Hopf's Little Mathematics

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1 Hopf's little map

1.1 The most primary spaces involved

Structure 1 This is a particular map

$$H \ : \ \mathbb{S}^3 \longrightarrow \mathbb{S}^2 \ . \tag{1}$$

Structure 2 There is an associated 'Cartesian Hopf map'

$$H_{\mathrm{C}} \ : \ \mathbb{R}^4 \longrightarrow \mathbb{R}^3 \ . \tag{2}$$

This an be viewed as a composition of (1) with some more basic maps, as follows.

$$\mathbb{R}^4 \ \xrightarrow{U_3} \ \mathbb{S}^3 \ \xrightarrow{H} \ \mathbb{S}^2 \ \xrightarrow{C_2} \ \mathbb{R}^3 \ . \tag{3}$$

For U_p the p-sphere *unit map*

$$\boldsymbol{x} \ \mapsto \ \hat{\boldsymbol{x}} := \frac{\boldsymbol{x}}{||\boldsymbol{x}||} \ . \tag{4}$$

And C_r the r-sphere *coning map*

$$C_r \ := \ P_r \circ q \ . \tag{5}$$

Where in turn P_r is the product map

$$\mathbb{S}^r \times \mathbb{R}_0 \ = \ \mathbb{S}^r \times [\,0, \infty\,) \ . \tag{6}$$

And q is the quotient map that squashes $\mathbb{S}^r \times \{0\}$ into a single *cone point* alias *apex* 0 . Overall,

$$C\,(\mathbb{S}^r) \ = \ (\,\mathbb{S}^r \times [\,0, \infty\,)\,)\,/^{\sim} \ . \tag{7}$$

Where the tilde indicates the above identification.

Remark 1 This topological-level cone construct produces an 'onion' of concentric layers. With a special point 0 sitting in the middle, where the 'shell radius' has collapsed to 0 . See Fig 1.

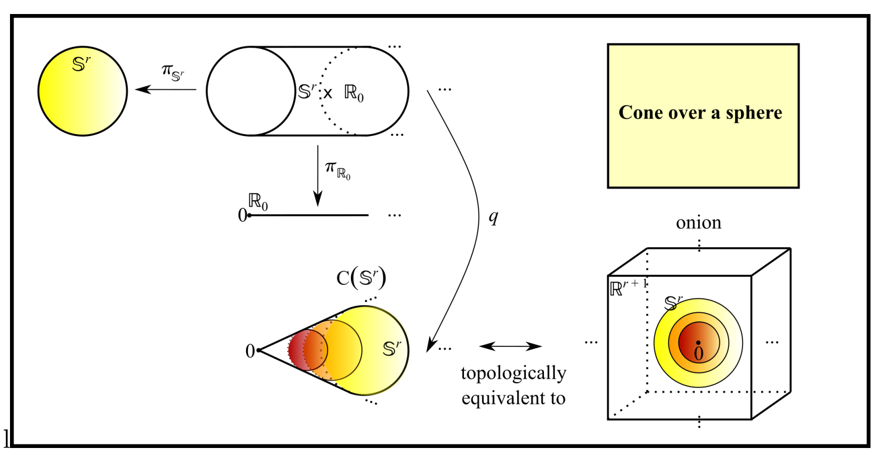

Figure 1:

Remark 2 The unit sphere map explains how to equably 'sit' $\mathbb{S}^2$ within $\mathbb{R}^3$. This is a basic result that has long been known. 'Equably' here means making equal use of the coordinates. It readily extends to how to equably 'sit' $\mathbb{S}^r$ within $\mathbb{R}^{r+1}$. This 'sitting' amounts to placing a lower-dimensional space within an *ambient* higher-dimensional space. If various technical niceties hold, Mathematicians then use the technical term [11] 'embed' rather than our conceptual and yet informal word 'sit'.

1.2 The maps themselves

Remark 1 Is there a way of equably sitting $\mathbb{S}^2$ within $\mathbb{R}^4$? Hopf [1] found an answer to this, provided that the $\mathbb{R}^4$ is viewed as

$$\mathbb{R}^2 \times \mathbb{R}^2 . \tag{8}$$

Structure 0 The following does the trick.

Firstly,

$$U = 2\,\underline{\boldsymbol{x}} \cdot \overline{\boldsymbol{y}} , \tag{9}$$

$$V = 2\,\underline{\boldsymbol{x}} \times \overline{\boldsymbol{y}} , \tag{10}$$

$$W = ||\boldsymbol{x}||^2 - ||\boldsymbol{y}||^2 . \tag{11}$$

Where the 2-d version of cross product is in play. Which can also be envisaged as the perpendicular component of a fiducial 3-d cross product,

$$(\,\underline{\boldsymbol{x}} \times \overline{\boldsymbol{y}}\,)_\perp . \tag{12}$$

(9-11) map from 2 $\mathbb{R}^2$ vectors $\boldsymbol{x}, \boldsymbol{y}$. Which jointly form a $\mathbb{R}^4$ vector

$$\boldsymbol{\xi} := \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} . \tag{13}$$

To a single $\mathbb{R}^3$ vector

$$\boldsymbol{U} := \begin{pmatrix} U \\ V \\ W \end{pmatrix} . \tag{14}$$

This is the Cartesian Hopf map. Let us thus subsequently refer to $\boldsymbol{U}$ as the *Cartesian Hopf* 3*-vector*. And to its components U, V, W as the *Cartesian Hopf coordinates.*

Proposition 1

$$||\boldsymbol{U}||^2 = ||\boldsymbol{\xi}||^4 . \tag{15}$$

<u>Proof</u> [Vector Algebra].

$$||\boldsymbol{U}||^2 = 4 \Big[\, (\underline{\boldsymbol{x}} \cdot \overline{\boldsymbol{y}})^2 + ||\underline{\boldsymbol{x}} \times \overline{\boldsymbol{y}}||^2 \, \Big] + \left(||\boldsymbol{x}||^2 + ||\boldsymbol{y}||^2 \right)^2$$

$$= 4 \Big[\, (\underline{\boldsymbol{x}} \cdot \overline{\boldsymbol{y}})^2 + ||\boldsymbol{x}||^2 \, ||\boldsymbol{y}||^2 - (\underline{\boldsymbol{x}} \cdot \overline{\boldsymbol{y}})^2 \, \Big] + ||\boldsymbol{x}||^4 - 2\,||\boldsymbol{x}||^2 \, ||\boldsymbol{y}||^2 + ||\boldsymbol{y}||^4$$

$$= ||\boldsymbol{x}||^4 + 2\,||\boldsymbol{x}||^2 \, ||\boldsymbol{y}||^2 + ||\boldsymbol{y}||^4 = \left(||\boldsymbol{x}||^2 + ||\boldsymbol{y}||^2 \right)^2 = ||\boldsymbol{\xi}||^4 . \tag{16}$$

Where the first step uses (9-11) and takes out a common factor of $2^2 = 4$. The second step uses the Lagrange identity alongside expanding the last term. The third step cancels terms, the fourth factorizes and the fifth uses (13). $\square$

Proposition 2 If $\boldsymbol{\xi}$ is a unit vector, then so is $\boldsymbol{U}$.

<u>Proof</u> Set $\boldsymbol{\xi}$ to be a unit vector using the unit map U_4 . Thus

$$||\xi||^4 = 1^2 = 1 . \tag{17}$$

Where the first step uses the on-$\mathbb{S}^3$ condition. But then (15) gives that

$$||\boldsymbol{U}||^2 \ = \ 1 \ . \tag{18}$$

I.e. the on-$\mathbb{S}^2$ condition. $\Box$

Remark 2 Thus we have arrived at Hopf's little map itself.

Remark 3 In summary, given 2 2-vectors, the following triple of functions turns out to have higher Mathematical significance. Twice the dot, twice the cross, and the difference of the two magnitudes.

Exercise 1$^-$ Repeat the above working in components. Also remove the hole in the above derivation of equability by showing that the result is unchanged if we define the difference of the two magnitudes the other way around.

2 Hopf's little mathematics

2.1 Some interesting properties

Remark 0 Cartesian Hopf coordinates are indeed Cartesian coordinates on $\mathbb{R}^3$.

Remark 1 H is a *surjection* [1] and a fortiori a *projection*. By which π is a common notation for it. This was useful in its day [1] for Algebraic Topology [7] computation.

Structure 3 Hopf's little map extends to

$$\mathbb{S}^1 \ \hookrightarrow \ \mathbb{S}^3 \ \longrightarrow \ \mathbb{S}^2 \tag{19}$$

Which is a *fibre bundle* [4, 10, 6]

$$(\,\mathfrak{F},\,\mathfrak{T},\,\mathfrak{B}\,) \ . \tag{20}$$

For *total space* $\mathfrak{T}$, *base space* $\mathfrak{B}$, and *fibre* $\mathfrak{F}$.

This is furthermore a *principal fibre bundle*. For its the *structure group* is

$$U(1) \ \cong \ SO(2) \ . \tag{21}$$

Which coincides with the fibres $\mathfrak{F}$ at the level of manifolds by

$$U(1) \ \stackrel{m}{=} \ \mathbb{S}^1 \ . \tag{22}$$

Structure 4 (19) is furthermore an example of a *fibration* [7, 6]. This is related to, and yet not identical with, it being a fibre bundle.

Structure 5 Every $\mathbb{S}^1$ fibre is linked once to every other. In other words, the *linking number* of any pair of fibres is 1 . The theory of links [5] is of topological interest, in some ways similarly to the theory of knots [5].

Remark 2 The Hopf bundle's *base space* $\mathfrak{B} \ = \ \mathbb{S}^2$ can be thought of in Differential-Geometric terms but also in stereographic projection terms. A complex formulation of the latter is useful. With the magnitude of the single complex coordinate playing the role of stereographic radial variable $\mathcal{R}$. To its phase playing the role of polar angle ϕ .

Structure 6 Projecting along the fibres, one can establish that our $\mathbb{S}^2$ carries the standard spherical metric [3, 12]. The azimuthal coordinate θ is related to the stereographic radial coordinate via the following incarnation of a venerable substitution.

$$\mathcal{R} \ = \ \tan\frac{\theta}{2} \ . \tag{23}$$

2.2 Further reading

Lyons [8] and Urbantke [9] cover a number of further interesting properties of Hopf's little mathematics. Penrose [2] devised a fine visualization for what is perhaps the main structure of Hopf's Little Mathematics that has not yet been included in the previous subsection's list. These and the first portion of [3] make for good background reading for this Summer School.

References

[1] H. Hopf, "Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche", ("On the Images of $\mathbb{S}^3$ on $\mathbb{S}^2$"), Math. Ann. (Berlin) Springer **104** 637 (1931). Reproduced in *Heinz Hopf, Collected Papers: Gesammelte Abhandlingen* ed. B. Eckmann (2001). For an Engligh translation, see Hopf's Collected Papers.

[2] R. Penrose, "The Geometry of the World', in *Mathematics Today* ed. L.A. Steen (Springer, N.Y. 1978).

[3] H. Gluck, Herman, F. Warner and W. Ziller, "The Geometry of the Hopf Fibrations." Enseign. Math. **32** 173 (1986).

[4] M. Nakahara, *Geometry, Topology and Physics* (I.O.P., London 1990).

[5] C. Livingston, *Knot Theory* (M.A.A, 1993).

[6] D. Husemoller, *Fibre Bundles* (Springer, New York 1994) Advanced Reading.

[7] A. Hatcher, *Algebraic Topology* (C.U.P., Cambridge 2001). Advanced Reading.

[8] D.W. Lyons, "An Elementary Introduction to the Hopf Fibration", Math. Magazine **76** 87 (2003), arXiv:2212.01642.

[9] H.K. Urbantke, "The Hopf Fibration Seven Times in Physics", J. Geom. Phys. **46** 125 (2003) Advanced Reading.

[10] T. Frankel, *The Geometry of Physics: An Introduction* (C.U.P., Cambridge 2011).

[11] J. Lee, *Introduction to Smooth Manifolds* (Springer, New York 2013).

[12] E. Anderson, "The Hopf Map 32 times in Geometry and Physics", forthcoming 2024.