Triangle, but not most Quadrilateral, Matrices are Combinatorial

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Abstract

A further reason is given for why the matrix theory of triangles is simpler and neater than that for quadrilaterals. Namely, that it is purely in terms of combinatorial matrices. The observed full commutation of the triangle matrices is then but a simple consequence. Also sets of invertible combinatorial matrices multiplicatively form commutative groups. Suppose that some invertability is dropped, as occurs for the triangle matrices since these include a Lagrange matrix: a type of projector. Then commutative monoid structure persists, now also just on combinatorial matrix grounds.

New combinatorial algebra characterizations are also given for the Lagrange and Apollonius matrices of the N-body problem and the triangle respectively. Alongside an alternative generalization for the latter. And arenas of algebraically-distinguished combinatorial matrices.

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1 Introduction

1.1 Motivation

Recent works [17, 23, 25, 26] have developed much of the theory of triangles from three matrices. Namely, the Lagrange [3] projector $\boldsymbol{P}$ [18, 24, 15], the Apollonius [1] involutor $\boldsymbol{J}$ [17] and the fundamental triangle matrix $\boldsymbol{F}$ [23, 25, 26], previously a.k.a. Heron [2] matrix [11]. In contrast, generalizing to quadrilaterals [18] produces many more matrices [20, 31, 33, 34] with large amounts of multiplicative incompatibility. And less results more generally.

Various reasons for triangles' exceptionally neat linear algebra results have been given.

1) Flat geometry. That for triangles separations coincide with sides. Whereas for quadrilaterals 2 of the 6 separations are diagonals, rather.

2) Linear algebra. That for $N = 3$ the eigenclustering vectors [28, 27] come in the right number – 2 – to constitute a spatial basis. As part of the diagonal of d-dimensional N-body problems with $N = d + 1$, Which has so far usually been called 'simplex-' or 'Casson-' [12] rather than just 'basis-diagonal' [19].

3) Topology and differential geometry. That in particular the triangle contains a 'hidden angel' [6, 9, 17, 37], namely 'Hopf's little map' [5, 8, 14]

$$_H \ : \ \mathbb{S}^3 \longrightarrow \mathbb{S}^2 \ . \tag{1}$$

For all that the theory of quadrilaterals involves a larger Hopf map

$$H \ : \ \mathbb{S}^5 \longrightarrow \mathbb{CP}^2 \ . \tag{2}$$

For the latter is both not quite as mathematically special. Nor known to be tied to area formulae or matrices encoding these. Whereas there are close ties [17, 25] between Hopf's little map and the quadratic form formulation of Heron's formula. Incidentally, the $\mathbb{S}^2$ and $\mathbb{CP}^2$ are realized as [12] the shape spaces modulo similarities of triangles and quadrilaterals respectively.

4) Flat geometry. There is a large increase in the number of area formulae upon passing from triangles to quadrilaterals [4, 21, 20, 38].

5) Distance geometry, representation theory (...) In any case, quadrilateral area formulae do not generalize all facets of [10, 16, 18, 20, 36] the surprisingly deep Heron's formula.

The purpose of this note is to add one more reason.

6) Algebraic combinatorics. Namely, that $\boldsymbol{P}$, $\boldsymbol{J}$ and $\boldsymbol{F}$ are all *combinatorial matrices* [7, 22]. In particular, this by itself guarantees that sets of such will both brackets-commute and multiplicatively form at least a commutative monoid. So these results of [25, 26] for triangle matrices in fact hold for any compatible set of combinatorial matrices. In contrast, none of the compatible fragments of the linear algebra of quadrilaterals enjoy the property of consisting purely of combinatorial matrices.

1.2 Motivation

In section 2, combinatorial matrices are defined. Two new notations are provided for these, which would rather simplify [23, 25]'s presentation. One is for blocks and the other for their trace-tracefree irreducible decomposition: more nuanced from a representation theory point of view.

Section 3, serves to point out that combinatorial matrices of compatible size always commute. And to find all combinatorial matrices that are projectors or involutors.

Section 4 then serves to consider arenas of algebraically-distinguished combinatorial matrices. In the process, there is cause to introduce trace-reversed blocks, and a corresponding third notation in terms of these. Arenas are configuration spaces, each comprising the totality of some given type of mathematical object. Work included on these here can be viewed as mapping the layout of algebraically-distinguished combinatorial matrices within the overall arena [22] of combinatorial matrices. This is now available as a background in which the matrix theory of triangles can be embedded, and the matrices that [25] opens up with as well.

Section 5 finally surveys the more patchy extent to which combinatorial matrices have so far been found within the maelstrom of quadrilateral linear algebra considered so far.

2 Combinatorial matrices

2.1 Definition

Definition 1 A *combinatorial matrix* [7] is a square matrix of the following form.

$$\begin{pmatrix} x + y & x & \dots & x \\ x & & & \vdots \\ \vdots & & & x \\ x & \dots & x & x + y \end{pmatrix} = y\mathbb{I} + x\mathbb{1} \, . \tag{3}$$

Remark 1 The second expression here follows [26]'s use of coordinate-free notation for arrays. And so happens to be a coordinate-free rendition of a componentwise expression used by Knuth [7]. Where in the standard basis, $\mathbb{I}$ is the corresponding identity matrix. And $\mathbb{1}$ is the 'block' matrix whose entries are all 1's. So Knuth used the Kronecker delta expression for the first of these. And no symbol at all for the second, it being absorbed by the identity relation.

2.2 Linearity and notation

Lemma 1 The combinatorial matrices for a given K form a 2-d vector space under matrix addition and multiplication by a field-valued scalar.

Notation 1 Since we shall be seeing many combinatorial matrices and properties thereof, we introduce the following further shorthands. Let us denote (3) by

$$(y, x)_K \tag{4}$$

in the *block basis*. And by

$$[x + y, x]_K \tag{5}$$

in the *irreducible basis*. I.e. now expanding in terms of $\mathbb{I}$ and $\mathbb{T}$: the tracefree part of $\mathbb{1}$ in [26]'s notation. A priori, this second notation comes with representation theory benefits. It subsequently turns out to be more practical to use as well.

The K-subscript denotes that the square matrix in question is of size K , i.e. $K \times K$. In the event of dealing with a set or space of compatible matrices, we can afford to drop this subscript.

Lemma 2 The above symbols enjoy the following properties.

i) *Homogeneous-linearity*

$$(\lambda y, \lambda x)_K = \lambda (y, x)_K \, , \tag{6}$$

$$[\lambda a, \lambda b]_K = \lambda [a, b]_K \, . \tag{7}$$

ii) *Additivity of compatible matrices*

$$(y, x)_K + (w, v)_K = (y + w, x + v)_K \, , \tag{8}$$

$$[y, x]_K + [w, v]_K = [y + w, x + v]_K \, . \tag{9}$$

2.3 Some basic algebra and representation theory examples

Example 0 In our notation, the zero matrix $\mathbb{0}$ is

$$(\,0,\,0\,)_K \;=\; [\,0,\,0\,]_K \;. \tag{10}$$

Example I The identity matrix $\mathbb{I}$ is

$$(\,1,\,0\,)_K \;=\; [\,1,\,0\,]_K \;. \tag{11}$$

Remark 1 The only matrices for which our two symbols' entries coincide are

$$(\,x,\,0\,)_K \;=\; [\,x,\,0\,]_K \;. \tag{12}$$

By homogeneous linearity, these are proportional to the identity matrix. This includes both of the above examples as the $x = 0$ and 1 subcases.

Example 1 The matrix of 1's a.k.a. block matrix $\mathbb{1}$ is

$$(\,0,\,1\,)_K \;=\; [\,1,\,1\,]_K \;. \tag{13}$$

Example T The tracefree part $\mathbb{T}$ of the block matrix is

$$(\,-1,\,1\,)_K \;=\; [\,0,\,1\,]_K \;. \tag{14}$$

These amount to finding each basis' sparse binary concomitant of the identity.

2.4 Some recently discussed examples from flat geometry

Example 3.P For a triangle, dropping the 3-subscripts,

$$\boldsymbol{P} \;=\; \left(1\,,\;\frac{\text{-}1}{3}\right) \;=\; \frac{1}{3}\,[\,2,\,\text{-}1\,]\;. \tag{15}$$

Example 3.J

$$\boldsymbol{J} \;=\; \left(\text{-}1\,,\;\frac{2}{3}\right) \;=\; \frac{1}{3}\,[\,\text{-}1,\,2\,]\;. \tag{16}$$

Example 3.F

$$\boldsymbol{F} \;=\; (\,\text{-}2,\,1\,) \;=\; [\,\text{-}1,\,1\,]\;. \tag{17}$$

Example N.F

$$\boldsymbol{F}(N) \;=\; (\,\text{-}2,\,1\,)_N \;=\; [\,\text{-}1,\,1\,]_N\;. \tag{18}$$

Example N.P

$$\boldsymbol{P}(N) \;=\; N^{-1}\,(\,N,\,\text{-}1\,)_N \;=\; N^{-1}\,[\,n,\,\text{-}1\,]_N\;. \tag{19}$$

Example N.M

$$\boldsymbol{M}(N) \;=\; N^{-1}\,[\,\text{-}1,\,n\,]_N\;. \tag{20}$$

Remark 1 Above

$$n \;:=\; N \,-\, 1\;. \tag{21}$$

Whose N-body problem interpretation is as the dimension of the label space for relative space.

2.5 A first few applications

Application 1 In particular, the 'fundamental LD for triangles' [26]

$$\boldsymbol{F} \;=\; \boldsymbol{J} \;-\; \boldsymbol{P} \tag{22}$$

becomes (dropping the 3-subscripts)

$$[\text{-}1,\, 1\,] \;=\; 3^{-1}\,[\text{-}1,\, 2\,] \;-\; 3^{-1}\,[\,2,\, \text{-}1\,]\;. \tag{23}$$

Then also

$$N^{-1}\;\{\;[\text{-}1,\, n\,]_N \;-\; [\,n,\, \text{-}1\,]_N\;\} \;=\; N^{-1}\,[\text{-}N,\, N\,]_N \;=\; N\,N^{-1}\,[\text{-}1,\, 1\,]_N \;=\; [\text{-}1,\, 1\,]_N\;. \tag{24}$$

By which (22) generalizes to

$$\boldsymbol{F}(N) \;=\; \boldsymbol{M}(N) \;-\; \boldsymbol{P}(N)\;. \tag{25}$$

This explains why $\boldsymbol{M}(N)$ features in [25], though we shall also be providing an alternative in section 3.

Lemma 3 i)

$$\mathrm{tr}(\,y,\, x\,)_K \;=\; K\,(\,x \;+\; y\,)\;. \tag{26}$$

ii) [7, 22]

$$\det(\,y,\, x\,)_K \;=\; y^k\,(\,y \;+\; K\,x\,)\;. \tag{27}$$

For $k \;:=\; K \;-\; 1$.

Exercise 1 Prove this.

Remark 1 Two Corollaries for this, in reverse order of current interest, are as follows.

Corollary 1 For a combinatorial matrix to be tracefree,[1]

$$x \;+\; y \;=\; 0\;. \tag{28}$$

The only tracefree possibilities are then the matrices $\propto \mathbb{T}$.

Corollary 2 For a combinatorial matrix to be singular, one or both of the following must hold.

$$y \;=\; 0\;, \tag{29}$$

$$y \;=\; \text{-}K\,x\;. \tag{30}$$

If the first equation holds alone, we have

$$[\,x,\, x\,]_K\;, \tag{31}$$

of nullity k .

If the second equation holds alone, we have

$$x\,[\,-k,\, 1\,]_K \;\propto\; \boldsymbol{P}(K)\;, \tag{32}$$

of nullity 1 .

Finally if both hold at once, we have

$$[\,0,\, 0\,]_K \;=\; \mathbb{0}\;: \tag{33}$$

the zero matrix of nullity K .

Remark 1 So the arbitrary-N equal-masses Lagrange projectors can also be characterized as that N's unique projector onto a $\geq$ 1-d linear space. This is for nontrivial N -body problems – $N \;\geq\; 3$ – since below this no room is left.

[1] This is subject to the assumption that the underlying field is $\mathbb{R}$. Characteristic-k offers other possibilities.

3 Some algebra of combinatorial matrices

3.1 Multiplication and commutativity

Lemma 4 (Multiplication rule)

$$[x_1 + y_1, x_1]_K \; [x_2 + y_2, x_2]_K =$$

$$[K x_1 x_2 + x_1 y_2 + x_2 y_1 + y_1 y_2 \, , \; K x_1 x_2 + x_1 y_2 + x_2 y_1]_K \; . \tag{34}$$

Proof Expand and then regroup each of the following.

$$\prod_{i=1}^{2} (x_i + y_i) + k x_1 x_2 \; . \tag{35}$$

$$\sum_{i=1}^{2} (x_i + y_i) x_j + (k - 1) x_1 x_2 \; . \tag{36}$$

Where $j \neq i$. □

Corollary 3 (Squares)

$$[x + y, x]_K{}^2 = [x(Kx + 2y) + y^2 \, , \; x(Kx + 2y)]_K \; . \tag{37}$$

Proof Set $1 = \text{blank} = 2$ in Lemma 4. □

Corollary 4 All compatible combinatorial matrices commute with each other.

Proof Lemma 4's expression is $1 \leftrightarrow 2$ invariant. □

Remark 1 Thus that the three triangle matrices commute can be explained simply from their being of the conceptual type 'combinatorial matrix'. Without any reference to geometry.

Remark 2 Matrix multiplication is associative, Lemma 4 gives that combinatorial matrices close, and Corollary 4 that they commute. Example I provides the multiplicative identity combinatorial matrix. Given a set of combinatorial matrices, if any singular such are present, the inverse property fails. One is thus left with [22] a commutative monoid [13]. Remark 1's manner of underpinning explanation then plays out once more. If none are singular, the inverses are themselves combinatorial (**Exercise 2**). In such cases, a commutative group ensues.

3.2 Combinatorial projectors

Definition 1 A matrix $\boldsymbol{P}$ is a *projector* if it obeys

$$\boldsymbol{P}^2 = \boldsymbol{P} . \tag{38}$$

Remark 1 $\boldsymbol{P}$ is a *nontrivial projector* if both its image and its kernel are of dimension ≥ 1. For if its kernel is of dimension 0, nothing is projected out and one has the identity matrix $\mathbb{I}$. While if its image is of dimension 0, everything is projected out. And one has the zero matrix $\mathbb{0}$ acting as a total annihilator. By the Rank–Nullity Theorem, both of the above can only simultaneously occur if the overall dimension is zero.

Proposition 1 i) The only possible combinatorial projectors are

$$[0, 0]_K , \quad K^{-1}[1, 1]_K , \quad K^{-1}[k, \text{-}1]_K , \quad [1, 0] . \tag{39}$$

ii) Among which, the nontrivial ones are the second and third, provided that $K \geq 2$.

<u>Proof</u> i) Equating non-diagonal and diagonal elements in turn, combinatorial projectors obey the following pair of simultaneous quadratic equations.

$$x(Kx + 2y) = x , \tag{40}$$

$$x(Kx + 2y) + y^2 = x + y . \tag{41}$$

Use (41) $-$ (40) in place of (41). I.e.

$$y^2 = y \quad \Rightarrow \quad y(y - 1) = 0 . \tag{42}$$

So

$$y = 0 \text{ or } 1 . \tag{43}$$

Thus

$$x(Kx \mp 1) = 0 . \tag{44}$$

So either

$$x = 0 , \tag{45}$$

returning our first and last targets.

Or

$$x = \pm K^{-1} . \tag{46}$$

Returning our second and third targets. Using factorization in both cases, and use of the definition of k in the latter.

ii) The first and fourth are the zero and identity matrices. For $K = 1$, the second is 1 : of trivial kernel. While the third is 0 : of trivial image. Such triviality cannot however occur for $K \geq 2$. □

Remark 2 For $N = 2$, the two nontrivial projectors have images of equal dimension 1.

Remark 3 However for $N \geq 3$ – the nontrivial N-body problems – we have the following new combinatorial-algebraic characterization.

Principle 1 *The Lagrange matrix is the sole nontrivial largest-image combinatorial projector.*

Remark 4 Aside from the trivial identity, projectors are singular. The projector property fixes the constant of proportionality previously found in characterizing the singular combinatorial matrices.

3.3 Combinatorial involutors

Definition 1 A matrix $\boldsymbol{J}$ is a *projector* if it obeys

$$\boldsymbol{J}^2 = \mathbb{I} . \tag{47}$$

Remark 1 It is a *nontrivial involutor* if it is not the identity. So that 2 is the minimum power returning the identity.

Proposition 2 i) The only possible combinatorial involutors are

$$\pm[1, 0]_K , \quad \pm 2K^{-1}\left[1 - K2^{-1}, 1\right]_K . \tag{48}$$

ii) Among which, the last three are the nontrivial ones.

<u>Proof</u> i) Equating non-diagonal and diagonal elements in turn, combinatorial projectors obey the following pair of simultaneous quadratic equations.

$$x(Kx + 2y) = 0 , \tag{49}$$

$$x(Kx + 2y) + y^2 = 1 . \tag{50}$$

Use (50) - (49) in place of (50). I.e.

$$y^2 = 1 \Rightarrow (y + 1)(y - 1) = 0 . \tag{51}$$

So

$$y = \pm 1 . \tag{52}$$

In either case,

$$x = 0 \tag{53}$$

is possible. Factorizing, this gives the first two targets.

If not,

$$x = \mp 2K^{-1} . \tag{54}$$

Factorizing, this gives the last two targets.

ii) The first target is the identity, and is thus discarded. . □

Remark 1 For $N = 1$, the above four targets collapse to just two: ± 1 .

Remark 2 For $N = 2$, they are distinct but the last 2 become

$$\pm\mathcal{T} . \tag{55}$$

I.e. the *transpositor* (transposition matrix) up to sign.

Principle 2 For $N \geq 2$, *the matrix* $\boldsymbol{J}(N)$ *is, up to sign, the sole nontrivial combinatorial involutor.*

Remark 3 The $N = 2$ case, as the transposition, admits a more basic permutation interpretation.

The $N = 3$ case is consequently the first for which some other characterization is desirable. Flat geometry obliges: this smallest nontrivial combinatorial involutor encodes Apollonius' sides-medians length exchange.

For $N \geq 4$, we have now motivated an alternative to [25]'s $\boldsymbol{M}(N)$ matrices. The $\boldsymbol{M}(N)$ are such that the difference (25) holds. But our $\boldsymbol{J}(N)$ possess the involution property (47) itself. At the partial cost that now

$$\boldsymbol{J}(N) - \boldsymbol{P}(N) = -N^{-1}[1, 1]_N = -N^{-1}\mathbb{1} . \tag{56}$$

4 Arenas of combinatorial matrices

4.1 Algebraically special subarenas

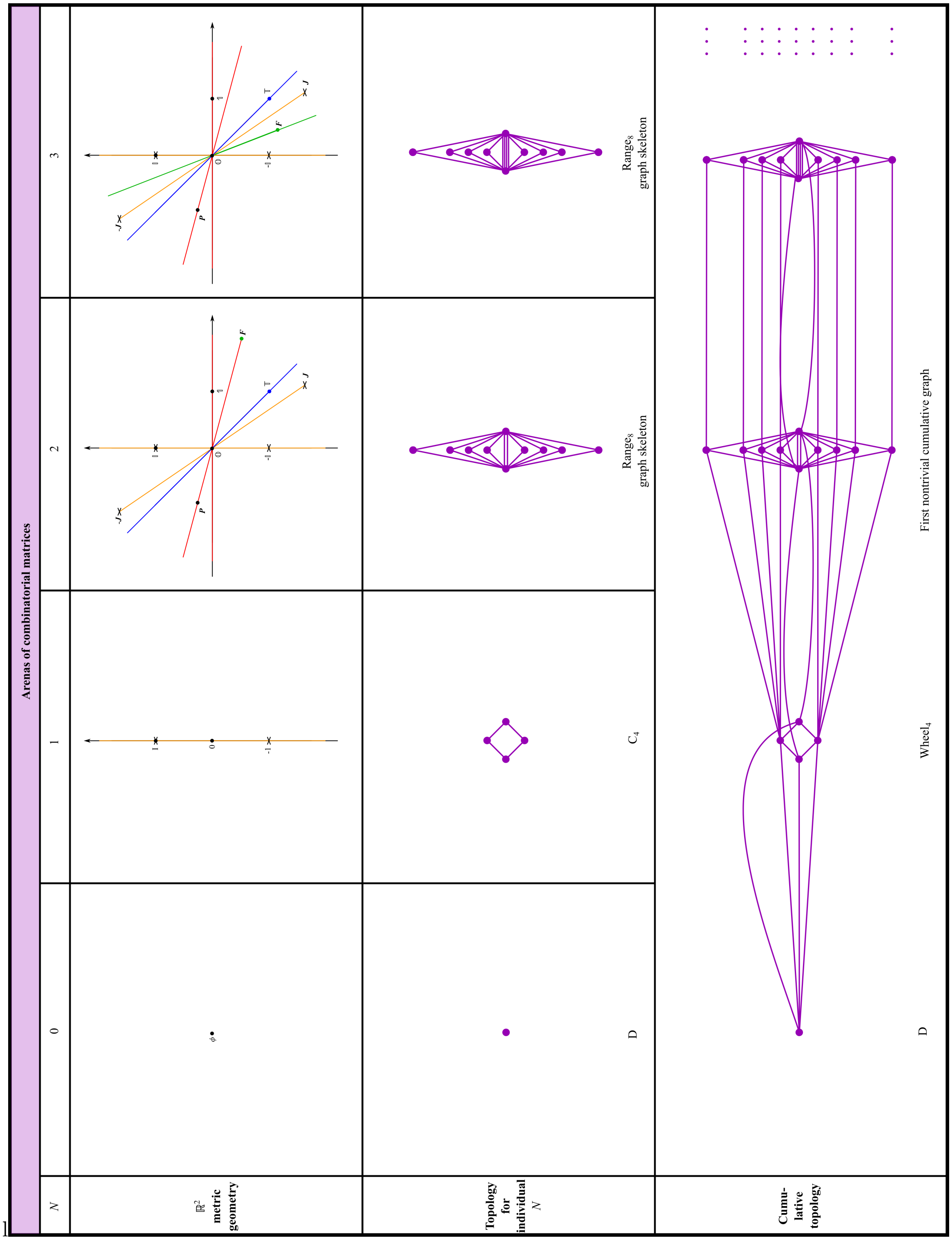

Figure 1:

Remark 1 We plot these $\mathrm{CM}(N)$ in the first row of Fig 1. Including the intersecting pair of lines of singularity, and the intersecting pair of lines of proportionality to an involution. Viewing the first of these as a determinant condition gives an invariant reason to include the zero-trace line as well. Anderson furthermore explained that the $\boldsymbol{F}(K)$ are also algebraically distinguished, as *trace-reversed* matrices. The general combinatorial matrices that enjoy this property are

$$x\,[\,-1,\,1\,]_K \;=\; x\,\boldsymbol{F}(K)\ . \tag{57}$$

These form yet another line.

All six of these lines share a common intersection point: $\mathbb{0}$. The next most special point is the identity $\mathbb{I}$, since this is both a trivial involutor and a trivial projector. There are additionally three other projectors and three other involutors, all six of which are realized as further distinct points for $N \geq 2$. And the irreducible $\mathbb{T}$ point. $\boldsymbol{F}$ is itself the final special point marked. Algebraically, this is the difference of the two irreducibles, to the block projector being their sum.

Notation 1 Expanding in terms of $\mathbb{I}$ and $\boldsymbol{F}(K)$: the trace-reversal of $\mathbb{1}$, we have

$$\langle\, 2\,x \;+\; y,\, x\,\rangle_K \tag{58}$$

in the *trace-reversed basis*. Squared brackets are nicest – irreducibles – while round and angled brackets are next-best reflections thereabout.

Exercise 3 Show that Lemma 2 extends to this, and express all of the matrices in subsections 2.3-4 in terms of this basis.

Remark 2 The first three columns of the figure present non-generic simplifications manifested by $N < 3$. $N = 2$ has 2 coincident lines. $N = 1$ is overall $\mathbb{R}$ rather than $\mathbb{R}^2$. And $N = 0$ is just a point representing the empty set. Also for $N = 3$, $\boldsymbol{F}$ exceptionally additionally enjoys the difference property (22); this is a metric-level property.

Remark 3 In formulating topological arenas $\mathrm{TopCM}(N)$ for combinatorial matrices' algebraically-distinguished subcases, the choice of including the point at infinity has been made. For $N \geq 2$, this corresponds to viewing the metric-level arenas' parameter spaces as $\mathbb{C}$ and then taking the Riemann sphere model for this. $N = 1$ then involves the corresponding compactification to a circle. Floorplan edges encode path-connectedness to stairwells' matrix padding. Such padding is a well-defined operation for combinatorial matrices. For it corresponds to adding a single row-and-column border of x 's. Excepting the placement of an $x + y$ in the corner.

4.2 Graph and order theory analysis

Remark 1 The overall order structure formed is not a poset since it contains triangles.

Remark 2 The padding operation by itself produces posets which are additionally trees. With specifics presented in Fig 2. These stabilize to homeomorphs adding 1 vertex everywhere.

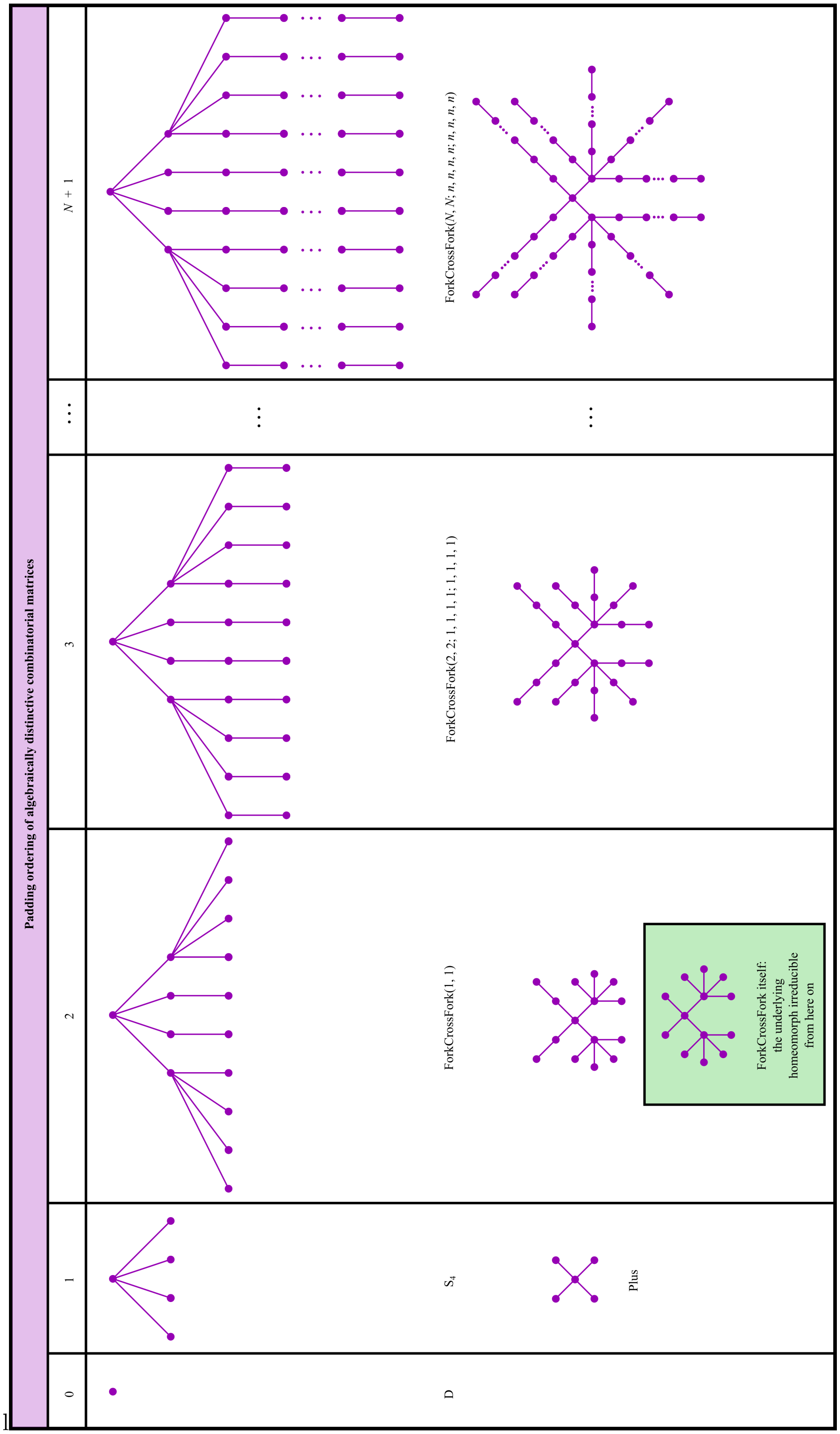

Figure 2:

Remark 3 The TopCM(N) are all planar. And stably settle down to a fixed form for $N \geq 3$. $N = 2$ suffices however for the cumulative arena TopCM[N] to be nonplanar (Fig 3).

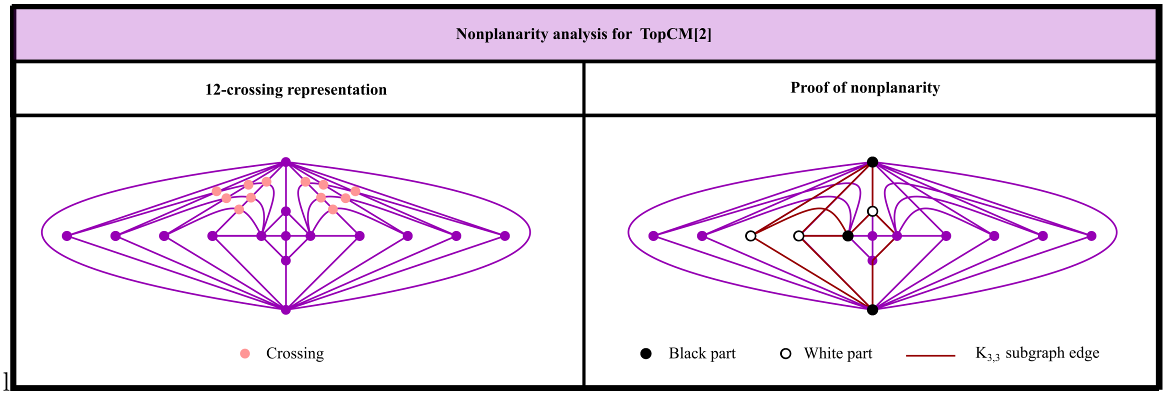

Figure 3:

5 Contrast with quadrilateral matrices

Remark 1 In contrast, in the matrix theory of cyclic quadrilaterals, the Lagrange projector [31] is compatible with a cycle of 3 Ptolemy matrices. These are not individually combinatorial. Though their sum is, returning the irreducible matrix $\mathbb{T}$. Nor are any of the accompanying Brahmagupta matrices [33] combinatorial. At the level of separations, rather, the corresponding Ptolemy involutor [30] is not combinatorial either.

Remark 2 In the matrix theory of convex quadrilaterals, the Brahmagupta matrices are supplanted by the Bretschneider involutors [34]. These are however also not combinatorial.

Remark 3 While the Ptolemy matrices are involutors, they are not the most direct analogues of the Apollonius involutor $\boldsymbol{J}$. These are instead eigenclustering length-exchange matrices [28, 29]. Though for $N = 4$ these carry 3-indices, 6-indices or both, and so are not compatible with sides matrices. Among these, $\boldsymbol{F}$ [32] and $\frac{1}{2}\boldsymbol{F}(6)$ [29] are combinatorial. Furthermore, these form a poset rather than a chain series [27].

Remark 4 In the theory of quadrilaterals, combinatorial matrices provide but an incomplete cover. Zero commutators are then not guaranteed, and indeed are in shorter supply.

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