a larger amount even for $N = 4$ than that which features in Generalizations 1 and 2. With, on the other hand, eigenclustering separation lengths totalling s unknowns [the bounding values for the count of which are supplied in (43)]. For our generalization proceeds very differently from the above two. Via observing that the Cevian (triangle cotransversal), whose length Stewart's Theorem computes, is a first instance of nontrivial eigenclustering magnitude. Thus imbued with Linear Algebra, and more specifically with its intersection with Spectral Theory, we generalize to arbitrary eigenclustering networks. This gives us a Wedderburn–Etherington number $w(N)$ of generalizations for each N, rather than just Bottema's 1.

Difference 2 And that the coefficients involved in our formulae can be interpreted in terms of mass ratios (rather than the above areas or hypervolumes). Nor does circumradius play a role in our formulae.

Difference 3 In our generalization, both cotransversals and transversals are capable of entering. The 4-body problem already serves as a minimum example of both entering. For the K's last stroke and the H's Newton line-interval are a cotransversal and a transversal respectively. This is to be contrasted with the above two generalizations seeking to compute the length of a single, Geometrically-natural generalization of a Cevian for a tetrahaedron. (Whether or not further extending likewise to each N-simplex).

Difference 4 Since our approach is moments-based, it is furthermore not restricted to any particular spatial dimension or to convex figures.

End-Note 1 By these differences, it is clear that our generalization of Stewart's Theorem is distinct from the above two from the previous literature.

A Notation

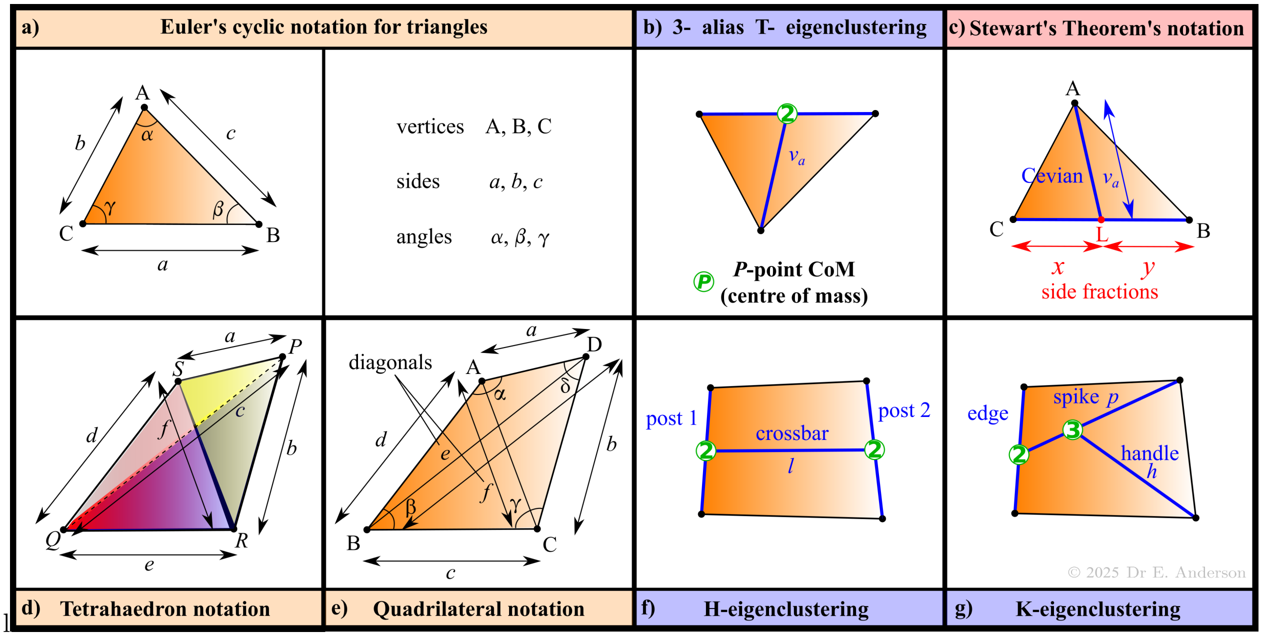

Figure 3:

Notational Remark 1 See Fig 3.a) for Euler's subsequently standard 3-cycle notation for the triangle. And a corresponding notation for a tetrahaedron (including the quadrilateral as a subcase). Alongside the sole eigenclustering network supported by triangles: T-shaped. And the 2 eigenclustering networks supported by tetrahaedra: H- and K-shaped.

Naming Remark 6 The names 'post' and 'crossbar' are with reference to a set of Rugby goal-posts [46]. While the names 'edge', 'spike' and 'handle' are with reference to an axe [46, 59].

Notational Remark 2 Conway's notation in the weaker sense [2]. involves using not (length) but (length)^2 variables. Denoted by the capital letter versions of each lower-case length variable.

Remark 3 See Fig 4.a) for the reduced masses in the arbitrary 3-body problem and H- and K-eigenclustering choices for the arbitrary 4-body problem. And Subfig b) for their equal-mass counterparts.

Naming Remark 7 Truer names for K and H themselves are straight- and bent-P_3 . With reference to the 2 possible rootings of the 3-path. This correspond to the more convenient *AMB* representation (*at-most-binary*) [61] that arises from the URBT representation by defoliation. Which is an equivalent representation for all nontrivial ELETs. For URBT, the vertices and all the participating nontrivial-subsystem CoMs form the tree's vertices. While for AMB, the vertices are removed, so the participating nontrivial-subsystem CoMs alone enter. This also gives a further name to Stewart's Theorem: the P_2-AMELET (which only has 1 possible root, and thus requires no further descriptor).

[2]See e.g. [66, 62] for the stronger case

Reduced masses			
Eigenclustering / Vertex mass type	$N = 3$	$N = 4$ H	$N = 4$ K
Arbitrary 'masses'	$\mu_1 = \frac{BC}{B + C}$ $\mu_2 = \frac{(B + C)A}{M}$	$\mu_1 = \frac{AB}{A + B}$ $\mu_2 = \frac{CD}{C + D}$ $\mu_3 = \frac{(A + B)(C + D)}{M}$	$\mu_1 = \frac{AB}{A + B}$ $\mu_2 = \frac{(A + B)C}{A + B + C}$ $\mu_3 = \frac{(A + B + C)D}{M}$
Equal masses	$\mu_1 = \frac{1}{2}$ $\mu_2 = \frac{2}{3}$	$\mu_1 = \frac{1}{2}$ $\mu_2 = \frac{1}{2}$ $\mu_3 = 1$	$\mu_1 = \frac{1}{2}$ $\mu_2 = \frac{2}{3}$ $\mu_3 = \frac{3}{4}$

Figure 4:

B EMELETs (Equal-Mass Eigenclustering Length-Exchange Theorems)

The smallest nontrivial Equal-Mass Eigenclustering Length-Exchange Theorems (EMELETs)		
Case	**Length evaluation form**	**Squared-length variables form**
Apollonius' Theorem = 3-EMELET	$m_a{}^2 = \frac{2(b^2 + c^2) - a^2}{4}$	$M_A = \frac{2(B + C) - A}{4}$
Euler's 4-Body Theorem = H-EMELET	$4l^2 = a^2 + b^2 + c^2 + d^2 - e^2 - f^2$	$4L = A + B + C + D - E - F$
K-EMELET	$8p^2 + 9h^2 = 3(b^2 + c^2 + d^2 + e^2 + f^2 - a^2)$	$8P + 9H = 3(B + C + D + E + F - A)$

Figure 5:

Remark 1 Any 3-body configuration supports an Euler triangle 3-cycle of Apollonius Theorems as per row 1 of Fig 5.

Remark 2 Euler's distinct 3-cycle of Euler's 4-Body Theorem is over the possible ways of pairing the vertices. The most commonly encountered case – in row 2 of our last Figure – has the diagonals contributing negatively. The other two cases pick out this or that opposite-side pairs instead.

Naming Remark 8 The names 'side' and 'diagonal' are quadrilateral terminology. As is the more commonly encountered name 'Euler Quadrilateral Theorem' For all that the Euler 3-cycle is itself realized regardless of dimension. And the Theorem also holds for line configurations and tetrahaedra [36, 58, 66].

Naming Remark 9 It is in the diagonal case that the Newton line-interval [5, 10, 13, 15, 43] itself features for equal masses. When we say that this is doubly-generalized, we mean the following. Once by Euler's 3-cycle. And in a second independent way by the current Article's passage to arbitrary vertex masses.

Remark 3 The K-EMELET counterpart is displayed in row 3 of our last Figure.

Naming Remark 10 The name 'transversal' was introduced in Carnot's [9] projective treatise; see also e.g. [11, 15]. Menelians [2, 18, 20, 34, 47] and Newton lines are examples of transversals. While Cevians [3, 4, 18, 20, 34, 47] are examples of cotransversals, as are the K-eigenclustering's spike and handle. [Transversal] [57, 58, 66] is then a portmanteau name covering both transversals and cotransversals. Eigenclustering lengths are a very special case of [transversal] lengths.

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